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MATRICULATION ALGEBRA

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PREFACE TO THE THIRD EDITION.

In presenting the third edition of Matriculation Algebra to the literary public the authors beg leave to submit that in compliance with the wishes of some teachers of H. E. Schools both in Calcutta and Muffusil, the book has been divided into two parts for the convenience of the students—the 1st part being intended for the 4th and 3rd classes and the 2nd for the preparatory and Matriculation classes. The authors also express their indebtedness to the heads of institutions who have uniformly accorded their kind patronage and support to this publication and to deserve which the authors have spared no pains in this revised edition to make it more attractive and better adapted to the capacities of Indian students. The two parts have been separately priced at Re 1. each—cheap enough for a volume of 490 pages of condensed matter considering the enormous increase in the cost of production—a fact which will also recommend it to the school authorities.

CALCUTTA :
Dated the 7th December, 1918. } AUTHORS.

CONTENTS.

MATRICULATION ALGEBRA

CHAPTER I.

DEFINITIONS.

1. A **quantity** is anything that is capable of being divided into parts, for example, the length of a room, the height of a man, the area of a floor and so on.

2. Suppose we want to measure a quantity, say, the length of a bench. For this purpose we take a rod of *convenient* length and suppose we can lay it 6 times in succession along the bench. Then it follows that the length of the bench is 6 times the length of the rod ; and if the rod is one foot, the length of the bench is 6 feet or if the rod is one yard, the length of the bench is 6 yards. The statement that the length of a bench is 6 feet or 6 yards will satisfy any one who wants to know its length. Quantities are thus measured and represented by numbers which express how many times a standard quantity of the *same kind* is contained in them.

A quantity which is used to measure other quantities of the same kind is called a **unit quantity** or simply, a **unit.**

The number (integral or fractional) which indicates how often the unit is contained in a given quantity is called the **measure** of that quantity.

Thus, since, 3 yards $=9$ feet $=108$ inches, if there be a tree 3 yards high, the measure of the height of the tree

$=3$ when the unit is one yard,

$=9$ when the unit is one foot,

$=108$ when the unit is one inch ; and so on.

It thus follows that a quantity will have different measures when different units are chosen, and that for smaller units, measures will be greater and *vice-versa.*

Note. 1. If we say that the measure of a quantity is 7, it means that the quantity contains 7 units. If the unit be one seer, the quantity contains 7 seers ; if the unit be one mile, the quantity contains 7 miles ; and so on.

Note. 2. The student should observe that a quantity and the unit employed to measure it must be of the *same kind.* A length, for example, cannot be measured by a weight but must be measured by a length. The unit in any case is to be chosen from practical considerations. Thus, in measuring rice in a gunny bag we may use one seer as unit, but in measuring a cartload of coal we are likely to take one maund or even ten maunds as our unit.

Ex. 1. Find the measure of 2 mds., the unit of weight being 5 seers.

We are to answer the question—"How many times is 5 seers contained in 2 mds. or 80 seers." ? Hence the measure required $= 80 \div 5 = 16$.

Ex. 2. The measure of the area of a room is 50 when the unit is 3 sq. yds., find the area.

Area of the room = 50 units

$= 50 \times 3$ sq. yds.

$= 150$ sq. yds.

Ex. 3. If the measure of 3 ft. 4 in. be 10, what is the unit ?

We have 3 ft. 4 in. = 40 in. ; hence by the question 10 units = 40 in. or one unit = 4 in.

EXERCISE I.

1. What is a quantity ? What is a unit ? What is the relation between a quantity and the number representing it ? What is meant by saying that the measure of a quantity is 10 ?

2. Find the measure of Rs. 100, when Rs. 10 is the unit of money.

3. Find the measure of 7 hrs. 8 min. 24 sec., when the unit of time is 3 min. 24 sec.

4. Find the measure of 3 cwt. 3 qrs. 14 lbs. when the unit of weight is 2 cwt. 1 qr. 20 lbs.

5. How high in miles is a mountain whose height is represented by 96 when the unit of length is 55 yds. ?

6. The weight of a body is represented by $11\frac{2}{3}$ when the unit is $1\frac{2}{7}$ seers, how heavy is it ? By what number is the weight represented when the unit is 4 seers ?

7. An area of one acre is represented by 44, what is the unit in square yards ?

8. The sum of Rs. 111 3 as. 2 p. is represented by 50, find the unit of money.

3. Algebra is the science which treats of numbers represented by figures and letters.

The numbers in Arithmetic are represented by *figures* which have definite values, but the numbers in Algebra are also represented by the *letters* of English and Greek alphabets, *viz.*, $a, b, c, d, \ldots\ldots, \alpha, \beta, \gamma, \delta, \ldots\ldots,$ each of which may have any value we choose to assign to it.

Thus the letter a stands for a units just as 2 does for 2 units or 5 for 5 units, but with this difference that whereas 2 or 5 has a definite value, the letter a has no fixed value assigned to it. In Algebra we reason about numbers represented by the letters without supposing any particular values assigned to the letters and the results arrived at in this science are therefore more general than those in Arithmetic. For this reason Algebra has been called by Newton **Generalised Arithmetic.**

Note. In Algebra the word quantity is often used in the sense of number. The letters used in Algebra are called *signs of quantity* (also known as *algebraical symbols*).

4. The four signs *viz.* $+, -, \times, \div$, are used in the same sense in Algebra as in Arithmetic. They are called **operators** or **signs of operation**, as they are indicators of certain operations to be performed upon quantities (called **operands** or **subjects** of operations) before which they are placed.

5. The sign +. We know that $3+4$ means that 4 is to be added to 3, so $3+4+5$ means that 4 is to be added to 3 and then 5 added to the result. Similarly $a+b$ (read 'a plus b') means that the number represented by b is to be added to the number represented by a, also $a+b+c$ (read 'a plus b plus c') means that the number represented by b is to be added to the number represented by a, and then the number represented by c added to the result.

Thus if $a=2$, $b=3$, $c=4$, then $a+b=2+3=5$, also $a+b+c=2+3+4=9$.

6. The sign −. We know that $9-4$ means that 4 is to be subtracted from 9, so $9-4-3$ means that 4 is to be subtracted from 9 and 3 to be subtracted from the result. Similarly $a-b$ (read 'a minus b') means that the number b is to be subtracted from the number a, also $a-b-c$ (read 'a minus b minus c') means that the number b is to be subtracted from the number a and the number c to be subtracted from the result.

Thus if $a=12$, $b=9$, $c=2$, then $a-b=12-9=3$, also $a-b-c=12-9-2=1$.

Note. It is to be noted that when quantities are connected by the signs + and −, the order of operations is from *left to right*. Thus $a-b+c-d$ means that b is to be subtracted from a, c added to the result and d subtracted from this last result.

7. The sign $\times$. We know 4×5 means that 4 is to be multiplied by 5, also $4 \times 5 \times 6$ means that 4 is to be multiplied by 5 and the result to be multiplied by 6. Similarly $a \times b$ (read 'a into b') means that the number a is to be multiplied by the number b, also $a \times b \times c$ (read 'a into b into c') means that the number a is to be multiplied by the number b and the result to be multiplied by c.

When the multiplications are supposed to be performed in $a \times b$, $a \times b \times c$, $3 \times x \times y \times z$ etc. they are respectively written as ab, abc, $3xyz$ etc. the signs being dropped and the letters placed consecutively. The notation here is therefore different from that in Arithmetic, where for example, 5×3 cannot be written 53 which means *fifty-three*.

Note 1. The sign $\times$ is sometimes replaced by a dot both in Algebra and Arithmetic; thus $a \times b$ is written as $a.b$, 5×3 as 5.3. The student should distinguish between 5.3 (which is the same as 5×3) and $5{\cdot}3$ (which is 5 *decimal* 3), the dot in the latter case being a decimal point and placed a little higher up.

Note 2. The beginner is reminded that $0 \times$ any quantity is zero. Thus $0 \times 5 = 0$, $3 \times 0 \times 7 = 0$, and so on. Also $0 \times 0 = 0$.

8. The sign $\div$. We know $50 \div 5$ means 50 is to be divided by 5, $50 \div 5 \div 2$ means 50 is divided by 5 and the result to be divided by 2; so $a \div b$ (read 'a divided by b) means that the number a is to be divided by the number b, also $a \div b \div c$ (read 'a divided by b divided by c') means that the number a is to be divided by the number b and the result to be divided by c.

The student should note that $a \div b$ is also written as $\frac{a}{b}$ or a/b.

It is easier to print the last form; hence it is coming more and more into use.

Note 1. When quantities are connected by the signs $\times$ and $\div$ the order of operations is from *left to right*. Thus $a \div b \times c \div d$ means that a is to divided by b, the result to be multiplied by c and the result so obtained to be divided by d.

Note 2. The student will not fail to see that there it a distinction between $a \div b \times c$ and $a \div bc$. In the former, as we have seen, a is to be divided by b and the result multiplied by c, and in the latter b *is* to be first multiplied by c and a divided by the result. This is because in bc the multiplication is supposed to be already performed by the suppression of $\times$.

Note 3. The beginner is reminded that $0 \div$ any finite quantity $= 0$, thus $0 \div 5 = 0$. He is cautioned that as yet he does not know the meanings of $0 \div 0$ and any finite quantity $\div 0$, which will be given in their proper places.

9. A collection of letters and figures connected by the signs of operation is called an Algebraical **expression**, and the parts of an expression connected by the signs + and − are called its **terms**.

Thus $2x+3y$, $3a-b\times c+d\div e$ are expressions of which the first has two terms, *viz.*, $2x$, $3y$ and the second has three terms, *viz.*, $3a$, $b\times c$, $d\div e$.

An expression is called **simple** or **compound** according as it contains one or more terms.

Simple expressions are also called **monomials**, and compound expressions are called **binomials, trinomials** or **polynomials** according as they respectively contain two, three or more terms.

Thus $3a$, $4a\div b$ are simple or monomial expressions; $3a+2b$, $a\div b-c\div d$ are binomial expressions; $3a+2b+5c$, $4a+2b\times 3c-7d\div e$ are trinomial expressions.

10. When all the signs of operation $+, -, \times, \div$ occur connecting a number of quantities, we are first to perform the operations indicated by $\times$ and $\div$ and afterwards those indicated by + and −, *i.e.*, each term must be treated as a whole and found out first and afterwards additions or subtractions performed. Thus in $a\times b-c\times d$ we are to determine the term $a\times b$ by multiplying a by b, then the term $c\times d$ by multiplying c by d, and finally subtract the second term from the first.

SUBSTITUTION.

11. The question of evaluating or finding the numerical value of an expression when arithmetical numbers are substituted for the symbols involved in it is an important one, and the method of procedure in such cases is shown below.

Note. We shall use the signs $\therefore$ and $\because$ to indicate *therefore* and *because* respectively.

Ex. 1. If $a=8$, $b=4$, $c=2$, $d=3$, evaluate $3a-5b+4c-2d$.

Required value $=3\times 8-5\times 4+4\times 2-2\times 3$

$=24-20+8-6$

$=4+8-6=12-6=6.$

Ex. 2. If $a=7$, $b=5$, $c=2$, $d=1$, evaluate

(i) $30a\div 2d-3c\times 2b$ (ii) $30a\div 2d\times 3c\div 2b$

(iii) $10a\div b\times c$ (iv) $10a\div bc$

We have $30a=30\times 7=210$, $2d=2\times 1=2$,

$3c=3\times 2=6$, $2b=2\times 5=10$, $10a=10\times 7=70$.

(i) $30a\div 2d-3c\times 2b=210\div 2-6\times 10$

$=105-60=45.$

(ii) $30a \div 2d \times 3c \div 2b = 210 \div 2 \times 6 \div 10$
$= 105 \times 6 \div 10 = 630 \div 10 = 63.$

(iii) $10a \div b \times c = 70 \div 5 \times 2 = 14 \times 2 = 28.$

(iv) $10a \div bc = 70 \div (5 \times 2) = 70 \div 10 = 7.$

Ex. 3. If $l = 15$, $m = 5$, $n = 4$, $p = 0$, evaluate

(i) $3lm - 4mn + 2np + 5pl$ (ii) $\frac{5nl}{2m} - \frac{6m}{l} + \frac{np}{ml}$

(i) Reqd. value $= 3 \times 15 \times 5 - 4 \times 5 \times 4 + 2 \times 4 \times 0 + 5 \times 0 \times 15$
$= 225 - 80 + 0 + 0$, $\because$ $0 \times$ any quantity $= 0$
$= 145.$

(ii) Reqd. value $= \frac{5 \times 4 \times 15}{2 \times 5} - \frac{6 \times 5}{15} + \frac{4 \times 0}{5 \times 15}$,
$= \frac{300}{10} - \frac{30}{15} + \frac{0}{75}.$
$= 30 - 2 + 0$, $\because$ $0 \div$ any quantity $= 0$.
$= 28.$

Ex. 4. If $x = \frac{4}{5}$, $y = \frac{3}{7}$, $z = \frac{2}{9}$, evaluate

(i) $3x - 5y - z$. (ii) $xy + 3yz - \frac{z}{x}$.

(i) $3x - 5y - z = 3 \times \frac{4}{5} - 5 \times \frac{3}{7} - \frac{2}{9}$
$= \frac{12}{5} - \frac{15}{7} - \frac{2}{9} = \frac{756 - 675 - 70}{315} = \frac{11}{315}.$

(ii) $xy + 3yz - \frac{z}{x} = \frac{4}{5} \times \frac{3}{7} + 3 \times \frac{3}{7} \times \frac{2}{9} - \frac{2}{9} \times \frac{5}{4}$
$= \frac{12}{35} + \frac{2}{7} - \frac{5}{18}$
$= \frac{216 + 180 - 175}{630} = \frac{221}{630}.$

EXERCISE II.

If $a = 7$, $b = 5$, $c = 4$, $d = 2$, evaluate

1. $3a + 5b$.
2. $\frac{3b}{10} + \frac{2c}{3}$.
3. $4c - 3d$.
4. $10d - a$.
5. $12a - 5b + 7c - 9d$.
6. $7a + 8b - 3c - 5d$.
7. $5ab - \frac{3}{2}cd$.
8. $\frac{2}{5}bc - \frac{3}{7}da$.
9. $3bc + 4ad - 2cd$.
10. $2ac + 5bd - 3bc$.
11. $7ab + 3bc - 2cd + 4ad$.
12. $5ac - 2bd + 7ad - 3cd$.

If $a=18$, $b=6$, $c=3$, $d=2$, find the value of

15. $3a\times 2b$. **16.** $3a\div 2b$. **17.** $2ab\times 3cd$. **18.** $2ab\div 3cd$.

19. $12a\div 2b\times c\div d$. **20.** $12a\times 2b\div c\times d$; $12a\times 2b\div cd$.

21. $12a\div 2b\div c\times d$. **22.** $12a\div 2b\div c\div d$.

23. $\dfrac{3a+2b}{3c-2d}$. **24.** $\dfrac{5b-2c}{2a-5d}$. **25.** $\dfrac{6a-3b-2c}{9b-2c+d}$.

26. $\dfrac{3ac+5ab-ad-2bc}{3ac+5ab+ad+2bc}$. **27.** $\dfrac{abc+bcd+acd-abd}{abc+bcd+acd+abd}$.

If $a=6$, $b=3$, $c=2$, $d=1$, $x=4$, $y=5$, $z=0$, find the value of

28. $ax+by+cz$. **29.** $abx+bcy+acz$.

30. $5abcx-2bcdy+3acdz-4abdx+8dxyz$.

If $a=8$, $b=5$, $c=4$, $d=2$, find the value of

31. $\dfrac{ab}{cd}+\dfrac{bc}{ad}$. **32.** $\dfrac{ac}{bd}-\dfrac{cd}{ab}$.

33. $\dfrac{3a-2b}{c}+\dfrac{5b-6c}{d}+\dfrac{7d-2b}{a}$. **34.** $\dfrac{5b+3c}{a+3d}+\dfrac{2a+3d-3}{2b+c}$.

35. $\dfrac{a+b}{b+c}+\dfrac{b+c}{c+d}+\dfrac{c+d}{d+a}$. **36.** $\dfrac{3a+4d-8c}{2b+c+d}$.

If $a=\frac{1}{2}$, $b=\frac{2}{3}$, $c=\frac{3}{4}$, evaluate

37. $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}$. **38.** $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$.

39. $12a-4b+5c$. **40.** $10ab+6bc-2ca$.

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12. Factors. When we multiply one quantity by another, the result by a third and so on, the final result is called the **product** of the quantities and the quantities are called the **factors** of the product.

Thus abc is the product of the factors a, b and c.

Note. It will be proved afterwards that the factors of a product may be taken in any order; thus, we can write $a\times b\times c$ as abc or bca or cab etc.

13. Co-efficient. When a quantity is regarded as the product of two factors, each is called the **co-efficient** (or **co-factor**) of the other.

Thus we may regard $2\,abc$ as the product of 2 and abc or of $2a$ and bc or of $2ab$ and c, and so on.

Hence in the quantity $2abc$, 2 is the co-efficient of abc, $2a$ is the co-efficient of bc and $2ab$ is the co-efficient of c.

A co-efficient is *numerical* or *literal* as it is a figure or a letter. Thus in $3x$ the co-efficient of x *viz.*, 3 is numerical and in ax the co-efficient of x *viz*, a is literal.

A numerical co-efficient is always placed first, thus we write $3a$ and not $a3$. If a numerical co-efficient is a fraction greater than 1, it is kept in the form of an improper fraction, thus $\frac{5}{3}ab$, $\frac{13}{7}x$.

Note. The co-efficient 1 is understood; thus a means $1a$, ab means $1ab$.

14. Powers. When a quantity is multiplied any number of times by itself, the product is called a **power** of that quantity, and the quantity is called the **base** of the power.

Thus aa is called the *second power* or *square* of a and is written a^2 (read 'a squared') ; aaa is called the **third power** or **cube** of a and is written a^3 (read 'a cubed') ; $aaaa$ is called **fourth power** of a or simply "a **fourth**" and is written a^4 ; and generally, if a is multiplied together n times, the product is called the **nth power** of a or simply a **nth**, and is written a^n.

The **index** or **exponent** of any power of a quantity is the number which indicates how many times the quantity is to be multiplied together to produce the power. This number is placed above and to the right of the quantity.

Thus in a^5 the index is 5, in a^n the index is n.

Note 1. The first power of a quantity is the quantity itself so that $a^1=a$. Thus in a the index 1 is to be understood. Therefore a is really $1a^1$.

It is evident that 1 raised to any power is 1, for 1 multiplied any number of times by itself is 1. Also *zero* raised to any power gives 0.

Note 2. The student should carefully distinguish between *power* and *index*. Thus a^3 is the third power of a, while 3 is the index of the power a^3.

15. Root. A quantity which when squared is equal to a given quantity is called the **square root** of the given quantity.

The square root of a is represented by $\sqrt[2]{a}$ or more often by $\sqrt{a}$, so that $\sqrt{a}\times\sqrt{a}=a$. Thus $\sqrt{4}=2$, for $2^2=4$.

A quantity which when cubed is equal to a given quantity is called the **cube root** of the given quantity.

The cube root of a is represented by $\sqrt[3]{a}$, so that $\sqrt[3]{a}\times\sqrt[3]{a}\times\sqrt[3]{a}=a$. Thus $\sqrt[3]{27}=3$, for $3^3=27$.

Similarly a quantity which when raised to the fourth power is equal to a is called the **fourth root** of a and is written $\sqrt[4]{a}$; and in general, a quantity which when raised to the nth power is equal to a is called the **nth root** of a and is written $\sqrt[n]{a}$.

Note. The symbol $\sqrt{}$ was originally the initial letter of the word *radix* which means a root and is often called the radical sign.

It is evident that any root of 1 is also 1.

Ex. 1. Find the value of (i) a^n (ii) $\sqrt[n]{a}$ when $n=3$, $a=8$.

(i) $a^n=8^3=8\times8\times8=512$.

(ii) $\sqrt[n]{a}=\sqrt[3]{8}=2$, for $2^3=8$.

Ex. 2. If $a=16$, $b=9$, $c=4$, $d=1$, evaluate

(i) $a^2+2b^2-3c^2+4d^2$ (ii) $\sqrt{a}+2\sqrt{b}-3\sqrt{c}+4\sqrt{d}$.

(i) Reqd. value $=16^2+2\times9^2-3\times4^2+4\times1^2$

$=256+162-48+4=374$.

(ii) Reqd. value $=\sqrt{16}+2\sqrt{9}-3\sqrt{4}+4\sqrt{1}$

$=4+2\times3-3\times2+4$

$=4+6-6+4=8$.

EXERCISE III.

If $a=4$, $b=3$, $c=2$, $d=5$, evaluate

1. a^5. 2. $\frac{16}{9}a^2b^3$. 3. $2a^2b^3c^4$. 4. $\frac{27}{25}b^4c^5$.

5. $3a^2b^3+2a^3b^2$. 6. $a^2b^2-b^2c^2+c^2d^2+a^2d^2$.

7. $a^3-bc^2+2a^2d-c^3$. 8. $3a^2bc-2ab^2c+5c^2ad$.

9. $\dfrac{a^3+b^3+c^3-3abc}{a+b+c}$. 10. $\dfrac{b^2+d^2}{a+c}+\dfrac{a^2+c^2}{b+d}$.

If $a=25$, $b=16$, $c=4$, $d=3$, evaluate

11. $\sqrt{a}\times\sqrt{b}$. 12. $3\sqrt{a}+2\sqrt{c}-\sqrt[4]{b}$. 13. $\sqrt[3]{8}$.

14. $\sqrt{c^4}$. 15. $a\sqrt{a}-b\sqrt{b}+c\sqrt{c}$. 16. $\sqrt[4]{b^2}$.

If $a=\frac{4}{5}$, $b=\frac{1}{2}$, $c=\frac{2}{3}$, evaluate

17. $\frac{3}{4}a^2+\frac{2}{3}b^2$. 18. $\dfrac{b^2}{c}-\dfrac{a^2}{b}$. 19. a^2b+b^2c. 20. $b^3+c^3-a^3$.

16. Brackets. The signs ( ), { }, [ ] are called **brackets** and they are respectively distinguished as *parantheses*, *braces* and *crotchets*. When a number of quantities connected by the signs of operation are to be treated *as a whole* they are enclosed within brackets. Thus $(a+b)\times c$ or $(a+b)c$ means that $a+b$ as a whole is to be multiplied by c; and this is different from $a+b\times c$ which

means that b is to be multiplied by c and the product added to a. Similarly $(a+b)\div(c+d)$ means that the result of adding b to a is to be divided by the result of adding d to c, and this is different from $a+b\div c+d$.

Instead of the brackets enclosing a number of quantities a straight line (called the *vinculum*) drawn over them is sometimes used to serve the same purpose. Thus $a-\overline{b+c}$ means the same as $a-(b+c)$; so $\sqrt{a+b}$ is the same as $\sqrt{(a+b)}$.

Note 1. Observe that $\frac{a+b}{c}$ means $(a+b)\div c$ which may also be written $(a+b)/c$. We may state that $a+\frac{b}{c}$, $a+b/c$ and $a+b\div c$ mean the same thing.

Note 2. The student should distinguish between $\sqrt{}a+b$ and $\sqrt{(a+b)}$; in the former, square root of a is added to b, while in the latter b is added to a and the square root of the result taken. Similarly $\sqrt{}ab$ (in which the square root of a is to be multiplied by b) is different from $\sqrt{(ab)}$ (in which the square roet of ab is meant). We may note that it is sufficient to denote $\sqrt{(81)}$ by $\sqrt{81}$, as the latter cannot mean anything else.

17. We have already defined an algebraical expression. There is a distinction between an expression and a term which should be noted. Thus $3abc\,(a+b+c)$ is an expression of *one* term, of which there are four factors. *viz.*, 3, a, b, c, $a+b+c$, the last factor itself being an expression of 3 terms. Again $3a(b+c)\div cd+bcd(c+d)$ is an expression containing two terms, of which the first is $3a(b+c)\div cd$ and the second is $bcd(c+d)$.

We give below more examples on numerical substitution to make the student familiar with the subject. To evaluate any expression as we have already seen we are to find out the values of each term first, before proceeding to additions or subtractions.

Ex. 1. If $a=10$, $b=18$, $c=3$, $d=2$, evaluate

$$a+b\times c\div d-b\div(c\times d)+b\div c\times d.$$

$$\begin{aligned}\text{Reqd. value} &= 10+18\times3\div2-18\div(3\times2)+18\div3\times2\\ &= 10+54\div2-18\div6+6\times2\\ &= 10+27-3+12=46.\end{aligned}$$

Ex. 2. If $a=6$ $b=3$, $c=2$, evaluate

$$3a(b+c)\div bc+5a^2\times2b-(a^2+b^2+c^2)(c+1).$$

There are 3 terms in the expression.

Now $3a(b+c)\div bc=3\times6\times(3+2)\div(3\times2)=3\times6\times5\div6=15.$

$5a^2\times2b=5\times6^2\times2\times3=5\times6\times6\times2\times3=1080.$

$$(a^2+b^2+c^2)(c+1)=(6^2+3^2+2^2)(2+1)$$
$$=(36+9+4)(2+1)=49\times 3=147.$$

$\therefore$ Reqd. value $=15+1080-147=948$.

Ex. 3. If $a=5$, $b=3$, $c=10$, $x=1$, $y=2$, $z=4$, find the value of (i) $\sqrt{(3b+7x)}+\sqrt{(3b)}+7x$.

(ii) $\sqrt{(4ax+2cz)}+5b^3y^4\div 3x^2z$.

(i) Reqd. value $=\sqrt{(3\times 3+7\times 1)}+\sqrt{(3\times 3)}+7\times 1$
$$=\sqrt{16}+\sqrt{9}+7=4+3+7=14.$$

(ii) Reqd. value.
$$=\sqrt{(4\times 5\times 1+2\times 10\times 4)}+(5\times 3^3\times 2^4)\div(3\times 1^2\times 4)$$
$$=\sqrt{(20+80)}+(5\times 27\times 16)\div(3\times 1\times 4)$$
$$=\sqrt{100}+2160\div 12$$
$$=10+180=190.$$

Ex. 4. If $a=1$, $b=2$, $c=4$, $d=5$, evaluate
$$\frac{b^2c^2+c^2a^2+a^2b^2}{(bd)^2}+\frac{7(c+d)-3a(b+c)}{5a+2b+4c}.$$

Reqd. value $=\dfrac{2^2.4^2+4^2.1^2+1^2.2^2}{(2.5)^2}+\dfrac{7(4+5)-3\times 1(2+4)}{5\times 1+2\times 2+4\times 4}$
$$=\frac{4\times 16+16\times 1+1\times 4}{10^2}+\frac{7\times 9-3\times 1\times 6}{5+4+16}$$
$$=\frac{64+16+4}{100}+\frac{63-18}{25}=\frac{84}{100}+\frac{45}{25}$$
$$=\frac{21}{25}+\frac{45}{25}=\frac{66}{25}=2\frac{16}{25}.$$

EXERCISE IV.

If $x=4$, $y=3$, $z=2$, evaluate

1. $\frac{1}{24}(2x+3y^2)$. 2. $(2x)^2+(3y)^2$. 3. $\frac{5}{6}(5x-3y)(3y-2z)$

4. $\frac{3}{2}\sqrt{(3xyz^2)}$. 5. $\sqrt[3]{(3xy^2z)}$. 6. $\sqrt[5]{(3x^2y^4z)}$.

7. $\frac{5}{12}(xy)^2$. 8. $\sqrt{x}+5+\sqrt{(x+5)}$. 9. $\sqrt{9}x+\sqrt{(9x)}$.

10. $\sqrt{(x+y+z)}-\sqrt{x}+\sqrt{(6yz)}$.

11. $\sqrt{(x^2+y^2)}\sqrt{(y^2+z^2+xy)}\div 5xyz$.

12. $x^x+y^y+z^z$ 13. $(x^2+y^2+z^2)+(x+y+z)^2-(xy+yz+zx)$.

14. $\left(\dfrac{x+1}{y+1}\right)^2\times\dfrac{x^2+z^2}{x+z}+\dfrac{z^3+5}{y^3-7}\div\dfrac{x}{y}$.

15. $\sqrt{\left(\dfrac{2x}{2}\right)}+\sqrt[3]{\left(\dfrac{16z}{x}\right)}+\sqrt[4]{\left(\dfrac{4x}{27y}\right)}$.

If $a=12$, $b=9$, $c=6$, $d=2$, evaluate

16. $a-b+(c-d)$.
17. $a-(b-c-d)$.
18. $6a-2(5b-2c)-4d$.
19. $4a-(bc-ad)$.
20. $a^2+b^2-(c^2+d^2)$.
21. $3ab-2(ac-bd)+3(a^2-b^2)$.
22. $a^2\div b^2\times c^2\div d^2$.
23. $(3a-2b)(3b-2c)+(3c-2d)(7d-a)$.
24. $a^2/b+b^2/c+c^2/d$.
25. $(3a-2b)\div(3b-2c)+(3c-2d)\div(7d-a)$.
26. $a^2\div bc\times d^2+a^2\div(bc\times d^2)+b^2\times c^2\div ad$.
27. Find the square root of $s(s-a)(s-b)(s-c)$ where $a=9, b=40, c=41, s=45$.
28. If $a=3$, $b=4$, $c=5$, $s=6$, prove that
$$2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)=60.$$
29. Prove that the expression $x^4-10x^3+35x^2-50x+24$ vanishes when $x=1,2,3$ or 4.
30. Show that the expressions $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$ and $a^3+b^3+c^3-3abc$ are equal when $a=\frac{5}{2}$, $b=\frac{4}{3}$, $c=\frac{3}{5}$.
31. If $a=\frac{2}{3}$, $b=\frac{1}{2}$, $c=\frac{4}{5}$, evaluate
    (i) $(a+b-c)(ab-bc+ca)$.
    (ii) $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.
32. Indicate the following operations in symbols ;—
    (1) b added to a and c subtracted from the result.
    (2) The difference of a and b multiplied by c, and d added to the result.
    (3) a divided by b added to c multiplied by d, and the result subtracted from x.
    (4) a divided by b and the result divided by c.
    (5) a divided by b and the result multiplied by c.
    (6) b multiplied by c, and a divided by the result (do not use brackets).

QUESTIONS FOR EXAMINATION.

1. Define quantity and explain how a quantity is represented.
2. Distinguish between power and index, term and expression, co-efficient and factor. Give illustrations.
3. Distinguish between power and root of a quantity. Give examples.
4. Classify expressions and give examples of each class.
5. Why has Algebra been called *Generalised Arithmetic*? Explain the significance of the title.

CHAPTER II.

POSITIVE AND NEGATIVE QUANTITIES.

1. All concrete quantities can be regarded in two opposite aspects. Thus, a sum of money may be either a *gain* or a *loss*, either a *receipt* or a *payment*, either a *possession* or a *debt*; a length may be measured from a point either to the *right* or to the *left*, either *upwards* or *downwards*; similarly in case of other concrete quantities.

2. If I *gain* Rs. 300 then my income is *increased*, but if I *lose* Rs. 300, my income is *decreased*. Hence in calculating my income a *gain* of Rs. 300 may be denoted by + Rs. 300 and a loss of Rs. 300 by — Rs. 300, in accordance with the additive and subtractive meanings respectively of the signs of + and —.

Again, if I *gain* Rs. 300 then my debt is *decreased*, but if I *lose* Rs. 300, my debt is *increased*. Hence in calculating my debt a *loss* of Rs. 300 may be denoted by + Rs. 300 and a gain of Rs. 300 by — Rs 300 in accordance with the ordinary notions of the signs + and −.

In this way we can represent the *character* of a sum of money (*i.e.* whether it is a gain or loss) by prefixing the sign + and − before it. It is immaterial whether a sum *gained* is prefixed by the sign + or the sign —, for we may regard it either as increasing my income or decreasing my debt: but what is important to observe is that when a gain is prefixed by the sign +, a loss in the same investigation must be prefixed by the sign —, and *vice versa*.

3. The signs + and — are thus used to indicate the *character* of the quantities before which they are placed. In this sense they are called *signs of affection*, and must be regarded as inseparable from the quantities to which they are prefixed, so that if a quantity is moved it must carry its + or − sign before it to proclaim its character.

4. The student will mark the apparently double use of the signs + and −, *first*, as signs of addition and subtraction respectively, *secondly*, as marks of distinction between quantites of opposite or contrary nature; and he will not fail to see that the second signification is not inconsistent with but rather arises from the additive and subtractive meanings of these signs. The operations of addition and subtraction are two *opposite* processes as quantities like gain and loss are of *opposite* nature—what is done by one is undone

by the other. It is therefore proper that such quantites should be differentiated by the signs + and—.

5. Quantities to which the signs + and – are prefixed quite irrespective of the actual processes of addition and subtraction are respectively called **positive** and **negative** quantities.

If +5 refers to the number of rupees *gained*, then — 5 must refer to the number of rupees *lost*. Here +5 is a positive quantity and —5 is a negative quantity.

Thus a negative quantity – 5 may stand by itself as a positive quantity + 5 does, and – 5 means 5 units of a character opposite to the character of the 5 units denoted by +5.

6. The sign + is generally omitted before a positive quantity, so that when no sign is prefixed to a quantity the sign + is to be understood ; but the sign — must never be omitted before a negative quantity.

The signs + and – are respectively called *positive* and *negative* signs, and by the *sign* of a quantity is always understood the sign + or the sign – prefixed to it. When we are asked to change the sign of a quantity we are to change + into – and – into +.

Two quantities are said to have *like* signs when both are positive or both are negative, and to have *unlike* signs when one is positive and the other negative. Thus +3, +5 have like signs, so also –3, –5 ; while +3 – 5 have unlike signs, so also —3, +5.

7. We shall now consider other illustrations to distinguish between positive and negative quantities.

Suppose $X'OX$ is a straight road in which O is a fixed mark, and suppose a person walks along it from the *left* up to the mark O. If now he moves two miles to the *right* of O, the total distance walked by him is *increased*, but if he moves 2 miles to the *left* of O the total distance walked by him is *decreased*. Hence if we denote the distance described to the right of O by +2 miles, the distance described to the left of O is to be denoted by – 2 miles.

Again, take a centigrade thermometer. When placed in melting ice the level of the mercury stands at zero of the graduation. If now the ice is heated, the level of the mercury rises, say, to 5 degrees *above* zero, but if the ice is cooled, the level falls, say, to 5 degrees *below* zero. In the former case we may say the temperature is + 5° and in the latter case it is —5°.

Similarly, if + 10 refers to the number of minutes *past* noon, then – 10 will indicate 10 minutes *before* noon : if + 30 refers to 30 degrees of north latitude, then – 30 will refer to 30 degrees of

south latitude ; if $+500$ refers to 500 years *B. C.*,—500 will denote 500 years *A. D.* ; and so on in other cases.

8. The **absolute** or **numerical** value of a quantity is its value without reference to its positive or negative character.

Thus $+5$ has the same absolute value as -5, or $+5$ and -5 are *numerically* equal but of *opposite* character.

9. The terms positive and negative are used in connection with algebraical symbols as with arithmetical numbers : thus $+a$, $+b$ are positive ;$-a$, $-b$ are negative. $-a$ will denote a units of a character opposite to the character of the a units denoted by $+a$.

QUESTIONS FOR EXAMINATION.

1. Clearly explain the distinction between positive and negative quantities, and give illustrations.

2. If in a transaction I gain Rs. 10, then lose Rs. 14, and then gain Rs. 2, show how to express algebraically my financial position at the end of the transaction.

3. A thermometer falls to -5° and then rises to $+10^\circ$, what is meant by this statement ?

3. What is meant by each of the following :—
   (i) a gain of—Rs. 50. (ii) a loss of $-$ Rs. 50.

5. If I travel 5 miles to the north, then 7 miles to the south and next 2 miles to the north, express algebraically the distance travelled.

CHAPTER III.

PLOTTING OF A POINT.

1. The exact situation of a point in a plane is easily determined by measuring its distances from two straight lines in the plane at right angles to each other.

Suppose that the adjoining figure represents a rectangular garden whose sides are OX and OY. We can determine the position of a point and locate it exactly if we know its distances from OX and OY. Thus if a point is known to be at distances of 2 and 3 yards respectively from the lines OY and OX its position is easily determined by finding a point N in OX at a distance of 2 yds. and then going 3 yds. upwards parallel to OY (the point P, fig. 1). Again suppose we are to determine the position of a given point in the garden, say Q. For this we draw the perp. QN_1 on OX and measure ON_1 and QN_1 *i.e.* the distances of the point from the two lines OY and OX respectively, in this case 5 yds. and 6 yds. These distances determine the point Q in the garden.

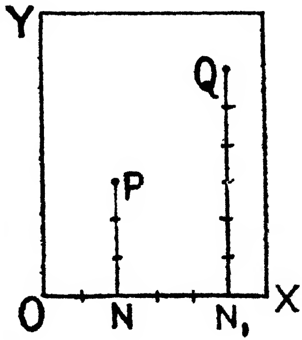


Fig. 1.

2. Let XOX′, YOY′ be two fixed straight lines in the plane of the paper intersecting at right angles at the point O and dividing the plane into four spaces XOY, YOX′, X′OY′, Y′OX called respectively the first, second, third and fourth **quadrants** (marked I, II, III, IV in figure 2). We can determine the position of a point in the plane with reference to the lines XOX′ and YOY′ which are called **axes**. The former is called the *axis of x* (*x-axis*) and the latter, the *axis of y* (*y-axis*); and their point of intersection *viz.*, O, is called the **origin**.

When there are 3 letters a b, c we shall write their sums two at a time in cyclic orders as $b+c$, $c+a$, $a+b$; so their difference two at a time as $b-c$, $c-a$, $a-b$; and their products two at a time as bc, ca, ab.

Ex. 1. Prove that

(i) $(b-c)+(c-a)+(a-b)=0$

(ii) $a(b-c)+b(c-a)+c(a-b)=0$

(iii) $(b+c)(b-c)+(c+a)(c-a)+(a+b)(a-b)-0$

(i) This is proved by mere addition.

(ii) Left-hand side $=ab-ac+bc-ab+ac-bc$, (multiplying out) $=0$.

(iii) Left-hand side $=b^2-c^2+c^2-a^2+a^2-b^2=0$.

These indentities, though easy, are important and should be remembered by the student.

Ex. 2. Find the value of

$$(b-c)(b+c-ma)+(c-a)(c+a-mb)+(a-b)(a+b-mc)$$

Here first term $=(b-c)(\overline{b+c}-ma)=(b^2-c^2)-ma(b-c)$

Second term $=(c-a)(\overline{c+a}-mb)=(c^2-a^2)-mb(c-a)$,

Third term $=(a-b)(\overline{a+b}-mc)=(a^2-b^2)-mc(a-b)$.

$\therefore$ by adding column by column, the given expression

$=(b^2-c^2+c^2-a^2+a^2-b^2)-m\{a(b-c)+b(c-a)+c(a-b)\}$

$=0-m\times 0=0$.

Ex. 3. Prove that $-(b-c)(c-a)(a-b)$

$$=a^2b-ab^2+b^2c-bc^2+c^2a-ca^2.$$

We have $-(b-c)(c-a)(a-b)=(b-c)\times\{-(c-a)(a-b)\}$

$=(b-c)\times\{(a-c)(a-b)\}$, $\because$ $-(c-a)=a-c$

$=(b-c)\{a^2-a(b+c)+bc\}$

$=a^2(b-c)-a(b^2-c^2)+bc(b-c)$

$=(a^2b-a^2c-ab^2+ac^2+b^2c-bc^2)$

$=(a^2b-ab^2+b^2c-bc^2+c^2a-ca^2)$.

We can write the result in any one of the equivalent forms

$a^2(b-c)+b^2(c-a)+c^2(a-b)$,

$bc(b-c)+ca(c-a)+ab(a-b)$,

$-\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}$.

See ex. 5, exercise XXXI.

Ex. 4. Prove that $(b+c)(c+a)(a+b)$

$$=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc.$$

We have $(b+c)(c+a)(a+b)=(b+c)\{(a+c)(a+b)\}$

$$=(b+c)\{a^2+a(b+c)+bc\}$$
$$=a^2(b+c)+a(b^2+2bc+c^2)+bc(b+c)$$
$$=a^2b+a^2c+ab^2+2abc+ac^2+b^2c+bc^2$$
$$=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc$$

The product may be written as $P+2abc$, where P stands for any one of the equivalent forms of ex. 4, exercise XXXI.

Ex. 5. Prove that $(bc+ca+ab)(a+b+c)$

$$=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+3abc.$$

We have $(bc+ca+ab)(a+b+c)=\{bc+a(b+c)\}\{a+(b+c)\}$

$$=abc+bc(b+c)+a^2(b+c)+a(b^2+2bc+c^2)$$
$$=abc+b^2c+bc^2+a^2b+a^2c+ab^2+2abc+ac^2$$
$$=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+3abc.$$

The product may be written as $P+3abc$, where P stands for any one of the equivalent forms of ex. 4, exercise XXXI.

Ex. 6. Prove that $(b+c)(c+a)(a+b)$

$$=(bc+ca+ab)(a+b+c)-abc.$$

This follows from examples 4 and 5, or thus independently :—

Let $a+b+c=s$, then $b+c=s-a$, $c+a=s-b$, $a+b=s-c$.

$\therefore\quad (b+c)(c+a)(a+b)$

$=(s-a)(s-b)(s-c)$

$=s^3-(a+b+c)s^2+(bc+ca+ab)s-abc$ [formula. VIII]

$=s^3-s.s^2+(bc+ca+ab)(a+b+c)-abc$ putting s for $a+b+c$ in the 2nd term and $(a+b+c)$ for s in the 3rd term].

$=(bc+ca+ab)(a+b+c)-abc.$

Ex. 7. Prove that

$$(a+b+c)(a+b+d)+(a+c+d)(b+c+d)-(a+b+c+d)^2=ab+cd.$$

Let $a+b+c+d=s$, then $a+b+c=s-d$, $a+b+d=s-c$.

$$a+c+d=s-b,\ b+c+d=s-a.$$

Hence the left-hand side $=(s-d)(s-c)+(s-b)(s-a)-s^2$

$$=s^2-(c+d)s+cd+s^2-(a+b)s+ab-s^2$$
$$=s^2-(a+b+c+d)s+ab+cd.$$
$$=s^2-s.s+ab+cd=ab+cd.$$

Ex. 8. Prove that $(b-c)(x-b)(x-c)+(c-a)(x-c)(x-a)$

$$+(a-b)(x-a)(x-b)=bc(b-c)+ca(c-a)+ab(a-b)$$

We have $(x-b)(x-c)=x^2-x(b+c)+bc$, Formula VI.

$$\therefore \quad (b-c)(x-b)(x-c)=x^2(b-c)-x(b^2-c^2)+bc(b-c)$$

Similarly $(c-a)(x-c)(x-a)=x^2(c-a)-x(c^2-a^2)+ca(c-a)$

$$(a-b)(x-a)(x-b)=x^2(a-b)-x(a^2-b^2)+ab(a-b)$$

Adding up column by column we find that co-efficient of $x^2=0$, so also co-efficient $x=0$; and the left-hand side of the given expression

$$=bc(b-c)+ca(c-a)+ab(a-b).$$

Ex. 9. If $2s=a+b+c$, prove that

$$s(s-a)(s-b)+s(s-b)(s-c)+s(s-c)(s-a)-(s-a)(s-b)(s-c)=abc.$$

We have $(s-a)(s-b)=s^2-s(a+b)+ab$, Formula VI.

$$\therefore \; s(s-a)(s-b)=s^3-s^2(a+b)+s.ab$$

Similarly $s(s-b)(s-c)=s^3-s^2(b+c)+s.bc$

$$s(s-c)(s-a)=s^3-s^2(c+a)+s.ac$$

Also $-(s-a)(s-b)(s-c)=-s^3+s^2(a+b+c)-s(bc+ca+ab)+abc$, by formula VIII.

$\therefore$, adding column by column, $s(s-a)(s-b)+\ldots\ldots\ldots$

$=2s^3-s^2(a+b+c)+s.0+abc$

$=2s^3-s^2.2s+abc$, putting $2s$ for $a+b+c$

$=abc.$

Ex. 10. If $2s=a+b+c$, prove that

$$2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)=abc.$$

Multiplying out, (see formulæ VIII and VI)

$2(s-a)(s-b)(s-c)=2s^3-2(a+b+c)s^2+2(bc+ca+ab).s-2abc.$

$a(s-b)(s-c)=a\{s^2-s(b+c)+bc\}=a.s^2-a(b+c)\,s+abc$

$b(s-c)(s-a)=b\{s^2-s(c+a)+ca\}=b.s^2-b(c+a).s+abc.$

$c(s-a)(s-b)=c\{s^2-s(a+b)+ab\}=c.s^2-c(a+b).s+abc.$

Adding up and observing that on the right

the co-eff. of $s^2=-2(a+b+c)+a+b+c=-(a+b+c)$,

the co-eff. of $s=2(bc+ca+ab)-a(b+c)-b(c+a)-c(a+b)=0.$

we have $2(s-a)(s-b)(s-c)+\ldots\ldots$

$=2s^3-(a+b+c)s^2+0.s+abc$

$=2s^3-2s.s^2+abc$, putting $2s$ for $a+b+c$

$=abc.$

EXERCISE XLI.

Prove that

1. $(b-c)(x+a)+(c-a)(x+b)+(a-b)(x+c)=0.$
2. $a(1+bc)(b-c)+b(1+ca)(c-a)+c(1+ab)(a-b)=0.$
3. $(b-c)(b+c+ma+n)+(c-a)(c+a+mb+n)$
$$+(a-b)(a+b+mc+n)=0.$$

Prove that

4. $(b-c)(1+ca)(1+ab)+(c-a)(1+ab)(1+bc)$
$$+(a-b)(1+bc)(1+ca)=a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2).$$
5. $(b-c)(x+a)^2+(c-a)(x+b)^2+(a-b)(x+c)^2$
$$=a^2(b-c)+b^2(c-a)+c^2(a-b).$$
6. $a(b-c)^3+b(c-a)^3+c(a-b)^3$
$$=a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3).$$
7. $a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3$
$$=a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3).$$
8. $(ax+k)(by-cz)+(by+k)(cz-ax)+(cz+k)(ax-by)=0.$
9. $a^3(b-c)+b^3(c-a)+c^3(a-b)$
$$=bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$$
$$=-\{a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)\}.$$
10. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$
$$=a^2b^2(a-b)+b^2c^2(b-c)+c^2a^2(c-a)$$
$$=-\{a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3)\}.$$
11. $(a+b)(a+c)-a^2=(b+c)(b+a)-b^2=(c+a)(c+b)-c^2$
$$=bc+ca+ab.$$
12. $(a+b)(a+c)+(b+c)(b+a)+(c+a)(c+b)-(a+b+c)^2$
$$=bc+ca+ab.$$
13. $(b-c)^2+(a-b)(a-c)=(c-a)^2+(b-c)(b-a)$
$$=(a-b)^2+(c-a)(c-b)=a^2+b^2+c^2-bc-ca-ab.$$

If $2s=a+b+c$, prove that

14. $s(s-a)+(s-b)(s-c)=bc.$
15. $(s-a)^2+(s-b)^2+(s-c)^2+s^2=a^2+b^2+c^2.$
16. $2(s-a)(s-b)+2(s-b)(s-c)+2(s-c)(s-a)$
$$=2s^2-(a^2+b^2+c^2)=2(bc+ca+ab-s^2)$$

17. $a(s-a)^2+b(s-b)^2+c(s-c)^2+2(s-a)(s-b)(s-c)=abc.$

18. Prove that

$$(a+b+c)(a-b+c)(a+b-c)(-a+b+c)$$
$$=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4.$$

19. If $a+b+c+d=2s$, prove that

(i) $(s-a)^2+(s-b)^2+(s-c)^2+(s-d)^2=a^2+b^2+c^2+d^2.$

(ii) $(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-d)+(s-d)(s-a)$
$$=ab+bc+cd+da.$$

Prove the following

20. $(a+b+c)(a+c+d)+(a+b+d)(b+c+d)-(a+b+c+d)^2$
$$=ac+bd \quad [\text{put } a+b+c+d=s].$$

21. $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2.$

22. $(a^2+b^2+c^2+d^2)(x^2+y^2)$
$$=(ax+by)^2+(cx+dy)^2+(ay-bx)^2+(cy-dx)^2.$$

23. $(a^2+b^2+c^2)(x^2+y^2+z^2)$
$$=(ax+by+cz)^2+(ay-bx)^2+(cx-az)^2+(bz-cy)^2.$$

24. $(b+c)(c+a)(a+b)=a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc.$

25. $(a+b+c)^3+a^3+b^3+c^3-(b+c)^3-(c+a)^3-(a+b)^3=6abc.$

26. $(a+b+c)(a^2+b^2+c^2)+3abc$
$$=a^3+b^3+c^3+(a+b+c)(bc+ca+ab).$$

27. $(x^2-yz)^2-(y^2-zx)(z^2-xy)=x(x^3+y^3+z^3-3xyz).$

28. $(b+c)^2+(c+a)^2+(a+b)^2-(b+c)(c+a)-(c+a)(a+b)$
$$-(a+b)(b+c)=a^2+b^2+c^2-bc-ca-ab.$$

29. $(a^2+b^2+c^2)^2-(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$
$$=2(a^4+b^4+c^4).$$

30. If $x-a=y-b=z-c$, prove that
$$x^2+y^2+z^2-yz-zx-xy=a^2+b^2+c^2-bc-ca-ab.$$

31. If $s=a+b+c$, prove that
$$s(s-2a)(s-2b)+s(s-2b)(s-2c)+s(s-2c)(s-2a)$$
$$=(s-2a)(s-2b)(s-2c)+8abc.$$

---

16. We shall now consider some *conditional identities i.e.*, relations which are true when the values of the letters involved are limited by given conditions.

Ex. 1. If $a+b+c=0$, prove that
$$a^2-bc=b^2-ca=c^2-ab=\tfrac{1}{2}(a^2+b^2+c^2).$$

Here $a+b+c=0$, $\therefore -a=b+c$, whence $a^2=(b+c)^2$......(1)

Now $2(a^2-bc)=a^2+a^2-2bc$

$=a^2+(b+c)^2-2bc$, from (1)

$=a^2+b^2+c^2$.

Similarly $2(b^2-ca)=a^2+b^2+c^2$, $2(c^2-ab)=a^2+b^2+c^2$.

$\therefore\ a^2-bc=b^2-ca=c^2-ab=\frac{1}{2}(a^2+b^2+c^2)$.

Note. Observe that when $a+b+c=0$, we obtain the following results by transposition :—

$$\begin{aligned} a+b&=-c, & a&=-(b+c)\\ b+c&=-a, & b&=-(c+a)\\ c+a&=-b, & c&=-(a+b)\end{aligned}$$

Ex. 2. If $a+b+c=0$, prove that $a^2+b^2+c^2=-2(bc+ca+ab)$.

We have $(a+b+c)^2=a^2+b^2+c^2+2(bc+ca+ab)$

$\therefore$ putting $a+b+c=0$, we get

$0=a^2+b^2+c^2+2(bc+ca+ab)$

$\therefore$ by transposition, $a^2+b^2+c^2=-2(bc+ca+ab)$,

Ex. 3. If $a+b+c=0$, prove that $a^3+b^3+c^3=3abc$.

We have $(a+b)^3=a^3+b^3+3ab(a+b)$.........(1)

Now $\because a+b+c=0$, $\therefore a+b=-c$; hence from (1)

$(-c)^3=a^3+b^3+3abc(-c)$,

or $-c^3=a^3+b^3-3abc$.

$\therefore$ by transposition, $a^3+b^3+c^3=3abc$.

This important result may be verbally stated thus :—

If the sum of three quantities be zero, then the sum of their cubes is equal to three times their product.

Obs. The results of the preceding examples are true if for a, b, c we substitute any three quantities whose sum is zero ; and in this way we can deduce many important identities.

Thus, let $a=y-z$, $b=z-x$, $c=x-y$, then evidently $a+b+c=0$. Hence from the examples 1, 2, 3 we respectively get the following results :—

(i) $(y-z)^2-(z-x)(x-y)=(z-x)^2-(x-y)(y-z)$

$=(x-y)^2-(y-z)(z-x)$

$=\frac{1}{2}\{(y-z)^2+(z-x)^2+(x-y)^2\}$

(ii) $(y-z)^2+(z-x)^2+(x-y)^2$

$=-2\{(y-z)(z-x)+(z-x)(x-y)+(x-y)(y-z)\}$

(iii) $(y-z)^3+(z-x)^3+(x-y)^3=3(y-z)(z-x)(x-y)$.

EXERCISE XLII.

If $a+b+c=0$, prove that

1. $a(a+b)(a+c)=b(b+a)(b+c)=c(c+a)(c+b)$
2. $b^2+bc+c^2=c^2+ca+a^2=a^2+ab+b^2=\frac{1}{2}(a^2+b^2+c^2)$
3. $a(b^2+c^2-a^2)=b(c^2+a^2-b^2)=c(a^2+b^2-c^2)$.
4. $ab(a+b)+bc(b+c)+ca(c+a)+3abc=0$,
5. $(bc+ca+ab)^2=b^2c^2+c^2a^2+a^2b^2=\frac{1}{4}(a^2+b^2+c^2)^2$.

[See ex. 2, art. 4.]

FORMULÆ.

The following list of the formulæ already established should be committed to memory by the students:—

I $(a+b+c+\ldots)m=am+bm+cm+\ldots$

II. $(a+b)^2=a^2+2ab+b^2$

III. $(a-b)^2=a^2-2ab+b^2$

IV. $(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab$

V. $(a+b)(a-b)=a^2-b^2$

VI. $(x+a)(x+b)=x^2+(a+b)x+ab$

VII. $(ax+b)(cx+d)=acx^2+(bc+ad)x+bd$.

VIII. $(x+a)(x+b)(x+c)$
$$=x^3+(a+b+c)x^2+(bc+ca+ab)x+abc.$$

IX. $(a+b)(a^2-ab+b^2)=a^3+b^3$

X. $(a-b)(a^2+ab+b^2)=a^3-b^3$

XI. $(a+b)^3=a^3+3a^2b+3ab^2+b^3$
$$=a^3+b^3+3ab(a+b)$$

XII. $(a-b)^3=a^3-3a^2b+3ab^2-b^3$
$$=a^3-b^3-3ab(a-b)$$

XIII. $(a+b+c)^3=a^3+b^3+c^3+3(b+c)(c+a)(a+b)$

XIV. $(a+b+c)^3=a^3+b^3+c^3$
$$+3(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc)$$

XV. $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)=a^3+b^3+c^3-3abc$.

XVI. $(a+b+c)(a-b+c)(a+b-c)(-a+b+c)$
$$=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4.$$

XVII. $-(b-c)(c-a)(a-b)$

$$=a^2b-ab^2+b^2c-bc^2+c^2a-ca^2$$
$$=a^2(b-c)+b^2(c-a)+c^2(a-b)$$
$$=bc(b-c)+ca(c-a)+ab(a-b)$$
$$=-\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}$$

XVIII. $(b+c)(c+a)(a+b)$

$$=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc$$
$$=a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$$
$$=bc(b+c)+ca(c+a)+ab(a+b)+2abc$$
$$=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc.$$

[For formulæ XVII and XVIII see examples 3 and 4, art. 15.]

CHAPTER XIII.

FACTORS.

1. In the last chapter we obtained a series of products by multiplying together two or more factors; and we here propose the inverse question of *resolving* a product into its factors. For this purpose we are to read the formulæ already established *from right to left.*

2. Expressions of the form am + bm + cm + ...

Factorization of expressions of the form $am+bm+cm+\ldots$ was considered in Chap. XII and we need only add a few examples for exercise.

EXERCISE XLIII.

Resolve into factors

1. $3a^2b^3-10ab^3$. **2.** $2xyz-6xy^2$. **3.** $8a^2bc-2abc^2$.

4. $3a^3b^4c^2-4a^2b^3c^3+5a^3b^2c^2$. **5.** $3x^4y^4-9x^3y^5+12x^5y^3$.

6. $5a^3b^2c^3d^2-10a^2b^3c^2d^3+15a^4b^2c^4d^2-20a^5b^3c^2d^3$.

7. $ab(x-a)-cd(x-a)$. **8.** $x^2(y^2+z^2)-2yz(y^2+z^2)$.

---

3. We can sometimes arrange the terms of an expression into groups, each group having the same compound factor common, and in that case the expression can be resolved into factors as in art. 2. This is a very powerful process but requires much practice for ready application.

Ex. 1. Resolve into factots $x^2+ax+bx+ab$

The expr. $=(x^2+ax)+(bx+ab)$

$=x(x+a)+b(x+a)$

$=(x+a)(x+b)$

Ex. 2. Resolve into factors $ac+bc+ad+bd$

The expr. $=(ac+bc)+(ad+bd)$

$=c(a+b)+d(a+b)$

$=(a+b)(c+d)$.

Ex. 3. Resolve into factors $10a^2-2ac-15ab+3bc$

The expr. $=(10a^2-2ac)-(15ab-3bc)$

$=2a(5a-c)-3b(5a-c)$

$=(5a-c)(2a-3b)$

EXERCISE XLIV.

Resolve into factors.

1. $ac-ad+bc-bd$.
2. $ac+ad-bc-bd$.
3. $x^2-ax+bx-ab$.
4. $a^2+ab+ac+bc$.
5. $a^2x^2+a^2+b^2x^2+b^2$.
6. $ax-bx-az+bz$.
7. $4x^2+2xy-2mx-my$.
8. $ax-2ay-bx+2by$.
9. $3x^4+3x^3-5x-5$.
10. $3a+3ax+b+bx$.
11. a^3-a^2+a-1.
12. $x^3+x^2y+xy^2+y^3$.
13. $a(x^2+y^2)+(a^2+1)xy$.
14. $ab(p^2+q^2)+pq(a^2+b^2)$.
15. $5ab^2-5abc-2b^2c+2bc^2$
16. $2px-9qy+6py-3qx$.
17. $ax-bx+cx-ay+by-cy$.
18. $a^2x+bc+abx+b^2y+ac+aby$.
19. $ab^2+abc+b^2c+bc^2+a^2b+a^2c$.
20. $(a+b)(l-mc)-(b+c)(l-ma)$.
21. $ax+by+cz+bx+cy+az+cx+ay+bz$.

4. Perfect squares. From formulæ II, III and IV we have respectively.

(i) $\mathbf{a^2+2ab+b^2=(a+b)^2}$

(ii) $\mathbf{a^2-2ab+b^2=(a-b)^2}$

(iii) $\mathbf{a^2+b^2+c^2+2bc+2ca+2ab=(a+b+c)^2}$.

We may factorize directly thus :

(i) $a^2+2ab+b^2=(a^2+ab)+(ab+b^2)$
$$=a(a+b)+b(a+b)$$
$$=(a+b)(a+b)=(a+b)^2$$

(ii) $a^2-2ab+b^2=(a^2-ab)-(ab-b^2)$
$$=a(a-b)-b(a-b)$$
$$=(a-b)(a-b)=(a-b)^2$$

(iii) $a^2+b^2+c^2+2bc+2ca+2ab$
$$=a^2+2a(b+c)+(b+c)^2$$
$$=\{a+(b+c)\}^2=(a+b+c)^2.$$

Ex. 1. Find the value of $4a^2+28ab+49b^2$, when $a=7$, $b=-2$.

The expr. $=(2a)^2+2.2a.7b+(7b)^2$
$$=(2a+7b)^2=\{2.7+7(-2)\}^2=0$$

Ex. 2. Find the value of $64x^2-48xy+9y^2$, when $x=4, y=10$.

The expr. $=(8x)^2-2.8x.3y+(3y)^2$
$$=(8x-3y)^2=(8\times 4-3\times 10)^2=2^2=4.$$

Ex. 3. Simplify $(y+z-x)^2+2(y+z-x)(z+x-y)+(z+x-y)^2$.

Putting $y+z-x=a$, $z+x-y=b$, the given expr.
$$=a^2+2ab+b^2=(a+b)^2$$
$=\{(y+z-x)+(z+x-y)\}^2$, restoring x, y, z.
$$=(2z)^2=4z^2.$$

Ex. 4. Simplify

$(3a+2b+c)^2-2(3a+2b+c)(a+2b+c)+(a+2b+c)^2$.

Putting $3a+2b+c=x$, $a+2b+c=y$.

the given expr. $=x^2-2xy+y^2=(x-y)^2$

$=\{(3a+2b+c)-(a+2b+c)\}^2$, restoring a, b, c.
$$=(2a)^2=4a^2.$$

Ex. 5. Evaluate $a^2+b^2+c^2-2bc+2ca-2ab$,

when $a=26$, $b=21$, $c=5$.

The expr. $=(a-b+c)^2$
$$=(26-21+5)^2=10^2=100.$$

Ex. 6. Prove that $(y+z)^2+(z+x)^2+(x+y)^2+2(y+z)(z+x)$
$+2(z+x)(x+y)+2(x+y)(y+z)=4(x+y+z)^2$.

Putting $y+z=a$, $z+x=b$, $x+y=c$,

the left-hand side $=a^2+b^2+c^2+2ac+2cb+2ab$
$=(a+b+c)^2$
$=(y+z+z+x+x+y)^2$
$=\{2(x+y+z)\}^2$
$=4(x+y+z)^2$.

Ex. 7. Prove that

$(a+b)^2+2(a+b)c+c^2=(b+c)^2+2(b+c)a+a^2$.

Putting $a+b=x$, the left side
$=x^2+2xc+c^2=(x+c)^2$
$=(a+b+c)^2$, substituting for x.

Again, putting $b+c=y$, the right side
$=y^2+2ya+a^2=(y+a)^2$
$=(a+b+c)^2$, substituting for y.
$\therefore$ the two sides are equal.

Ex. 8. Factorize $a^2-2ab+b^2-2a+2b$.

The expr. $=(a^2-2ab+b^2)-(2a-2b)$
$=(a-b)^2-2(a-b)$
$=(a-b)(a-b-2)$.

5. Supppose we want to determine the quantity which when added to the expression a^2+2ab makes it a perfect square. Now we know that a^2+2ab *plus* b^2 is a perfect square, *viz.* $(a+b)^2$; hence the quantity required is b^2 or *square of half the co-efficient of a.*

Ex. What must be added to a^2+3a to make it a perfect square?

$a^2+3a=a^2+2.a.\frac{3}{2}$; hence we must add $(\frac{3}{2})^2$.

Ex. 2. What must be added to $9x^2-5xy$ to make it a perfect square?

$9x^2-5xy=(3x)^2-2.3x.\dfrac{5y}{6}$; hence we must add $\left(\dfrac{5y}{6}\right)^2$.

EXERCISE XLV.

Find the factors of (mentally) :—

1. x^2+2x+1.
2. x^2-4x+4.
3. $9x^2+6x+1$.
4. $a^2-10ab+25b^2$.
5. $9-12x+4x^2$.
6. $9x^2-24xy+16y^2$.

Find the factors of (mentally) :—

7. $1-2xy+x^2y^2$. **8.** $x^4-6x^3y+9x^2y^2$. **9.** $a^4-2a^2bc+b^2c^2$.

10. $\frac{9x^2}{16}-2xy+\frac{16y^2}{9}$. **11.** $\frac{a^2}{9}+\frac{ab}{3}+\frac{b^2}{4}$.

12. $9a^6-18a^3b^3+9b^6$. **13.** $a^2x^2-2abxy+b^2y^2$.

14. $16x^2-24ax+9a^2$. **15.** $4x^2y^2z^2-4xyz+1$.

16. $(5x+y)^2+8(5x+y)+16$. **17.** $4(2a-3b)^2-4(2a-3b)+1$.

18. $(x+y)^2+2(x+y)(x-y)+(x-y)^2$.

Factorize

19. $a^2+2ab+b^2+2bc+2ca$. **20.** $4x^2-12xy+9y^2-2x+3y$.

21. $4-4a+a^2-2b+ab$. **22.** $25x^2-10x+1+5ax-a$.

Find the value of

23. $4x^2+20xy+25y^2$, when $x=8, y=-3$.

24. $4x^2y^2+12xyz+9z^2$, when $x=2, y=-5, z=10$.

25. $a^2+b^2+c^2+2bc-2ca-2ab$, when, $a=37, b=21, c=16$.

26. $4x^2+9y^2+16z^2-12xy+24yz-16zx$, when $x=16, y=4, z=2$.

27. $4x^2-4(xy-xz)+(y-z)^2$, when $x=19, y=57, z=12$.

28. If $x=b+c, y=c-a, z=a-b$, prove that

$x^2+y^2+z^2-2xy+2yz-2zx=4b^2$. (C. E. 1883).

Simplify

29. $(5a+7b)^2+2(5a+7b)(4a-9b)+(4a-9b)^2$.

30. $9(16x-5y)^2-12(16x-5y)(9x-2y)+4(9x-2y)^2$.

31. $(3a+7b-8c)^2+2(3a+7b-8c)(7c-3a-5b)+(7c-3a-5b)^2$

32. Prove that $(a^2+b^2)(c^2+d^2)-(ac+bd)^2$ is a perfect square.

33. Prove that $(x+3y)^2+2(x+3y)(x+5z)+(x+5z)^2$
$=(2x+z)^2+2(2x+z)(3y+4z)+(3y+4z)^2$.

34. Prove that $9(a+b)^2+12(a+b)(a+2c)+4(a+2c)^2$
$=16(b+c)^2-8(b+c)(b-5a)+(b-5a)^2$

35. Prove that $(a+2b)^2+(b+2c)^2+(c+2a)^2+2(a+2b)(b+2c)$
$+2(b+2c)(c+2a)+2(c+2a)(a+2b)=9(a+b+c)^2$

What must be added to each of the following to make it a perfect square ?

36. x^2+4x. **37.** x^2+7x. **38.** x^2+px.

39. $4x^2+7x$. **40.** $25x^2+13x$. **41.** a^2x^2+2abx.

From any point P in the plane draw PN perpendicular to XOX′. Then we can reach the point P by travelling from O (the starting point, hence called the origin) to N along the *axis* of x and then from N to P parallel to the *axis* of y, or the two roads ON, NP at right angles will lead us from O to P. Thus the lengths ON and NP fix the position of the point P in the plane and are called its **co-ordinates**, ON alone being called the **abscissa** and NP alone being called the **ordinate**, of the point.

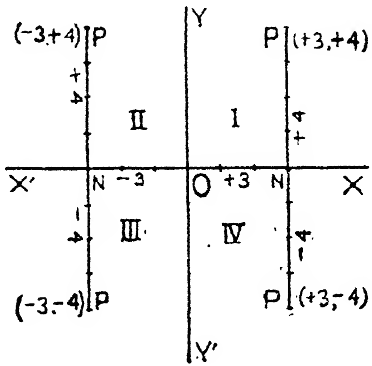


Fig. 2.

Now in reaching P we may have to describe the part ON of our journey along the *axis* of x either to the *right* or *left* of O, and the part NP either *upwards* or *downwards* parallel to the *axis* of y, as shown in fig. (2) in quadrants I, II, III and IV respectively.

To distinguish between these cases the following convention is used : ON is regarded positive when it is described (along the *axis* of x) to the *right* of O, as in Quadrants I and IV (fig. 2) ; and consequently ON is regarded negative when it is described to the *left* of O, as in Quadrants II and III (fig. 2). Also NP is regarded positive when it is described (parallel to the axis of y) *upwards*, as in Quadrants I and II (fig. 2), and consequently NP is regarded negative when it is described *downwards*, as in Quadrants III and IV (fig. 2).

Thus if the absolute values of ON and NP are 3 and 4 respectively, we have

the abscissa and ordinate of P in the first quadrant (fig. 2) $+3, +4$ respectively ;

the abscissa and ordinate of P in the second quadrant (fig. 2) $-3, +4$ respectively ;

the abscissa and ordinate of P in the third quadrant (fig. 2) $-3, -4$ respectively ;

the abscissa and ordinate of P in the fourth quadrant (fig. 2) $+3, -4$ respectively,

It must not be supposed that the straight lines XOX′ and YOY′ must be at right angles in order to be axes ; the axes of co-ordinates may be oblique *i.e.*, not at right angles. Distances parallel to XOX′ *i.e*, x-axis are called x (abscissæ) and distances parallel to YOY′ *i.e.*, y-axis are called y (ordinates) in all cases. In this book we shall everywhere consider axes to be rectangular.

3. In writing the co-ordinates of a point the abscissa is always written first and then the ordinate ; and the point whose abscissa and ordinate, are respectively, say 4 and 5 units, is briefly designated as the point (4, 5).

A point is named (a, b) when its abscissa $x=a$ and its ordinate $y=b$.

It is found from the preceding figure that the abscissa of a point is positive when it is in the first and fourth quadrants, and negative in the second and third ; while the ordinate is positive in the first and second quadrants and negative in the third and fourth.

Thus in the first quadrant x and y are both $+$, in the second x is $-$ and y is $+$.

In the 3rd quadrant x and y are both $-$ and in the 4th x is $+$ and y is $-$.

It is plain from the figure

(1) That the point (O, O) is the origin *i,e.*, the co-ordinates of the origin are (O, O).

(2) That if a point is on the x-axis *i.e.*, on XOX′, its $y=0$ *i.e.* ordinate is O.

(3) That if a point is on the y-axis *i.e.* on YOY′ its $x=0$ *i.e.* abscissa is O.

(4) That x-co-ordinate of a point on a line parallel to the axis of y at a distance a from it is a.

(5) That y-co-ordinate of point on a line parallel to the axis of x at a distance b from it is b.

4. When the co-ordinates of a point are given we can mark the position of the point with reference to the axes ; this process of marking of the position of a point is called **plotting** the point.

For the purpose of plotting we commonly use squared paper ruled by two sets of perpendicular lines at intervals of one tenth of an inch with every fifth or tenth line more thickly printed. The paper is thus divided into small squares of which each side is one-tenth of an inch. Any two of the thick lines which intersect may be taken as axes of reference and one or more sides of the small squares as the unit of length.

Ex. 1. Plot the following points :—

(i) (2, 3) (ii) (—3, 4), (iii) (4,—4). (iv) (—2,—5).

Use inch squared paper and take the unit of length to be one side of the small squares (*i.e.* one tenth of an inch).

(*i*) Here the abscissa of the point is +2 units and the ordinate is + 3 units. Hence we measure 2 units to the *right* of O along the axis of x, and then 3 units *upwards* parallel to the axis of y. We thus reach the point P (fig. 3) and this is the point (2, 3).

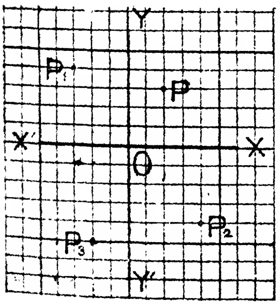


Fig. 3.

(*ii*) Here the abscissa of the point is—3 units and the ordinate is +4 units. Hence we measure along the axis of x, 3 units to the *left* of O and 4 units *upwards* (parallel to the axis of y). Then P_1 (fig. 3) is the point (—3, 4).

(*iii*) Here we measure 4 units along the axis of x to the *right* of O and then measure 4 units *downwards* parallel to the axis of y. We thus reach the point P_2 (fig.3) which is (4,—4).

(*iv*) Here we measure 2 units along the axis of x to the *left* of O and then 5 units *downwards* parallel to the axis of y. We thus reach the point P_3 (—2,—5) in (fig. 3.)

Ex. 2. Plot the points :—(*i*) (—3, 0), (*ii*) (0, – 3), (*iii*) (0, 0).

(*i*) Taking the same unit as in Ex. 1, here we measure OP along the axis of x equal to 3 units to the *left* of O and stop there without going up or down, as the ordinate is zero. Hence P is the required point on the axis of x (fig. 4).

Thus *a point whose ordinate is zero lies on the axis of x.*

(*ii*) We measure OP equal to 3 units, along the axis *y downwards* and stop there as the abscissa is o.

Then P is the point (o,—3) on the axis of y. (fig. 4).

Thus *a point whose abscissa is zero lies on the axis of y.*

(*iii*) The point whose abscissa is zero as well as the ordinate is zero is evidently the orgin O.

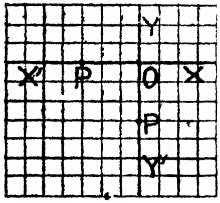


Fig. 4.

5. In plotting points on squared papers, when the co-ordinates of a point are not whole numbers, the unit of length is to be chosen properly. In case of fractional co-ordinates the unit of length may be conveniently taken equal to as many sides of the small squares as are equal to the L. C. M. of the denominators of the fractions. Similarly, when the co-ordinates are expressed to one place (or two places) of decimals it is convenient to take the unit of length to be ten sides of the small squares.

It is to be noted that in showing the relative positions of two or more points with reference to the axes they must be plotted with the same unit of length.

The unit of length should be always stated : thus, we write "scale 1″ = 1" or "scale ·1″ = 1" according as we take the unit to be 1″ or ·1″.

(1″ stands for 1 in., ·1″ stands for ·1 in.)

Ex. 1. Plot the points (*i*) $(1, -\frac{1}{4})$ (*ii*) $(-\frac{1}{2}, \frac{5}{8})$.

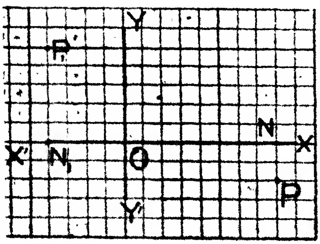


Fig. 5.

Take the unit of length to be 8 sides of the small squares, 8 being the L. C. M. of the denominators of the fractional co-ordinates above.

(*i*) Here abscissa = 1 unit = 8 sides, and ordinate = $-\frac{1}{4}$ unit = -2 sides. Hence take ON to the *right* of O equal to 8 sides of the squares and NP *downwards* equal to 2 sides. We thus get the point $P = (1, -\frac{1}{4})$ (fig. 5).

(*ii*) Here abscissa $= -\frac{1}{2}$ unit $= -4$ sides of the squares, and ordinate $= \frac{5}{8}$ unit $= 5$ sides of the squares. Hence take N_1 to

the *left* of O equal to 4 sides and N_1P_1 *upwards* equal to 5 sides. We thus get the point $P_1 = (-\frac{1}{2}, \frac{5}{8})$ (fig. 5).

Ex. 2. Plot the points (*i*) (1·7, – 1·3). (*ii*) (·9, ·35)

Take the unit of length to be ten sides, that is, let the scale be 1″ = 1.

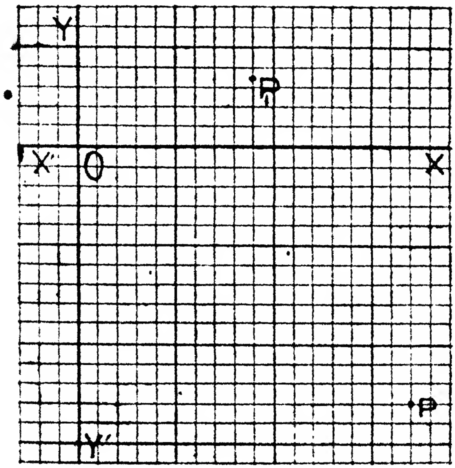


FIG. 6.

(*i*) Measure along the axis of x to the *right* of O, 1·7 units *i.e*, 17 sides of the small squares, and then measure downwards 1·3 units or 13 sides of the small squares. Then we reach the required point P (fig. 6).

(*ii*) Measure on to the right of O along the axis of x equal to ·9 unit (or 9 sides of the small squares) and then measure upwards ·35 unit (or 3·5 sides of the small squares). Then we reach the required point P_1 (fig. 6).

Note. In measuring 3·5 sides of a square we take 3 sides and (by guess) half of a side.

6. The following are some examples on the use of squared paper.

Ex. 1. If I walk 3 miles to the east and then 4 miles to the north, find how far I am from the starting point.

Let O represent the starting point and take OX, OY drawn to the east and north respectively as the axes of x and y. Let each division of the paper denote one mile. Then my first position is N and final position P (fig. 7). Draw an arc of a circle with centre O and radius OP, cutting OX at Q. Then $OP = OQ = 5$ miles. Hence I am 5 miles from the starting point.

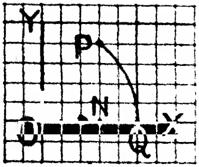


Fig. 7.

Ex. 2. Plot the points $(\frac{5}{9}, \frac{2}{3})$ and $(-\frac{7}{9}, \frac{1}{9})$ and find their distance apart.

Take the unit of length equal to 9 divisions of the paper. Then the abscissa and ordinate of the first point $=5$ and 6 divisions of the paper respectively; while the abscissa and ordinate of the second point $=-7$ divisions and 1 division respectively. Plot these points, A and B (fig. 8). With centre A and radius AB draw a circle cutting at C the line through A parallel to the axis of y. Then $AB = AC = 13$ divisions of the paper. Hence as 9 divisions $=1$ unit, the required distance $=\frac{13}{9}$ units $=1\frac{4}{9}$ units.

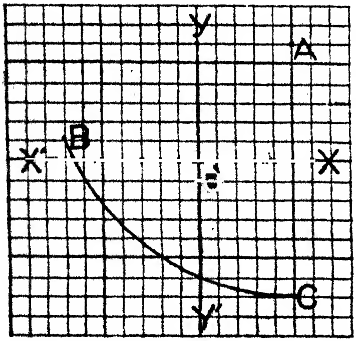


Fig. 8.

Ex. 3. Plot the points $(4, 4)$, $(-2, 4)$, $(-2, -1)$, $(4, -1)$. Prove that they are the angular points of a rectangle and find its area.

Taking one side of a small square $=1$, and plotting the points in order we find them to be A, B, C, D, (fig. 9), and $ABCD$ is evidently a rectangle. The area of the rectangle obtained by

counting squares is 30 units of area or ·30 square inch, since ·1″ = unit length and ∴ ·01 sq. in. = unit area.

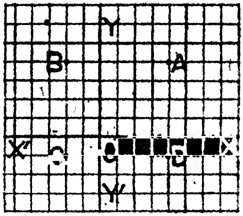


Fig. 9.

Ex. 4. The angular points of a triangle are (3, 3), (—5, 1) (6, − 3) ; plot the points and find the area of the triangle by counting squares of the paper.

Take the scale ·1″ = 1 and plot the points in order,—the points *A*, *B*, *C* respectively (fig. 10). In counting squares we take the part of a square which is greater than one-half to be *one* and the part which is less than one-half to be *zero*. We thus find the area to be 27 square units or ·27 sq. in.

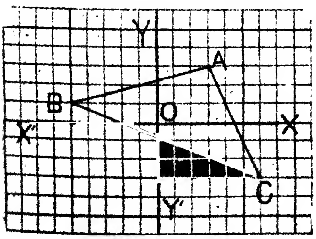


Fig. 10.

EXERCISE V.

1. Plot the following points on a black board ruled in squares :—

(2, 3) ; (−4, 5) ; (−2, 3) ; (−1, −4) ; (3, −3) ;
(−3, 0) ; (0, −5) ; (0, 4) ; (7, −2) ; (−5, 4) ;
(−6, 8) ; (−7, 6) ; (−5, −8) ; (5, 0) ; (−6, −6) ;
(2, −5) ; (5, −2) ; (−2, −5) ; (−5, −2) ; (−7, 1).

2. Name the co-ordinates of all points in the following diagram (fig. 11) :

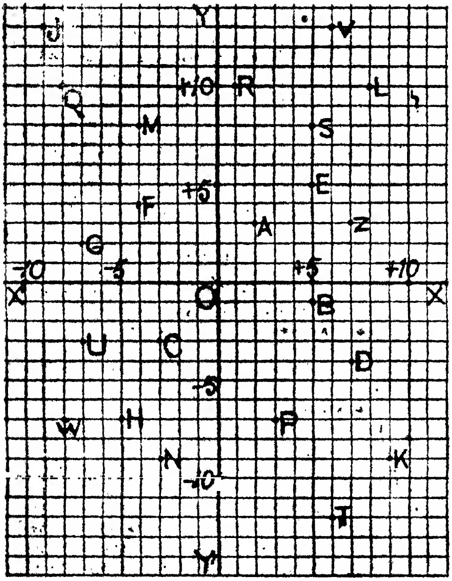


Fig. 11.

3. Plot the points $(1, 2)$, $(2, 4)$, $(3, 6)$, $(-4, -8)$, $(-5, -10)$ and show that they all lie on a straight line through the origin.

4. Plot the points $(0, 6)$, $(4, 0)$ $(2, 3)$, $(6, -3)$, $(8, -6)$, $(-2, 9)$, $(-4, 12)$ and prove that they all lie on a straight line.

5. Plot the following pairs of points and find the approximate distance between each pair :—

$(3, -4)$, $(-3, 5)$; $(2, 6)$, $(1, -5)$; $(7, 8)$, $(-6, 4)$.

6. A ship sails 50 miles to the north, then 60 miles to the east and next 40 miles to the south ; find approximately her final distance from the starting point.

7. Plot the points $(4, 5)$, $(-4, 5)$, $(4, -5)$, $(-4, -5)$. Prove that they are equally distant from the origin and form a rectangle of which the area is 80 square units.

8. Plot the following pairs of points and find the approximate distance between each pair :—

(1·5, 2·8), (2·6, −1·9) ; (−2·7, −1·8), (−1·8, 2·1) ; (·7, −1·4), (−1·7, 1·4) ; (·8, −·9), (1·1, 2·3).

9. Plot the following pairs of points and find the approximate distance between each pair :—

$$\left(\tfrac{1}{2}, -\tfrac{3}{4}\right), \left(-\tfrac{3}{2}, 2\right) ; \left(-\tfrac{2}{3}, \tfrac{1}{9}\right), \left(\tfrac{4}{9}, -\tfrac{5}{9}\right).$$

10. Plot the following series of points :—

(*i*) (6,0), (6,1), (6,2) (6,3), (6,−1), (6,−4) ;

(*ii*) (7,−3) (5,−3) (4,−3) (1,−3) (0,−3).

Show that they lie on two straight lines respectively parallel to the axis of y and x at the distances of 6, −3 respectively.

11. Explain by means of a diagram that the points (a,b) (a,c) (a,d), (a,e) and (a,f) lie on a straight line parallel to the axis of y at the distance of a from it ; and that the points (a,b), (c,b) and (d,b) lie on a straight line parallel to the axis of x at the distance of b from it.

12. A man walks 1·8 miles to the south and then 2·4 miles to the east, find his approximate distance from the starting point.

13. Plot the points (7,−3) (4,8), (−7,4) and find the area of the triangle formed by them by counting squares.

14. The sides of a right-angled triangle are 2·3 ft. and 2·9 ft., find the length of the hypotenuse by using squared paper.

CHAPTER IV.

GENERALISED ARITHMETIC.

1. In this chapter when we shall have occasion to illustrate positive and negative quantities, we shall do so by referring to gain and loss—representing gain as positive and consequently loss as negative.

ADDITION.

2. The process of finding the result when two or more quantities are taken together is called **addition**, and the result is called the **sum**. The quantities which are taken together are called *addends* or *summands*.

3. How to add any quantity. (I) A *gain* of Rs. 5 produces an *increase* of Rs. 5 in my income, or (+Rs. 5) *added* to my income increases it by Rs. 5.

Hence $+(+5)$ is equivalent to $+5$;......(i)
and similarly in other cases.

Thus when $+5$ is added to $+7$, we have

$$\left.\begin{array}{ll} & (+7)+(+5)=+7+5\text{.........}(1) \\ \text{Similarly,} & (-7)+(+5)=-7+5\text{.........}(2) \end{array}\right\}\text{...(A)}$$

(II) Again a *loss* of Rs. 5 produces a *decrease* of Rs. 5 in my income or (– Rs. 5) *added* to my income decreases it by Rs. 5.

Hence $+(-5)$ is equivalent to -5 ;......(ii)
and similarly in other cases.

Thus when -5 is added to $+7$, we have

$$\left.\begin{array}{ll} & (+7)+(-5)=+7-5\text{.........}(3) \\ \text{Similarly,} & (-7)+(-5)=-7-5\text{.........}(4) \end{array}\right\}\text{...(B)}$$

From the results (A) and (B) we get the following rule for adding *any* quantity :—

To add a quantity, positive or negative, simply attach it by its own sign to the quantity (positive or negative) to which it is to be added.

Def. The Algebraical sum of two (or more) quantities is the sum of those quantities taken with proper signs.

Thus the algebraical sum of 5 and -3 is $5+(-3)=5-3$, and this is the *arithmetical difference* (*i.e.*, the difference when the smaller is taken from the greater) of 5 and 3.

SUBTRACTION.

4. The process of finding the result when one quantity (called the **subtrahend**) is taken from another (called the **minuend**) is called **subtraction**, and the result is called **difference** or **remainder**.

Subtraction is thus the inverse of addition and to subtract one quantity from another is to find a quantity which when added to the first gives the second.

5. How to subtract any quantity. (I) A *gain* of Rs. 5 taken away from my income produces a *loss* of Rs. 5 in it.

Hence subtracting $+5$ is equivalent to adding -5

or $-(+5)=-5$......(iii)

Thus when $+5$ is subtracted from $+7$ we get

$$\left.\begin{array}{ll} & (+7)-(+5)=+7-5\ldots\ldots(1) \\ \text{Similarly,} & (-7)-(+5)=-7-5\ldots\ldots(2) \end{array}\right\}\ldots\ldots(C)$$

(II) Again, a *loss* of Rs. 5 taken away from my income produces a *gain* of Rs. 5 in it. Hence subtracting -5 is equivalent to adding $+5$

Or, $-(-5)=+5$........(iv).

Thus when -5 is subtracted from $+7$ we get

$$\left.\begin{array}{ll} & (+7)-(-5)=+7+5\ldots\ldots(3) \\ \text{Similarly,} & (-7)-(-5)=-7+5\ldots\ldots(4) \end{array}\right\}\ldots\ldots(D)$$

From the results (C) and (D) we get the following rule for the subtraction of *any* quantity :—

To subtract any quantity, positive or negative, add its opposite (*i.e.* attach it with its sign changed) to the quantity, positive or negative, from which it is to be subtracted.

6. If we use symbols in the result (i), (ii), (iii) and (iv) of arts 3 and 5, we get respectively.

$$+(+b)=+b\ldots\ldots\ldots(1)$$
$$+(-b)=-b\ldots\ldots\ldots(2)$$
$$-(+b)=-b\ldots\ldots\ldots(3)$$
$$-(-b)=+b\ldots\ldots\ldots(4)$$

In the above results we have assumed b to stand for any *positive* number. Now it is very necessary that algebraical symbols should not be restricted to positive values and we shall presently prove that the above results are true for negative values also of b.

Thus let b stand for -3, then $+b=+(-3)=-3$ and $-b=-(-3)=+3$. Hence (1) and (2) will be true for the negative value -3 of b if $+(-3)=-3$ and $+(+3)=+3$ are true, and the latter are so, for they respectively follow from (2) and (1) for the positive value 3 of b. Similarly the results (1) and (2) are true for all negative values of b; and so also the results (3) and (4).

We conclude from the above that the following *rule of signs* holds good in these cases :—

Two like signs in conjunction give plus, two unlike signs in conjunction give minus.

Note. In connection with algebraical symbols the student will observe that the terms positive and negative refer to their *apparent* and not *essential* characters. Thus if we suppose a to stand for 5, then $+a=+5$, $\therefore +a$ is positive both in appearance and reality; but if a stands for -5 then $+a=+(-5)=-5$, and $\therefore +a$ is positive in appearance but negative in reality.

When we say that a may be positive or negative, we mean that a may stand for a positive or a negative number.

7. If we use symbols in the results (A), (B), (C), (D) of articles 3 and 5 we respectively get

$$(+a)+(+b)=+a+b\ldots\ldots\ldots(1)$$
$$(+a)+(-b)=+a-b\ldots\ldots\ldots(2)$$
$$(+a)-(+b)=+a-b\ldots\ldots\ldots(3)$$
$$(+a)-(-b)=+a+b\ldots\ldots\ldots(4)$$

In the above results we have assumed b to stand for a positive number and a to stand for a positive or a negative number. In the manner of the last article we can however prove that b may also stand for a negative number. Hence the results (1), (2), (3), (4) are true for all positive and negative values of a as well as of b.

Note. The student will note that in Algebra there is no essential difference between addition and subtraction; for we may regard the subtraction of a negative quantity as the addition of a positive quantity, and *vice versa*. Thus $a-b$ may be regarded as the addition of the positive quantity $+a$ to the negative quantity $-b$ or as the subtraction of the positive quantity b from the positive quantity a.

8. The expressions $a+b$, $a-b$ on the right of (1), (2), (3), (4) in art 7 cannot be further simplified, and the results are regarded

complete algebraically. We can however simplify them when numerical values are given to the symbols, as shown below.

Since a *gain* of Rs. 5 and a *gain* of Rs. 3 make a *total gain* of Rs. 8, while a *loss* of Rs. 5 and a *loss* of Rs. 3 make a *total loss* of Rs. 8, we have

$$(+5)+(+3)=+8,$$
$$(-5)+(-3)=-8.$$

Hence, in adding two (or more) numbers having like signs we take the sum of their absolute values and prefix their common sign to the sum. (Rule I).

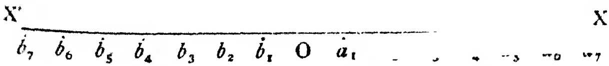
Again, since a *gain* of Rs. 5 and a *loss* of Rs. 3 yield a *net gain* of Rs. 2, while a *gain* of Rs. 5 and a *loss* of Rs. 9 yield a *net loss* of Rs. 4 we have

$$(+5)+(-3)=+2,$$
$$(+5)+(-9)=-4.$$

Hence, in adding two numbers having unlike signs we take the arithmetical difference of their absolute values and prefix the sign of the numerically greater number. (Rule II).

9. Graphical Illustration. The results of the last article can be graphically illustrated thus :

Let XOX' be a straight line and O a fixed point in it. Along OX set off $Oa_1=a_1a_2=a_2a_3=\ldots=1$ unit, and along OX' set off $Ob_1=b_1b_2=b_2b_3=\ldots=1$ unit. Now suppose distances measured



on the straight line from *left to right* are positive and consequently distances measured from *right to left* are negative. Let O be the starting point.

(i) $(+7)+(-3)$ means that we go from O to a_7 (7 units in the positive direction) and then from a_7 to a_4 (3 units in the negative direction) ; and this means that we are taken from O to a_4 (4 units in the positive direction).

$$\therefore\ (+7)+(-3)=+4.$$

(ii) $(-7)+(+3)$ means that we go from O to b_7 (7 units in the negative direction) and then from b_7 to b_4 (3 units in the positive direction) ; and this means that we describe the distance from O to b_4 (4 units in the negative direction)

$$(-7)+(+3)=-4.$$

(iii) $(-2)+(+5)$ means that we go from O to b_2 and then from b_2 to a_3; and this means that we are carried from O to a_3.

$$\therefore (-2)+(+5)=+3.$$

(iv) $(-4)+(-3)$ means that we go from O to b_4 and then from b_4 to b_7; and this means that we are carried from O to b_7.

$$\therefore (-4)+(-3)=-7.$$

EXERCISE VI.

Add together (mentally)

1. $+3, +4.$	**2.** $+5, -3.$	**3.** $-7, +2.$
4. $-2, +7.$	**5.** $-6, +6.$	**6.** $-4, -5.$
7. $+3, +2, +5.$	**8.** $-4, -3, -9.$	**9.** $-2, -7, -1.$

Subtract the second from the first (mentally)

10. $+7, +5.$	**11.** $-9, +4.$	**12.** $+12, -7.$
13. $+4, +3.$	**14.** $+5, +7.$	**15.** $-9, -4.$
16. $-2, -7.$	**17.** $-9, -9.$	**18.** $+5, -3.$

Simplify (mentally)

19. $(+6)+(-7).$	**20.** $(+4)-(+7).$	**21.** $-3-(+7).$
22. $(+4)-(+1).$	**23.** $-4-(-4).$	**24.** $(-5)-(+7).$
25. $+(-5)-(+5)$	**26.** $-(-5)+(+5).$	**27.** $-(-5)-(-5).$

Simplify graphically

28. $7-5.$	**29.** $-7+5.$	**30.** $-7-5.$
31. $+(+4)-(-2).$	**32.** $-(-3)-(-4).$	**33.** $-(-3)-(+5).$

Add together

34. $3\frac{1}{4}, -4\frac{1}{3}.$	**35.** $-2\frac{2}{3}, -3\frac{4}{5}.$	**36.** $5\frac{5}{6}, -3\frac{3}{4}.$
37. $5\frac{2}{7}, -7\frac{2}{5}.$	**38.** $9\frac{8}{9}, -2\frac{1}{3}.$	**39.** $-6\frac{1}{4}, 2\frac{2}{3}.$

---

10. Suppose in a transaction I *gain* Rs. 5, then *lose* Rs. 3, next *gain* Rs. 2 and afterwards *lose* Rs. 6; my financial worth in rupees will be represented by $+5-3+2-6$. Now it is evident that in whatever order these gains and losses take place, my final

position cannot change. Hence *in the algebraical sum of a set of numbers their order is immaterial, each number being preceded by its own sign.*

Again, in a series of gains and losses we may combine all the gains into a total gain and all the losses into a total loss, and the final effect can be calculated by considering this total gain and this total loss. Hence we get the following convenient rule for the algebraical sum of a set of numbers :—

Add together all the positive numbers, also add together all the negative numbers, and lastly add the aggregate of the positive numbers and the aggregate of the negative numbers (Rule III).

Thus, $+5-3+2-6-9+7$.

$= +5+2+7-3-6-9$, re-arranging

$= +14-18$, adding the positive and the negative numbers separately

$= -4$.

This and similar results can be illustrated graphically in the manner of the last art.

11. The **Algebraical difference** of two quantities a and b is $a-b$, the result of subtracting the second from the first ; and a is said to be **greater** or **less** than b according as the difference $a-b$ is positive or negative. Thus,

$+5$ is greater than $+2$, for $+5-(+2)=+3$, a positive quantity ;

$+2$ is greater than 0, for $+2-0=+2$, a positive quantity ;

0 is greater than -4, for $0-(-4)=+4$, a positive quantity ;

-4 is greater than -9, for $-4-(-9)=+5$, a positive quantity ; and so on.

We have thus a series of positive quantities *above* 0 *viz.*, $+1$, $+2$, $+3$,......in ascending order of magnitude, and a series of negative quantities *below* 0, *viz.*, -1, -2, -3,......in *descending* order of magnitude ; 0 being the boundary of positive and negative quantities.

$+3$
$+2$
$+1$
0
-1
-2
-3

It is evident that any positive quantity is greater than 0 and any negative quantity is less than 0. Evidently then any positive quantity is greater than any negative quantity.

Note 1. Strictly speaking we cannot compare positive and negative quantities, for they are not of the same character. Thus + Rs. 2 which I own cannot be said to be greater or less than − Rs. 3 which I owe.

Note 2. We have already defined the absolute or numerical value of a quantity (see art. 7, Chap. II). Thus $+5$ and -5 are *numerically* equal, but from the above $+5$ is *algebraically* greater than -5. Similarly -7 is *numerically* greater than 3 but *algebraically* less.

Obs. The following signs are sometimes used :—

$>$ for "greater than" ; $<$ for "less than" ;

$\ngtr$ for "not greater than" ; $\nless$ for "not less than" ;

$\neq$ for "not equal to" ; $\sim$ for "unknown difference".

By $a \sim b$ we mean $a-b$ or $b-a$ according as a is greater than b or b is greater than a. Also $a \pm b$ means "a plus b or a minus b".

EXERCISE VII.

Simplify (mentally)

1. $+10+(-5)+2-1$.

2. $-7-(+2)+(-5)+9$.

3. $-3-(+5)+1+(-3)$.

4. $(-3)+5+(-1)-(-2)$.

Simplify graphically

5. $+5-3+7-5$.

6. $-7+3-4+6$

7. $-2+1-5-3+4$.

8. $+5-(-2)+(-8)-(+4)$.

MULTIPLICATION.

12. In Arithmetic **Multiplication** is defined as the process of taking a given number a repeated number of times.

The given number is called the *multiplicand*, the number showing how many times the multiplicand is to be taken is called the *multiplier* and the result of multiplying is called the *product*.

13. The above definition of multiplication really assumes that the multiplier is a positive integer, and seems to be meaningless when the multiplier is a positive fraction or a negative quantity. But in these cases we may extend our ideas a little and interpret accordingly.

Thus when the multiplier is a positive fraction as in $\frac{4}{5}\times\frac{2}{3}$, we may take $\frac{4}{5}\times\frac{2}{3}$ to mean $\frac{4}{5}$ taken $\frac{2}{3}$ of *one time* (*i.e.* once), that is, to divide $\frac{4}{5}$ into 3 equal parts and repeat 2 of these parts.

$$\text{Thus } \frac{4}{5}\times\frac{2}{3}=\frac{4}{5\times 3}+\frac{4}{5\times 3}=\frac{4\times 2}{5\times 3}.$$

Again, if the multiplier is a negative quantity, we can also interpret as in cases (iii) and (iv) of the next article.

14. We shall now consider the product of any two numbers.

(i) $(+4)\times(+5)=+4$ repeated 5 times

$=+4+4+4+4+4$

$=+20=+(4\times5)$.

(ii) $(-4)\times(+5)=-4$ repeated 5 times

$=-4-4-4-4-4$

$=-20=-(4\times5)$.

(iii) $(+4)\times(-5)=+4$ repeated -5 times

$=+4$ repeated 5 times but in the opposite direction

$=+20$ with sign changed

$=-20=-(4\times5)$.

(iv) $(-4)\times(-5)=-4$ repeated -5 times

$=-4$ repeated 5 times but in the opposite direction

$=-20$ with sign changed

$=+20=(4\times5)$.

We can deduce similar results with fractional numbers, taking into account the extended meaning of multiplication as given in the preceding article. If in the above we use symbols we have the following results in Algebra :—

$(+a)\times(+b)=+(ab)$......(1), $(-a)\times(+b)=-(ab)$......(2).

$(+a)\times(-b)=-(ab)$......(3), $(-a)\times(-b)=+(ab)$......(4).

Hence we have the following **law of signs** for the multiplication of two quantities, positive or negative, integral or fractional :—

The product of two quantities is found by multiplying their absolute values and prefixing the sign + if both quantities are positive or both negative, and the sign — if one quantity is positive and the other negative i.e., two like signs give + and two unlike signs give —.

15. When one quantity is multiplied by a second, the product by a third and so on, the final result is called the *continued* product of all the quantities.

Ex. 1. Find the continued product of -3, -5, -4.

Product $=(-3)\times(-5)\times(-4)$

$=(+15)\times(-4)=-60$.

Note. By the repeated application of the above law of signs it appears that the continued product of a number of quantities is positive or negative

according as there is an even or an odd number of negative quantities multiplied together.

Ex. 2. Find by trial 2 numbers (i) whose algebraical sum is -4 and whose product is -12; (ii) whose algebraical sum is -7 and product is 12.

(i) The pairs of factors of -12 are $+3, -4$; $-3, +4$; $+2, -6$; $-2, +6$; $+1, -12$; and $-1, +12$.

Of these the pair $+2$ and -6 have their sum $= -4$.

Hence the required numbers are $+2, -6$.

(ii) The pairs of factors of $+12$ are $+3, +4$; $-3, -4$; $+2, +6$; $-2, -6$; $+1, +12$; $-1, -12$.

Of these the pair $-3, -4$ have their sum $= -7$.

Hence the required numbers are -3 and -4.

EXERCISE VIII.

Multiply (mentally)

1. $+3$ by $+5$.	2. -4 by $+5$.	3. $+5$ by -7.
4. $+2$ by -2.	5. -4 by -6.	6. $+9$ by -6.
7. -8 by $+6$.	8. 3 by -7.	9. -7 by -9.

Simplify (mentally)

10. $(+6)\times(-7)$.	11. $(-9)\times(+7)$.	12. $(-11)\times(-11)$.
13. $(-7)\times 0$.	14. $(-7)\times(-6)$.	15. $(+9)\times(-10)$.

Find the continued product of (mentally)

16. $+2, -3, +4$.	17. $-5, 7, -6$.	18. $3, -4, +5$.
19. $-3, -4, -5$.	20. $-6, -6, -6$.	21. $5, -5, 4$.

Find by trial 2 numbers whose (algebraical) sum and product are respectively :—

22. $5, 6$.	23. $-5, 6$.	24. $-3, -10$.
25. $3, -10$.	26. $-7, 10$.	27. $7, 10$.

DIVISION.

16. Division is defined as the process of finding how many times a given quantity (called the **divisor**) is contained in another quantity (called the **dividend**). The number of times the divisor is contained in the dividend is called the **quotient**.

Division is thus the inverse of multiplication, and to divide one quantity by another is to find a third quantity such that the product of this and the second may be equal to the first.

Thus, since we have

$$(+2)\times(-3)=-6,\ \therefore\ (-6)\div(+2)=-3\ ;$$
$$(-2)\times(-3)=+6,\ \therefore\ (+6)\div(-2)=-3.$$

Generally, since

$$(+a)\times(+b)=+ab,\ \therefore\ (+ab)\div(+a)=+b\ ;$$
$$(-a)\times(+b)=-ab,\ \therefore\ (-ab)\div(-a)=+b\ ;$$
$$(+a)\times(-b)=-ab,\ \therefore\ (-ab)\div(+a)=-b\ ;$$
$$(-a)\times(-b)=+ab,\ \therefore\ (+ab)\div(-a)=-b.$$

Thus we have the same **rule of signs** as in art. 15 for the division of one quantity by another :—

The quotient of one quantity divided by another is found by dividing the absolute value of the first by that of the second and prefixing the sign + if both the quantities are positive or both negative and the sign −, if one quantity is positive and the other negativa, i.e., two like signs give + and two unlike signs give—.

EXERCISE IX.

Divide (mentally)

1. $+12$ by $+3$.	**2.** -20 by $+5$.	**3.** -42 by -6.
4. $+72$ by -8.	**5.** -63 by $+9$.	**6.** 54 by -6.
7. -56 by $+7$.	**8.** -52 by -4.	**9.** -13 by $+13$.
10. 0 by -5.	**11.** -12 by $+4$.	**12.** -44 by -11.

SUBSTITUTION.

17. In Chap. I we considered substitution in which the letters were restricted to positive values. We shall now suppose this restriction to be removed and assume the letters to have any positive or negative values.

Ex. 1. If $a=-2$, $b=3$, $c=-4$, $d=5$, find the value of

(i) $2a^2$ (ii) $3c^3a$ (iii) $abcd$.

(i) $2a^2=2\times(-2)^2=2\times(-2)\times(-2)=2\times4=8$.

(ii) $3c^3a=3\times(-4)^3\times(-2)=3\times(-4)\times(-4)\times(-4)\times(-2)$
$=-12\times16\times(-2)=-192\times(-2)=384$.

(iii) $abcd=-2\times3\times(-4)\times5=-6\times(-20)=120$.

Ex. 2. With the same values of a, b, c, d as in Ex. 1., evaluate

(i) $(a+b)^2+(b+c)^2-(c+a)(d+a)$.

(ii) $\dfrac{a(b-c)}{b(c-a)}+\dfrac{c(a-d)}{d(a-b)}$.

(i) We have $a+b=-2+3=1$, $b+c=3+(-4)=-1$.

$c+d=-4+5=1$, $d+a=5+(-2)=3$.

$\therefore$ the expression $=1^2+(-1)^2-1\times 3$

$=1+1-3$, $\quad\because (-1)^2=(-1)\times(-1)=+1$.

$=-1$.

(ii) We have $b-c=3-(-4)=7$, $c-a=-4-(-2)=-2$,

$a-d=-2-5=-7$, $a-b=-2-3=-5$.

$\therefore$ the expression $=\dfrac{-2\times 7}{3\times(-2)}+\dfrac{(-4)\times(-7)}{5\times(-5)}$

$=\frac{7}{3}-\frac{28}{25}=\frac{91}{75}$.

EXERCISE X.

If $a=5$, $b=-3$, $c=-2$, $d=4$, evaluate

1. $2a+3b$.
2. $2b-3c$.
3. $5c-2d$.
4. $a-2b+3c$.
5. $a+2b-3c$.
6. $-a+2b+3c$.
7. $-a+(-b)-(-c)+d$.
8. $a-(-b)+(-c)-(-d)$.
9. $a^2(b+c)+b^2(c+a)+c^2(a+b)$.
10. $(a-b)^2+(b-c)^2+(c-a)^2$.
11. $\sqrt{(c^2d)}+\sqrt{(5ab^2)}-\sqrt{(-3bd)}$.
12. $a^2b^2c^2-b^2c^2d^2$.
13. $2a^2bc-3b^2cd+c^2ab$.
14. $b^a+c^d-d^a$.
15. $\dfrac{ac}{bd}-\dfrac{bc}{ad}$.
16. $\dfrac{a^2}{b^2}-\dfrac{c^2}{d^2}$.
17. $\dfrac{b-c}{bc}+\dfrac{c-a}{ca}+\dfrac{a-b}{ab}$.
18. $\dfrac{3a}{c-d}+\dfrac{2b}{a-d}-\dfrac{c}{b-d}$.

If $a=-12$, $b=6$, $c=-3$, $d=2$, evaluate

19. $2a\div b\times 2c\div 3d$.
20. $2a\div b\times(2c\div 3d)$.
21. $a^2\times b^2\div c^2\div d^2$.
22. $a^2\div b^2\times c^2\div d^2$.
23. Tabulate the values of $3x^2-4x+5$ when $x=-3, -2, -1, 0, 1, 2, 3$.
24. Tabulate the values of $-2x^2+5x-6$ when $x=-\cdot 1, -\cdot 2, -\cdot 3, -\cdot 4$.

25. Verify the following :—

(i) $a^3+b^3=(a+b)(a^2-ab+b^2)$, when $a=-2$, $b=-3$.

(ii) $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$, when
$(a=-1$, $b=-2$, $c=-3$, $d=1$.

26. If $a=5$, $b=-9$, $c=4$, prove that

(i) $a^3+b^3+c^3=3abc$

(ii) $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)=0$.

27. If $x=-\frac{3}{4}$, $y=\frac{1}{2}$, $z=-\frac{1}{3}$, prove that

(i) $(x+y+z)^2=x^2+y^2+z^2+2yz+2zx+2xy$.

(ii) $(x+y)(x-y)+(y+z)(y-z)+(z+x)(z-x)=0$.

28. If $a=4$, $b=2$, $c=-6$, prove that
$a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc=0$.

CHAPTER V.

ADDITION.

1. When two or more quantities do not differ at all or if they differ, they differ only in their numerical co-efficients, they are said to be **like** ; otherwise they are called **unlike**.

Like terms therefore contain same letter or letters.

Thus, $2a$, $5a$, $7a$ are like quantities ; so also $3a^2b$, $4a^2b$, $\frac{5}{2}a^2b$. But $3a$, $4a^2$, $9c$ are unlike quantities.

2. Addition of like quantities. We can add two or more like quantities, whether they are positive or negative, by the rules for adding positive and negative numbers given in Chap. IV. Thus, just as 5 *tens* and 3 *tens* give 8 *tens*, 5 things and 3 things give 8 things ; so $5a$ and $3a$ give $8a$ or $5ab$ and $3ab$ give $8ab$ **Hence the sum of a number of like quantities is a like quantity of which the co-efficient is the sum of the co-efficients of the quantities.** To find the sum of the co-efficients the student is referred to the rules I, II and III of articles 8 and 10, Chap. IV.

Ex. 1. Add together $4a$, $7a$, $9a$.

The sum of the co-efficients 4, 7, 9 is 20 ; hence the sum required $=20a$.

Ex. 2. Add together $-x$, $-5x$, $-10x$.

Note that $-x$ means $-1x$. The sum of the co-efficients -1, -5, -10 is -16; hence the sum required $= -16x$.

Ex. 3. Add together $7ab$, $-3ab$, $4ab$, $5ab$, $-9ab$, $-6ab$.

The sum of the co-efficients 7, -3, 4, 5, -9, -6 is -2; hence the sum required is $-2ab$.

Ex. 4. Add together $2(x+y)$, $-4(x+y)$, $5(x+y)$.

Here we have 3 **like expressions** and add them as in the case of like terms. The sum of the co-efficients 2, -4, 5 is 3; hence the sum required $=3(x+y)$.

3. Addition of unlike quantities. When we are required to add two or more unlike quantities we are to simply *write them down in succession with their signs unchanged* (see art. 3, Chap. IV) and regard the addition complete in this form. We cannot here combine the quantities into one quantity, as in the case of like quantities. Thus, just as we cannot find a single sum by adding 3 oranges and 6 apples but can simply write 3 oranges+6 apples, so if $3a$ is added to $6b$ the result can be written as $3a+6b$.

Ex. 1. Add $3a$, $-4b$.

The sum $=3a-4b$, and it cannot be simplified further.

Ex. 2. Add $-2ab$, $3bc$, $-5ca$.

Note that $3bc$ means $+3bc$ (see art. 6, Chap. II); hence the sum $= -2ab+3bc-5ca$.

EXERCISE XI.

Add together

1. $7x$, $-9x$, $3x$.

2. $4xy$, $7xy$, $-13xy$.

3. $\frac{2}{3}xyz$, $-\frac{4}{9}xyz$, $-\frac{7}{8}xyz$.

4. a^2bc, $-7a^2bc$, $\frac{3}{2}a^2bc$.

5. $-xy^2$, $\frac{8}{5}xy^2$, $-\frac{3}{5}xy^2$.

6. $\frac{5}{6}a^3$, $-\frac{2}{3}a^3$, $\frac{1}{6}a^3$.

7. $-2abx$, $7abx$, $9abx$, $-\frac{1}{2}abx$.

8. $\frac{5}{2}a^2b$, $-3a^2b$, $7a^2b$, $-\frac{9}{2}a^2b$.

9. $5abcd$, $-2abcd$, $7abcd$, $11abcd$, $-13abcd$.

10. $\frac{3}{4}\sqrt{(ab)}$, $-\frac{7}{2}\sqrt{(ab)}$, $\frac{2}{3}\sqrt{(ab)}$.

11. $\sqrt[3]{a}$, $-\frac{5}{7}\sqrt[3]{a}$, $\frac{4}{5}\sqrt[3]{a}$.

12. $7a^2c^2$, $-15a^2c^2$, $24a^2c^2$, $32a^2c^2$, $-10a^2c^2$, $15a^2c^2$.

13. $5(a^2-b^2)$, $-2(a^2-b^2)$, $6(a^2-b^2)$, $9(a^2-b^2)$.

Add together

14. $2a,\ 3b.$ **15.** $4x, -5y.$ **16.** $-5z,\ -2x.$

17. $-\frac{3}{2}a^2,\ \frac{2}{3}a,\ \frac{4}{5}.$ **18.** $-2ab,\ 3c^2, -4xy.$

19. $4b^3, -\frac{3}{4}c^3,\ 2b^2c.$ **20.** $5ab, -2xy,\ \frac{1}{4}c^2, -\frac{4}{5}d^2$

---

4. An expression like $a-2b+3c-4d$ may be put as an algebraical *sum* in the form $a+(-2b)+(+3c)+(-4d)$, that is, as the *sum* of the quantities $a,\ -2b, +3c, -4d.$ In this sense the quantities are called *terms* of the expression $a-2b+3c-4d$; and henceforth (see Art 9, Chap. I) we shall understand the terms of an expression to include the prefixed signs.

We may always suppose the terms of an expression to represent a series of gains and losses. Thus $a-2b+3c-4d$ represents a *gain* of a with a *loss* of $2b$ followed by a *gain* of $3c$ with a *loss* of $4d$, and in this view an expression always represents a *sum* of gains and losses.

5. Commutative Law. The terms of an algebraical expression may be arranged in *any order*, provided each term carries its sign with it. This follows in the manner of art. 10, Chap. IV. for a series of gains and losses does not affect one *finally* (from the mathematical point of view) in whatever order these gains and losses take place.

$$\text{Thus, } 3a-2b-4c+d-5e$$
$$=-2b+d-5e+3a-4c.$$
$$=d-2b+3a-4c-5e=\text{etc.}$$

This is called the **Commutative Law** for addition and subtraction, *viz., in a chain of additions and subtractions the order of operations is indifferent.*

6. Associative Law. The terms of an algebraical expression may be *grouped* in any manner *i.e.* any number of them may be taken *as a whole.* This follows from the fact that if we have to consider the final effect of a series of separate gains and losses we may do so by dividing these gains and losses into *groups* in any way, noting the effect of each group as giving one single gain or one single loss and lastly combining these effects.*

The grouping of terms may be effected by closing them within brackets, for that means that those terms are taken as a whole. Thus the expression $2a-3b+4c-5d$ is the *sum* of the

terms $2a$, $-3b$, $4c$, $-5d$ and we can group together any number of these terms, *i.e.* regard them as a *whole*. Hence

$$2a-3b+4c-5d$$
$$=(2a-3b)+5c-5d, \text{ grouping the first 2 terms}$$
$$=2a+(-3b+4c-5d), \text{ grouping the last 3 terms}$$
$$=\text{etc.}$$

This is called the **Associative Law** for addition and subtraction, *viz.*, *in a chain of additions and subtractions the terms may be grouped in any manner.*

Ex. Add together $-2a$, $3b$, $4c$, $-5b$, $-2c$, $4a$.

$$\text{Sum}=-2a+3b+4c-5b-2c+4a$$
$$=-2a+4a+3b-5b+4c-2c, \text{ [commutative law]}$$
$$=(-2a+4a)+(3b-5b)+(4c-2c), \text{ [associative law]}$$
$$=2a-2b+2c, \text{ by collecting like terms.}$$

7. Although the terms of an expression may be arranged in any order (art. 5), yet in many cases we follow some definite mode of arrangement.

When an expression contains terms having various powers of a letter, it is usual to arrange it in *ascending* or *descending* powers of that letter: thus $3x^3-2x^2+5x+2$ in descending powers of x or $2+5x-2x^2+3x^3$ in ascending powers of x is the proper form. Similarly, the expression $a^3-2a^2b+3ab^2-b^3$ which is in descending powers of a or ascending powers of b may be taken to be in proper form.

When the terms of an expression contain single letters it is usual to place the letters in alphabetical order: thus, $2a+7b-9c+d$.

8. Addition of compound expressions. A compound expression like $2a-3b+4c-5d$ may be regarded as the algebraical sum of the terms $2a$, $-3b$, $4c$, $-5d$. Hence to add a compound expression as a *whole* is to add in succession each term of the expression, and this we know (see art. 3, Chap. II) is done by simply attaching each term by its own sign to that to which the given expression is to be added.

Thus to add $b+c$ as a *whole* is to add b, c in succession, or $+(b+c)=+b+c$.

$$\therefore a+(b+c)=a+b+c.$$

Similarly, to add $b-c$ as a *whole* is to add b, $-c$ in succession, or $+(b-c)=b-c$.

$$\therefore a+(b-c)=a+b-c.$$

Ex 1. To $3a-5b+2c$ add $5a+2b-9c$.

To add $5a+2b-9c$ as a whole is to add $5a$, $2b$, $-9c$ separately in succession.

$$\text{reqd. sum}=3a-5b+2c+5a+2b-9c$$

$$=3a+5a-5b+2b+2c-9c, \text{ bringing like terms together by commutative law}$$

$$=8a-3b-7c, \text{ collecting like terms by associative law}$$

This method may be applied in a convenient form as stated in the following rule.

Place the expressions to be added in lines one under another (after re-arrangement of the terms if necessary) so that each set of like terms may be in the same column and then add each column beginning from one end.

Ex 2. Add together $4x^3-2+3x^2-x$, $3-2x^3+4x-2x^2$, and $5x^2-7+x^3-x$.

Arrange the expressions in descending powers of x and place them as in the rule; thus

$$\begin{array}{rrrr} 4x^3 & +3x^2 & -x & -2 \\ -2x^3 & -2x^2 & +4x & +3 \\ x^3 & +5x^2 & -x & -7 \\ \hline \end{array}$$

∴ Reqd. sum $=3x^3+6x^2+2x-6$, by adding up each column.

Ex. 3. Add together $2a^2b^2-3ab^3-5a^4$, $2ab^3+b^4-3a^3b$, $5a^3b+2a^4+4ab^3$.

Arrange the expressions in descending powers of a, leaving spaces where powers are wanting: thus

$$\begin{array}{rrrrr} -5a^4 & & +2a^2b^2 & -3ab^3 & \\ & -3a^3b & & +2ab^3 & +b^4 \\ +2a^4 & +5a^3b & & +4ab^3 & \\ \hline -3a^4 & +2a^3b & +2a^2b^2 & +3ab^3 & +b^4 \end{array} = \text{required sum.}$$

Ex. 4. Add together $3(a+b)^2-$[illegible]$(b+c)^2+2(c+a)^2$,

$$-[illegible](a+b)^2+[illegible](b+c)^2-[illegible](c+a)^2$$

$$-10(a+b)^2+[illegible](b+c)^2-[illegible](c+a)^2$$

∴ **Reqd. sum $=-3(a+b)^2+3(b+c)^2-10(c+a)^2$**

EXERCISE XII.

Add together.

1. $3ab,\ -4bc,\ 5ca,\ -2ab,\ 4bc.$ 2. $3a,\ -4b,\ 5c,\ -2a,\ -3c,\ 4b$
3. $2a^2,\ 3b^2,\ -c^2,\ -4c^2,\ 2b^2,\ -a^2,\ 4b^2,\ 5c^2,\ -a^2.$
4. $-5abx,\ 7bcy,\ -8abz,\ -4bcy,\ 2xyz,\ 7abx.$
5. $5a-7b,\ 2a+3b.$ 6. $2x+3y,\ -4y+5x.$
7. $\frac{1}{2}ax-\frac{2}{3}by,\ -\frac{1}{3}ax+\frac{4}{5}by.$ 8. $\frac{3}{4}a^2-\frac{4}{5}b^2,\ \frac{5}{8}b^2-\frac{7}{4}a^2.$
9. $3a-4b+7c,\ 2a+5b-9c.$ 10. $3y-4x+7z,\ 5x-3z-2y.$
11. $b-c,\ c-a,\ a-b.$ 12. $a+b-c,\ b+c-a,\ c+a-b$
13. $3x-4y-5z,\ -5x+3y+7z,\ 4x+2y-3z.$
14. $a+b+c+d,\ a-b+c-d,\ a+b-c-d,\ -a+b+c-d.$
15. $a^2-5a+7,\ -12+2a^2-3a,\ -3a^2+5+8a.$
16. $-3x^2+7xy-4y^2,\ 5y^2-9xy+2x^2,\ 4xy+3y^2-7x^2.$
17. $\frac{2}{3}a^2-\frac{3}{4}ax+\frac{1}{2}x^2,\ \frac{4}{5}x^2+\frac{1}{2}a^2+\frac{5}{2}ax,\ -\frac{5}{3}ax+\frac{3}{2}xy-\frac{1}{2}a^2.$
18. $12x^3-12x^2+5x-7,\ 5x^2-3x^3+6-9x,\ 10+x-3x^2+5x^3.$
19. $3b^4+2b^3-5b^2+7b+5,\ \ 5b-7+2b^4-b^3,\ \ 7b^2-12+5b-b^3-b^4,$
$2b^3-b^2+3b-3b^4$
20. $\frac{3}{4}xy-\frac{4}{5}yz,\ \frac{3}{4}yz-\frac{4}{5}zx,\ \frac{3}{4}zx-\frac{4}{5}xy.$
21. $\frac{1}{2}x^4-\frac{1}{3}x^3+\frac{1}{4}x^2-\frac{1}{5}x+\frac{1}{6},\ -\frac{1}{3}x^4+\frac{1}{4}x^3-\frac{1}{5}x^2+\frac{1}{6}x-\frac{1}{2},$
$\frac{1}{4}x^4-\frac{1}{5}x^3+\frac{1}{6}x^2-\frac{1}{2}x+\frac{1}{3}.$
22. $7a^2-5b^2+2ab,\ \ 3b^2-4c^2+3bc,\ \ 5c^2-2a^2+4ca,\ \ 3a^2+4b^2-5c^2,$
$2bc-3ca+4ab.$
23. $2a^4-3a^3b+5a^2b^2-7ab^3+5b^4,\ 3b^4+5a^4-7a^2b^2+5a^3b+9ab^3,$
$2a^2b^2-a^3b+ab^3,\ 5a^4-7b^4.$
24. $x^2-3xy+y^2,\ -2x^2+5xy-3y^2,\ 5y^2+2x^2-7xy,\ 2x^2-3y^2+9xy,$
$7xy-3x^2-7y^2.$
25. $\frac{1}{2}\sqrt{a}-\frac{1}{3}\sqrt{b}+\frac{1}{4}\sqrt{c},\ \frac{3}{4}\sqrt{b}-\frac{7}{4}\sqrt{c}+\frac{1}{3}\sqrt{a},\ \sqrt{c}-\sqrt{a}+\sqrt{b}.$
26. I get Rs. x from A, Rs. y from B and Rs. z from C, where $x=2a-3b+4c-5d$, $y=-5a+7b-9c+10d$, $z=4a-3b+6c-4d$, prove that I am in possession of Rs. $(a+b+c+d)$.

CHAPTER VI.

SUBTRACTION.

1. We know from art. 5, Chap. IV that *to subtract a quantity is to add it with its sign changed.* This result is used in the following examples.

Ex. 1. Subtract (i) $-7a$ from $4a$. (ii) $-4a$ from $+7a$.

(i) Here remainder $=4a-(-7a)=4a+7a=11a$.

(ii) Here remainder $=+7a-(-4a)=+7a+4a=11a$.

Ex. 2. Subtract (i) $2b$ from $3a$ (ii) $-5y$ from $7x$.

(i) Here remainder $=3a-2b$.

(ii) Here remainder $=7x-(-5y)=7x+5y$.

2. Subtraction of compound expressions. Any expression is the algebraical *sum* of a number of terms. Hence to subtract a compound expression *as a whole* is to subtract in succession each term of the expression, *i.e.*, to add each term with its sign changed.

Thus to subtract $b-c$ *as a whole*, we add $-b+c$; hence $-(b-c)=-b+c$.

$$\therefore\ a-(b-c)=a-b+c.$$

Similarly to subtract $b+c$ *as a whole*, we add $-b-c$; hence $-(b+c)=-b-c$.

$$\therefore\ a-(b+c)=a-b-c.$$

Note. Regarding subtraction as the inverse of addition we see that to subtract $b-c$ from a is to find the quantity which when added to $b-c$ gives a. Now the quantity which when added to $b-c$ gives zero is $-b+c$; hence the quantity which when added to $b-c$ gives a is $-b+c$ added to a or $a-b+c$, which is therefore the result of subtracting $b-c$ from a. This also gives the same result as obtained before.

Ex. 1. Subtract $3a-4b-7c+d$ from $4a+3b-5c+2d$.

To subtract $3a-4b-7c+d$ as a whole is to subtract $3a$, $-4b$, $-7c$, d or, add $-3a$, $4b$, $7c$, $-d$ in succession.

$$\begin{aligned}\therefore\ \text{reqd. result} &= 4a+3b-5c+2d-3a+4b+7c-d\\ &= 4a-3a+3b+4b-5c+7c+2d-d\\ &= a+7b+2c+d,\ \text{collecting like terms.}\end{aligned}$$

. The above method may be applied in a convenient form as stated in the following rule —

Place the subtrahend under the minuend after re-arrangement of terms of both, if necessary, so that like terms may be under like terms, change the sign of each term of the subtrahend, and add to the minuend.

Ex 2 Subtract $5x^4-3x^3+3x^2+5x+7$ from

$7x^4-4x^3+5x^2-9x-3$

$$7x^4-4x^3+5x^2-9x-3=\text{minuend.}$$
$$\underline{5x^4-3x^3+3x^2+5x+7}=\text{subtrahend.}$$
$$2x^4-x^3+2x^2-14x-10=\text{remainder.}$$

The operation of changing the sign of every term of the subtrahend should be a mental one. Thus in the above we mentally collect $7x^4$, $-5x^4$, $-4x^3$, $+3x^3$, $5x^2$, $-3x^2$, and so on

Ex. 3 Subtract $3xy+5yz-2zx$ from $2yz+7xy$

$$7xy+2yz \qquad =\text{minuend}$$
$$3xy+5yz-2zx=\text{subtrahend}$$
$$4xy-3yz+2zx=\text{remainder}$$

EXERCISE XIII.

Subtract

1. $9x$ from $4x$ 2 $\frac{3}{4}x$ from $\frac{4}{5}x$. 3. $-\frac{2}{3}a$ from $\frac{3}{7}a$.
4. $\frac{2}{7}a^2$ from $-\frac{1}{12}a^2$. 5. $\frac{3}{4}b^2$ from $-\frac{1}{8}b^2$. 6. $-\frac{9}{2}x$ from $-x$.
7. $\frac{1}{3}ab$ from $-\frac{2}{3}ab$. 8. $-\frac{1}{4}c$ from $\frac{3}{4}c$. 9. $\frac{2}{3}xy$ from $-\frac{5}{3}xy$.

From

10. $a+b$ take $a-b$. 11. $a-b$ take $c-d$.
12. $\frac{2}{3}x-\frac{2}{5}y$ take $\frac{1}{4}x+\frac{1}{4}y$. 13. $\frac{1}{5}a+\frac{3}{7}b$ take $-\frac{3}{5}a+\frac{2}{7}b$.
14. $5a-7b+11c$ take $3a-5b+$ [illegible]
15. [illegible] take [illegible]
16. [illegible]
17. [illegible]
18. [illegible]
19. [illegible]
20. [illegible]

From

21. $7a^3b+4b^4-2a^4+4ab^3-3a^2b^2$

take $5a^2b^2-2ab^3+3a^4-b^4-4a^3b$.

22. $3x^4y^2-5x^3y^3+2x^6-4x^2y^4+3y^6-7x^5y+xy^5$

take $2x^2y^4+3x^5y-3x^6+2x^4y^2-4x^3y^3+5xy^5+y^6$.

3. Symbolical Representation. Statements in words can be translated into algebraical language by means of signs and symbols, and we give below instances of this symbolical representation which the student will learn carefully.

Ex. 1. By how much is x greater than y?

Consider the question — "by how much is 9 greater than 5?" The answer is 4 which is $9-5$. Similarly in the present case the answer is $x-y$. The same question may be put thus—what is the excess of x over y?

Similarly, $y-x$ is answer to the question—by how much is x less than y or what is the defect of x from y.

Ex. 2. A purse contains a pounds, b shillings and c pence, what is the total amount of pence in it?

Since one pound contains 240 pence, $\therefore$ a pounds contain $240\times a$ or $240a$ pence. Again, since one shilling contains 12 pence $\therefore$ b shillings contain $12\times b$ or $12b$ pence.

Hence the total number of pence $=240a+12b+c$.

Ex. 3. What is the cost of a articles at b rupees each?

cost of one article is b rupees,

$\therefore$ cost of 2 articles is $2\times b$ rupees,

cost of 3 articles is $3\times b$ rupees,

Similarly, cost of a articles is $a\times b$ or ab rupees.

Ex. 4. If I walk m miles in n hours, how far can I walk in p hours?

I walk m miles in n hours,

$\therefore$ I walk $\frac{m}{n}$ miles in 1 hour,

$\therefore$ I walk $\left(\frac{m}{n}\times p\right)$ miles in p hours.

EXERCISE XIV.

1. By how much is x greater than 20? By how much is 20 greater than y?

2. What number is less than x by 15? What number is less than 15 by y?

3. If we divide 10 into two parts of which x is one, what is the other part? If we divide y into two parts of which 10 is one what is the other part?

4. Express a rupees b annas in pice.

5. By how much is $3x-4y+7z$ greater than $5x-7y+9z$?

6. What is the defect of $a-b+c-d$ from $a+b-c+d$?

7. What is the excess of the sum of a and b over the defect of c from d?

8. In a class of a boys there are x Mahomedans, y Hindus and z Christians; how many are there of other faiths?

9. What must be added to $5x^3-7x^2+3x-9$ to give

(i) $16x^3-24+7x-20x^2$?

(ii) $5-13x+27x^2-9x^3$?

10. What must be subtracted from $19a^3+15a^2b-26ab^2+17b^3$ to give

(i) $49ab^2-27a^3+32a^2b-47b^3$?

(ii) $65a^2b+71a^3-57b^3+21ab^2$?

11. To what must $3a^2-4ab+5b^2$ be added to give $7a^2-9ab+12b^2$?

12. To what must $a+b-c-d+e$ be added to give $c-d+e-b+a$? What again to give zero?

13. From what must $a^2-b^2+c^2+3bc-2ac$ be subtracted to give $c^2+a^2-2b^2-2bc+ab$?

14. From what must x^2+ax-a^2 be substracted to give x^2+ax-a^2?

15. By how much is the sum of $3a-2b$, $4b+5c$, $7c-11a$ greater than the sum of $7a$, $-9b$, $12c$, $2a-b$, $3b-9c$?

16. Subtract $5x-7y+9z$ from $12z-5x+7y$ and subtract the remainder from zero.

17. Subtract $9a^2-15ab+19b^2$ from $15a^2-30ab+17b^2$ and add the remainder to the sum of $3ab-2b^2+17a^2$ and $3b^2-4a^2+7ab$.

18. Add $3x^2-2xy+5y^2$ to $3y^2+4x^2+9xy$ and subtract the result from the difference of $2y^2-7xy+8x^2$ and $xy+x^2+y^2$.

CHAPTER VII.

MULTIPLICATION.

1. To prove that $a\times b=b\times a$, *i.e.*, a multiplied by b is equal to b multiplied by a, for all values of a and b.

(i) Let a and b be *positive integers.* Write down b rows of units there being a units in each row. Thus—

1 1 1 1......a units (first row)

1 1 1 1......a units (second row)

1 1 1 1......a units (third row)

....

1 1 1 1......a units (bth row).

Since there are a units in each row and there are b rows, the number of units $=a$ repeated b times $=a\times b$. Again considering columns instead of rows, we find there are b units in each column and there are a columns. Hence the number of units also $=b$ repeated a times $=b\times a$.

$\therefore\ a\times b=b\times a.$

(ii) Let a and b be *positive fractions*; say, $a=m/n$, $b=p/q$, where m, n, p, q are positive integers.

Then $\quad a\times b=\dfrac{m}{n}\times\dfrac{p}{q}=\dfrac{mp}{nq}$; $b\times a=\dfrac{p}{q}\times\dfrac{m}{n}=\dfrac{pm}{qn}$.

But by (i) $mp=pm$, $nq=qn$; hence
$a\times b=b\times a.$

(iii) Let one or both of a and b be *negative.* Suppose $a=-m$, $b=-n$ where m and n are positive.

Then $a\times b=-m\times(-n)=+mn$, by rule of signs,

$b\times a=-n\times(-m)=+nm$, by rule of signs.

But $mn=nm$ by (i) and (ii); hence $a\times b=b\times a$.

Thus in all cases $a\times b=b\times a$ or $ab=ba$.

Note. **The product of a by b or of b by a will be indiscriminately denoted by ab or ba, and we shall call it product of a and b. Also we shall use the words—multiply together two quantities when it is not intended to specify the multiplier or the multiplicand.**

2. To prove that $(ab)\times c = a\times(bc) = b\times(ac)$

Write down c rows of stars, there being b stars in each row, thus :—

* * *......b stars (first row),
* * *......b stars (second row),
* * *......b stars (third row),

... ...

* * *......b stars (cth row).

Now suppose each star represents a units. Then there being b stars in each row, we have ab units in each row ; and since there are c rows, the total number of units in the diagram $=(ab)\times c$. Again, since there are b stars in each row and there are c rows, we have altogether bc stars ; and since each star represents a units, the total number of units in the diagram $=a\times(bc)$.

Also, there are c stars and $\therefore$ ac units in each column ; hence since there are b columns, the total number of units in the diagram $=b\times(ac)$.

$\therefore\ (ab)\times c = a\times(bc) = b\times(ac)$.

We can now prove the above relations to be true for all values of a, b, c as in art. 1.

3. Commutative Law. We have $a\times b\times c$

$=(ab)\times c$ by definition	$=a\times(bc)$ by art. 2
$=(bc)\times a$ by art. 1	$=b\times c\times a$ by definition.

Similarly $a\times b\times c=(ab)\times c$ by def

$=b\times(ac)$ by art. 2

$=(ac)\times b$ by art. 1

$=a\times c\times b$ by def.

$\therefore\ a\times b\times c = b\times c\times a = a\times c\times b = \dots$

Similarly, $a\times b\times c\times d = b\times d\times a\times c = d\times c\times b\times a = \dots$

This proves the *commutative law* for multiplication, *viz.*, *in a chain of multiplication the factors may be taken in any order.*

In practice the letters in a product are arranged in alphabetical order, any numerical factor being placed first. Thus, $c\times 2\times b\times a$ is written as $2abc$.

4. Associative Law. In art. 2 we have proved that $a\times b\times c = (ab)\times c = a\times(bc) = b\times(ac)$......(1).

Similarly $a \times b \times c \times d$

$$= a \times (bc) \times d, \text{ by art. 2}$$
$$= (bc) \times d \times a, \text{ by commutative law}$$
$$= (bcd) \times a, \text{ by def.}$$

In the same manner $a \times b \times c \times d = (bcd) \times b = (abd) \times c = \ldots$

$$\therefore\ a \times b \times c \times d = (bcd) \times a = (cda) \times b = (abd) \times c \ldots (2)$$

The results (1) and (2) prove the *associative law* for multiplication *viz.*, *in a chain of multiplication, the factors of the product may be grouped in any manner, and conversely.*

The following examples illustrate the commutative and associative laws in multiplication.

Ex. 1. Multiply $-4x$ by $+5y$.

Product $= (-4x) \times (+5y)$

$$= -(4x \times 5y), \text{ by law of signs (art. 15, Chap. IV)}$$
$$= -(4 \times x \times 5 \times y), \text{ by associative law}$$
$$= -(4 \times 5 \times x \times y), \text{ by commutative law}$$
$$= -(20 \times xy), \text{ by associative law}$$
$$= -20xy.$$

Ex. 2. $2b \times 3a \times 4c$

$$= 2 \times b \times 3 \times a \times 4 \times c, \text{ by associative law}$$
$$= 2 \times 3 \times 4 \times a \times b \times c, \text{ by commutative law}$$
$$= 24 \times abc, \text{ by associative law}$$
$$= 24abc.$$

5. Index Law. To prove that $a^m \times a^n = a^{m+n}$, where m and n are positive integers.

Let us first take a particular case, say, $a^2 \times a^3$. We have by definition of a power, $a^2 = a \times a$,

$$a^3 = a \times a \times a.$$

$$\therefore\ a^2 \times a^3 = (a \times a) \times (a \times a \times a)$$
$$= a \times a \times a \times a \times a, \text{ by associative law}$$
$$= a^5 \text{ (by definition)} = a^{2+3}.$$

Generally, by definition of a power,

$$a^m = a \times a \times \cdots \text{to } m \text{ factors,}$$
$$a^n = a \times a \times \ldots \text{to } n \text{ factors.}$$
$$\therefore\ a^m \times a^n = (a \times a \times \cdots \text{to } m \text{ factors}) \times (a \times a \times \ldots \text{to } n \text{ factors})$$
$$= a \times a \times a \times \ldots \text{to } (m+n) \text{ factors}$$
$$= a^{m+n} \text{ by definition.}$$

This is called the *Index Law for multiplication* and states that *in multiplying two powers of the same base we take a power of the base of which the index is the sum of the indices of the two powers. The rule is therefore to add up the indices.*

Cor. $a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{m+n+p}$.

Similarly, $a^m \times a^n \times a^p \times a^q \times \ldots = a^{m+n+p+q+\ldots\ldots}$

Thus *the product of any number of powers of the same base is a power of that base of which the index is the sum of the indices of the given powers.*

Ex. 1. Multiply $5x$ by $3x^4$.

(Note that $5x$ means $5x^1$)

$$\text{Product} = 5x \times 3x^4$$
$$= 5 \times x \times 3 \times x^4 \text{ [associative law]}$$
$$= 5 \times 3 \times x \times x^4 \text{ [communative law]}$$
$$= (5 \times 3) \times (x \times x^4) \text{ [associative law]}$$
$$= 15x^5, \text{ for } x \times x^4 = x^{1+4} = x^5 \text{ by Index Law.}$$

Ex. 2. Multiply $-3x^3y^4$ by $\frac{2}{5}x^5y^9$.

$$\text{Product} = -(3x^3y^4 \times \tfrac{2}{5}x^5y^9) \text{ by rule of signs}$$
$$= -(3 \times x^3 \times y^4 \times \tfrac{2}{5} \times x^5 \times y^9) \text{ by associative law}$$
$$= -(3 \times \tfrac{2}{5} \times x^3 \times x^5 \times y^4 \times y^9) \text{ by commutative law}$$
$$= -(3 \times \tfrac{2}{5}) \times (x^3 \times x^5) \times (y^4 \times y^9) \text{ by associative law.}$$
$$= -\tfrac{6}{5}x^8y^{13} \text{ by index Law.}$$

Note. In multiplying the student will be careful not to add together indices of powers of *different* letters: thus, $3x^2 \times 4y^3 = 12x^2y^3$, and there is no addition of the indices 2 and 3 here.

Ex. Find the continued product of $-3x^2$, $2x^3$, $5x^4$.

$$\text{Product} = -3x^2 \times 2x^3 \times 5x^4$$
$$= (-3 \times 2 \times 5) \times (x^2 \times x^3 \times x^4)$$
$$= -30x^9 \text{ by Index Law (Cor.)}$$

Ex. 4. Prove that (*i*) $(x^3)^4=x^{3\times4}$ (*ii*) $(xy)^3=x^3y^3$.

(*i*) We have $(x^3)^4=x^3\times x^3\times x^3\times x^3$, by definition.

$=x^{3+3+3+3}$ (Index Law, Cor.)

$=x^{3\times4}$.

Generally, we have $(x^m)^n=x^{mn}$, where m and n are positive integers.

(*ii*) We have $(xy)^3=xy\times xy\times xy$ by definition

$=x\times y\times x\times y\times x\times y$ [associative law]

$=x\times x\times x\times y\times y\times y$ [commutative law]

$=(x\times x\times x)\times(y\times y\times y)$ [associative law]

$=x^3y^3$ [Index law].

[$\because x\times x\times x=x^{1+1+1}=x^3$, $y\times y\times y=y^{1+1+1}=y^3$]

Generally $(xy)^n=x^ny^n$, where n is a positive integer.

The general laws named in (*i*) and (*ii*) will be considered afterwards.

Ex. 5. Find the continued product of—

$$-3a^2bc,\ 2ab^3c^2,\ -4a^3b^2c^4,\ -abc.$$

Product

$=-(3a^2bc\times 2ab^3c^2\times 4a^3b^2c^4\times abc)$ by rule of signs.

$=-(3\times2\times4)\times(a^2\times a\times a^3\times a)\times(b\times b^3\times b^2\times b)\times(c\times c^2\times c^4\times c)$

by commutative and associative laws.

$=-24a^{2+1+3+1}b^{1+3+2+1}c^{1+2+4+1}$ by Index Law.

$=-24a^7b^7c^8$.

6. Multiplication of monomials. The following rule for multiplying together any number of monomial factors will be now evident :—

Write down the product of the numerical co-efficients and place after it (in alphabetical order) each of the letters involved in the factors raising it to a power of which the index is equal to the sum of the indices of its powers in the different factors.

This rule enables us to write down a product immediately. It will be seen that a **product is positive or negative according as there is an even or an odd number of negative factors.**

Ex. Write down the product of

$$\tfrac{1}{2}a^2bc^3,\ -\tfrac{1}{3}b^2c^3d^2,\ \tfrac{3}{4}c^2da, \text{ and } -2a^3db^4$$

By the rule, the product $=\frac{1}{4}a^6b^7c^8d^4$; for,
product of numerical co-efficients $=\frac{1}{2}\times(-\frac{1}{3})\times\frac{3}{4}\times(-2)=\frac{1}{4}$,
sum of indices of powers of $a=2+1+3=6$;
sum of indices of powers of $b=1+2+4=7$;
sum of indices of powers of $c=3+3+2=8$,
sum of indices of powers of $d=2+1+1=4$.

EXERCISE XVI.

Multiply together

1. $4a, -3b$. **2.** $5x, -3y$. **3.** $2ab, bc$.

4. $-2xy, 3xy$. **5.** $2x^ny, -\frac{1}{3}xy^n$. **6.** $3x^3, 5x^4$.

7. $3a^2b^3c^4, 4ab^2c$. **8** $-2a^2b^2x^3, xyz$. **9.** $9x^3y^4z^2, -4x^2y^3z^5$.

10. $3x^2, -2y^2, z^2$. **11.** $-2x, 3y^2, -4z^3$.

12. $4ab, -2ca, -3bc$. **13.** $\frac{2}{3}x^2y, -\frac{9}{10}x^3y^2, \frac{5}{4}xy$.

14. $2l^3m^3n^4, -3lm^2n^3, lmn$. **15.** $\frac{2}{3}a^4b^3c, -9b^2c^3a, \frac{1}{4}c^2a^3b$.

16. $3alb^m, -2b^mc^n, c^nal$. **17.** $2x^ay^bz^c, -x^by^cz^a, x^cy^az^b$.

Find the value of

18. $(x^3)^n$. **19.** $(x^ny)^3$. **20.** $(-3x^ny^2)^4$.

21. $(-2ab^3)^n$ **22.** $(-2a^4b^2)^3$. **23.** $(-\frac{2}{3}xyz)^3$.

Find the continued product of

24. $3x^2y, -2y^2z, 2z^2x, -5x^2y^2z^2$.

25. $\frac{2}{3}a^2b^n, -\frac{3}{5}b^3c^4, -\frac{7}{6}c^3a, \frac{9}{4}a^nb^2c^2$.

26. $2x^n, -3y^nz^n, -z^3x^n, 4xyz, -y^n$.

27. $2ab, -3c^2, \frac{1}{3}b^2c, -\frac{1}{3}ac^2, -b^2$.

28. $\frac{1}{2}x^2y^2z^4, -\frac{3}{2}xy^3z^2, \frac{4}{3}x^2yz^3, -\frac{1}{5}x^5y^4z^6, 3xyz$.

29. $3a^4b^4c^4d^2, -\frac{1}{2}b^4c^3d^2a, -4cd^3a^5b^2, 2d^2a^3b^4c^2, dcba$.

30. $-\frac{1}{3}(xy)^2, (-\frac{1}{3}xy)^2, (-3xy)^3$. **31.** $(x^2y)^2, (-xy^2)^2, -(xy)^4$.

32. $xy^2z, (2x^2yz)^2, (-3x^2yz^3)^3, (-xyz^2)^4$.

7. To prove that $(a+b)m=am+bm$, *for all values of* a, b *and* m.

(i) Let m be a *positive integer*. Then it is evident that any multiple of a *whole* is the sum of the same multiple of *each* part of the whole; hence whatever quantities a and b may be, $(a+b)$ as a whole multiplied by $m=a$ multiplied by m *plus* b multiplied by m,

or $(a+b)\ m=am+bm$.

[To understand the above we may observe that a *gain* of 5 followed by a *loss* of 7 may be represented by $5-7$. Then $(5-7)$ multiplied by any positive integral number, say 10, will represent a *gain* of 5 multiplied by 10 with a *loss* of 7 multiplied by 10 or $(5-7)\ 10=5.\ 10-7.\ 10$].

(ii) Let m be *a positive fraction*. Then it is evident that any part (or parts) of a *whole* is the sum of the same part (or parts) of *each* member of the whole; hence whatever quantities a and b may be we have as in (i),

$$(a+b)m=am+bm,$$

(iii) Let m be *negative*, say, $m=-x$, where x is positive.

Then $(a+b)m=(a+b)(-x)$

$=-(a+b)x$ by law of signs

$=-(ax+bx)$ by (i) and (ii), since x is positive

$=-ax-bx$ (see art. 2, Chap. VI)

$=a(-x)+b(-x)$

$=am+bm$.

Thus $(a+b)m=am+bm$......(1),

where a, b, m may be positive, negative, integral or fractional.

Again. $m(a+b)=(a+b)m$ by art. 1.

$=am+bm$ from (1) above

$=ma+mb$ by art. 1.

Thus $m(a+b)=ma+mb$......(2),

where a, b, m may be positive, negative, integral or fractional.

The results (1) and (2) are cases of the **distributive law** for multiplication. In (1) each term of the multiplicand is multiplied by the multiplier and the process is called *distributing the multiplicand*; while in (2) the multiplicand is multiplied by each term of the multiplier and the process is called *distributing the multiplier*.

Note. 1. Changing b into $-b$ in (1) and (2) we respectively get $(a-b)m=am-bm$, and $m(a-b)=ma-mb$.

Note. 2. Since $(a\pm b)m=am\pm bm$, we can conversely put $am\pm bm$ as the product of two factors. Thus, $am\pm bm=(a\pm b)m$.

8. Product of a binomial and a monomial. From art. 7 we deduce the following rule for multiplying together a binomial and a monomial :—

Multiply together each term of the binomial and the monomial, attending to the rule of signs, and add algebraically the partial products.

Ex. 1. Multiply $2x^2+3y^2$ by $4x$.

$$\text{Product}=(2x^2+3y^2)4x$$
$$=2x^2.4x+3y^2.4x$$
$$=8x^3+12xy^2.$$

Ex. 2. Multiply $5ab-3bc$ by $-2abc$.

Here the terms of the binomial are $5ab$, $-3bc$.

$$\therefore\ \text{product}=(5ab-3bc)\times(-2abc)$$
$$=5ab\times(-2abc)+(-3bc)\times(-2abc)$$
$$=-10a^2b^2c+6ab^2c^2.$$

9. *To prove that* $(a+b+c+\ldots\ldots)m=am+bm+cm+\ldots\ldots$, *where* m, a, b, c......*may be any quantities.*

In (1) art. 7 put $b+c$ for b, and we get

$(a+b+c)m=am+(b+c)m$. where m, a, b, c are any quantities.

$$=am+bm+cm.$$

In the last result put $c+d$ for c and we get

$$(a+b+c+d)m=am+bm+(c+d)m$$
$$=am+bm+cm+dm.$$

Similarly, $(a+b+c+d+\ldots\ldots)m=am+bm+cm+dm+\ldots\ldots$,

where m, a, b, c, d...are any quantities.

Thus to obtain the product of a polynomial and a monomial, we are to multiply together *each* term of the polynomial and the monomial, attending to the rule of signs, and add the partial products.

Note. The student is reminded that $a+b+c+d+\ldots$ represents *any* polynomial, the letters a, b, c, d...standing for any positive or negative quantities. Thus $3x^2-4xy+7y^2-z^2=3x^2+(-4xy)+7y^2+(-z^2)$ and is equal to $a+b+c+d$, if $a=3x^2$, $b=-4xy$, $c=7y^2$, $d=-z^2$

Ex. 1. Multiply $2a^2-3b^2+4c^2$ by abc.

Here the terms of the polynomial are $2a^2$, $-3b^2$, $4c^2$.

$$\begin{aligned}\therefore\ \text{product} &= (2a^2-3b^2+4c^2)\times abc\\ &= 2a^2\times abc+(-3b^2)\times abc+4c^2\times abc\\ &= 2a^3bc-3ab^3c+4abc^3.\end{aligned}$$

The above process may be shewn thus :—

$$\begin{array}{l}2a^2-3b^2+4c^2=\text{multiplicand}\\ abc \qquad\qquad\quad =\text{multiplier}\\ \hline 2a^3bc-3ab^3c+4abc^3=\text{product}\end{array}$$

We write the multiplier under the multiplicand, and multiply each term of the multiplicand by the multiplier, beginning from the left.

Ex. 2. Multiply $\frac{3}{2}xy-\frac{2}{3}yz+\frac{1}{2}zx$ by $-\frac{3}{4}xyz$.

The process may be shown thus :—

$$\begin{array}{l}\frac{3}{2}xy-\frac{2}{3}yz+\frac{1}{2}zx\\ -\frac{3}{4}xyz\\ \hline -\frac{9}{8}x^2y^2z+\frac{1}{2}xy^2z^2-\frac{3}{8}x^2yz^2=\text{product.}\end{array}$$

EXERCISE XVII.

Multiply together :—

1. $2x-3,\ 5$. **2.** $3b+4a,\ 2c$. **3.** $5x-2y,\ -7z$.

4. $2ab-3bc,\ abc$. **5.** $3x^2y^3-2y^2z,\ 3xyz^2$.

6. $5lm-7mn+2nl,\ -4$. **7.** $\frac{1}{2}x-2y-z,\ 5z$.

8. $2a^2-5b^2-c^2,\ -2a^2bc^3$. **9.** $ab^2x-bc^2y+ca^2z,\ -abxy$.

10. $\frac{2}{3}x^5-\frac{3}{4}x^4+\frac{1}{9}x^3-\frac{3}{5}x^2-\frac{4}{7}x+\frac{3}{2},\ -\frac{7}{2}x^2$.

11. $2xy^2zu^3-4x^2yz^3u+3x^4y^3z^2u^3-x^2y^2z^6u^2,\ -2xy^2zu$.

10. Product of compound expressions. For all values of a, b, x, y, we have

$$(a+b)(x+y)=(a+b)m \text{ [putting } m \text{ for } x+y]$$
$$=am+bm, \text{ distributing the multiplicand}$$
[see (1) art. 7]
$$=a(x+y)+b(x+y), \text{ restoring the value of } m$$
$$=ax+ay+bx+by\ldots\ldots(1)$$

In the above we might distribute the multiplier and proceed thus :—

$$(a+b)(x+y)=n(x+y) \text{ [putting } n \text{ for } a+b]$$
$$=nx+ny, \text{ distributing the multiplier}$$
[see (2) art. 7]
$$=(a+b)x+(a+b)y, \text{ restoring the value of } x$$
$$=ax+bx+ay+by$$
$$=ax+ay+bx+by, \text{ as in (1).}$$

Generally, $(a+b+c+\ldots\ldots)(x+y+z+\ldots\ldots)$

$$=(a+b+c+\ldots)m, \text{ where } m=x+y+z+\ldots$$
$$=am+bm+cm+\ldots$$
$$=a(x+y+z+\ldots)+b(x+y+z+\ldots)+c(x+y+z+\ldots)+\ldots$$
$$=ax+ay+az+\ldots+bx+by+bz+\ldots+cx+cy+cz+\ldots$$

From the above we get the following rule for finding the product of any two compound expressions :—

Multiply *each* term of one expression by *each* term of the other, attending to the rule of signs that two like signs produce + and two unlike signs produce −, and take the algebraic sum of these partial products.

This rule states in general form the *distributive law* for multiplication.

Ex. 1. Multiply $x+3$ by $x+4$.

Product $=(x+3)(x+4)$

$$=(x+3)x+(x+3)4, \text{ distributing the multiplier}$$
$$=x^2+3x+4x+12$$
$$=x^2+7x+12.$$

The following process is convenient in practice

$$\begin{array}{rll}
 & x+3 = \text{multiplicand} & \\
 & \underline{x+4 = \text{multiplier}} & \\
 & x^2+3x \qquad = \text{product of } (x+3) \text{ by } x & \\
 & \underline{\quad +4x+12} \quad = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ by } 4 & \\
\text{By adding,} & x^2+7x+12 \quad = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ by } x+4 &
\end{array}$$

Place the multiplier below the multiplicand, multiply the multiplicand by each term of the multiplier beginning from the left and write down the successive products in rows one under another so that like terms may be in the same column. Then the complete product is obtained by adding up these partial products.

Ex. 2. Multiply $2x-3y$ by $3x-4y$.

$$\begin{array}{l}
2x-3y \\
\underline{3x-4y} \\
6x^2-9xy \qquad\qquad = \text{product by } 3x. \\
\underline{\qquad -8xy+12y^2} = \text{product by } -4y. \\
6x^2-17xy+12y^2 = \text{complete product.}
\end{array}$$

Ex. Multiply $3x^2-2x+5$ by $2x^2+3x-4$.

$$\begin{array}{l}
3x^2-2x+5 \\
\underline{2x^2+3x-4} \\
6x^4-4x^3+10x^2 \\
\qquad +9x^3-6x^2+15x \\
\underline{\qquad\qquad -12x^2+8x-20} \\
6x^4+5x^3-8x^2+23x-20 \qquad \text{required product}
\end{array}$$

Note. In the above example we have the multiplier and the multiplicand arranged in descending powers of x. This arrangement of the multiplicand and the multiplier in ascending or descending powers of some common letter, though not necessary, is very convenient, as it facilitates the collection of like terms.

Ex. 4. Multiply $\frac{1}{2}a^2-\frac{2}{3}ab+\frac{3}{5}b^2$ by $\frac{1}{3}a-\frac{3}{2}b$.

$$\begin{array}{l}
\frac{1}{2}a^2-\frac{2}{3}ab+\frac{3}{5}b^2 \\
\underline{\frac{1}{3}a-\frac{3}{2}b} \\
\frac{1}{6}a^3-\frac{2}{9}a^2b+\frac{1}{5}ab^2 \qquad = \text{product by } \frac{1}{3}a. \\
\underline{\qquad -\frac{3}{4}a^2b+ab^2-\frac{9}{10}b^3} = \text{product by } -\frac{3}{2}b. \\
\frac{1}{6}a^3-\frac{35}{36}a^2b+\frac{6}{5}ab^2-\frac{9}{10}b^3 = \text{complete product.}
\end{array}$$

Ex. 5. Find by inspection the co-efficient of x^2 in the product

$$(3x^2+2x+4)(5x-7),$$

In finding the product we multiply $3x^2+2x+4$ by $5x$ and $3x^2+2x+4$ by -7, and add the results. In the first part the term containing x^2 is $2x\times 5x$ or $10x^2$ and in the second part the term

containing x^2 is $3x^2 \times (-7)$ or $-21x^2$. Hence the term containing x^2 in the product is $10x^2 - 21x^2$ or $-11x^2$.

Thus the required co-efficient is -11.

11. The following results in multiplication are very important and will be fully considered again.

Ex. 1. Find the values of :—

(i) $(a+b)(a-b)$. (ii) $(a+b)^2$ or $(a+b)(a+b)$.

(i)
$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \qquad -ab-b^2 \\ \hline a^2 \qquad\quad -b^2 \end{array}$$

(ii)
$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \qquad +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

$\therefore (a+b)(a-b)=a^2-b^2$ $\qquad \therefore (a+b)^2=a^2+2ab+b^2$.

Ex. 2. Find the values of :—(i) $(a^2-ab+b^2)(a+b)$.

(ii) $(a^2+ab+b^2)(a-b)$ (iii) $(a^2+ab+b^2)(a^2-ab+b^2)$.

(i)
$$\begin{array}{l} a^2-ab+b^2 \\ a+b \\ \hline a^3-a^2b+ab^2 \\ \quad +a^2b-ab^2+b^3 \\ \hline a^3 \qquad\qquad\quad +b^3 \end{array}$$

(ii)
$$\begin{array}{l} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 \\ \hline a^3 \qquad\qquad\quad -b^3 \end{array}$$

$\therefore (a^2-ab+b^2)(a+b)=a^3+b^3$. $\qquad \therefore (a^2+ab+b^2)(a-b)=a^3-b^3$.

(iii)
$$\begin{array}{l} a^2+ab+b^2 \\ a^2-ab+b^2 \\ \hline a^4+a^3b+a^2b^2 \\ \quad -a^3b-a^2b^2-ab^3 \\ \qquad\qquad +a^2b^2+ab^3+b^4 \\ \hline a^4 \qquad\quad +a^2b^2 \qquad +b^4 \end{array}$$

$\therefore (a^2+ab+b^2)(a^2-ab+b^2)=a^4+a^2b^2+b^4$.

Ex. 3. Find the value of $(a^2+b^2+c^2-ab-ac-bc)(a+b+c)$

We proceed by arranging in descending powers of a :

$$\begin{array}{l} a^2-ab-ac+b^2+c^2-bc \\ a+b+c \\ \hline a^3-a^2b-a^2c+ab^2+ac^2-abc \\ \quad +a^2b \qquad -ab^2 \qquad -abc+b^3+bc^2-b^2c \\ \qquad\quad +a^2c \qquad -ac^2-abc \qquad -bc^2+b^2c+c^3 \\ \hline a^3 \qquad\qquad\qquad\qquad -3abc+b^3 \qquad\qquad +c^3 \end{array}$$

$\therefore (a^2+b^2+c^2-ab-ac-bc)(a+b+c)=a^3+b^3+c^3-3abc$.

EXERCISE XVIII.

Multiply together :—

1. $x-5,\ x+8$.
2. $x+7,\ x-2$.
3. $x-4,\ x-9$.
4. $3x+5y,\ 2x-7y$
5. $3a-10b,\ a+3b$.
6. $5a-6b,\ 2a-3b$.
7. $2a+b,\ 3c-2d$.
8. $a-4b,\ 2x-3y$.
9. $2a+3b,\ 5l-7m$.
10. $2x+3y,\ 2x-3y$.
11. $3ab+4cd,\ 3ab-4cd$.
12. $\frac{3}{2}ax-\frac{5}{4}by,\ \frac{7}{3}ax-\frac{3}{4}by$.
13. $2a^3-5a^2b+7ab^2-3b^3,\ 3a-5b$.
14. $4x^2y-5xy^2+2x^3-2y^3,\ 4x-3y$.
15. $5x^3+6xy^2-3x^2y-3y^3,\ 2x-7y$.
16. $2a^4-3a^2b^2+4ab^3-5b^4+2a^3b,\ 3a-6b$.
17. $\frac{1}{3}a^2-\frac{2}{3}a+\frac{1}{2},\ \frac{3}{4}a-\frac{1}{3}$.
18. $\frac{1}{2}x^2-\frac{1}{3}xy+\frac{1}{4}y^2,\ \frac{2}{3}x-\frac{4}{5}y$.
19. $a-b+c,\ a+b-c$.
20. $3x^3-7x+2-4x^2,\ 3x-4-2x^2$.
21. $\frac{2}{3}x^2-\frac{3}{4}x-\frac{5}{2},\ \frac{1}{2}x-\frac{1}{3}+\frac{4}{3}x^2$.
22. $2x-3y+5z,\ 3x-2y-4z$.
23. $a^2-3ab+b^2,\ 3a^2+4ab-2b^2$.
24. $4x^2-5ax+3a^2,\ 4x^2+5ax+3a^2$.
25. $9x^2-11xy+2y^2,\ 5y^2+3xy+x^2$.
26. $9x^2+y^2+4+2y-6x+3xy,\ 3x-y+2$.
27. $2x^2-7x+3x^4+2,\ 4-3x+5x^2$.
28. $5a^3-3a^2+2a-1,\ a^3+$[illegible]$-4a+5$.
29. $1+2x^4-3x^3-4x+x^2,\ 1-2x+3x^2$.
30. $2x^3-4x^4+x^2-3x+2,\ -4+x^2-3x$.
31. $4a^2+9b^2+c^2+6ab-2ac+3bc,\ 2a-3b+c$.
32. $1-x+x^2-x^3+x^4,\ 1+x+x^2+x^3+x^4$.
33. $3x^3y-4xy^3-x^4+2x^2y^2+y^4,\ x^3-3x^2y+3xy^2-y^3$.
34. $x^5-17x^4+105x^3+19x^2+23x-41,\ x^3-99x^2+x-29$.

Find the value of :—

35. $(3x+4y)^2$.
36. $(5x-2y)^2$.
37. $(2a-3b+5c)^2$
38. $(3x+y)^2\times(3x-y)^2$.
39. Find by inspection the co-efficient of x in
    (i) $(4x-7)(3x+2)$.
    (ii) $(7x+2)(5x-3)$.
    (iii) $(2x+9)(9x-11)$.
    (iv) $(4x-5)(9x-11)$,
40. Find by inspection the co-efficient of x^2 in
    $(5x^2-7x+2)(3x-8)$ and $(3x^3-4x^2+7x-1)(2x-5)$.

12. Continued product. In finding the continued product of a number of expressions we multiply the first by the second, the result by the third, then *this* result by the fourth and so on.

Ex. 1. Find the continued product of $2x-1$, $3x+2$, $4x-3$.

$$\begin{array}{rl} 2x-1 & \\ 3x+2 & \\ \hline 6x^2-3x & \\ +4x-2 & \\ \hline 6x^2+x-2 & \end{array} \qquad \begin{array}{l} 6x^2+x-2 \\ 4x-3 \\ \hline 24x^3+4x^2-8x \\ \quad -18x^2-3x+6 \\ \hline 24x^3-14x^2-11x+6 = \text{reqd product.} \end{array}$$

Ex. 2. Find the value of $(a+b)^3$ or $(a+b)(a+b)(a+b)$.

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array} \qquad \begin{array}{l} a^2+2ab+b^2 \\ a+b \\ \hline a^3+2a^2b+ab^2 \\ \quad +a^2b+2ab^2+b^3 \\ \hline a^3+3a^2b+3ab^2+b^3 = \text{reqd. value.} \end{array}$$

Ex. 3. Find the continued product of $a-b$, $a+b$, a^2+b^2, a^4+b^4.

$$\begin{array}{l} a-b \\ a+b \\ \hline a^2-ab \\ \quad +ab-b^2 \\ \hline a^2 \qquad -b^2 \end{array} \qquad \begin{array}{l} a^2-b^2 \\ a^2+b^2 \\ \hline a^4-a^2b^2 \\ \quad +a^2b^2-b^4 \\ \hline a^4 \qquad -b^4 \end{array} \qquad \begin{array}{l} a^4-b^4 \\ a^4+b^4 \\ \hline a^8-a^4b^4 \\ \quad +a^4b^4-b^8 \\ \hline a^8 \qquad -b^8 = \text{reqd. value} \end{array}$$

Note.—In this example if the factors are multiplied in any other order the process will be tedious. A judicious arrangement of factors in a continued product often saves much trouble in multiplication.

EXERCISE XIX.

Find the continued product of :—

1. $x+1$, $x+2$, $x+3$.
2. $x+4$, $x-5$, $x+6$.
3. $3x-1$, $4x+2$, $5x-3$.
4. $4x-5y$, $6x-y$, $2x-5y$.
5. $b+c$, $c+a$, $a+b$.
6. $b-c$, $c-a$, $a-b$.
7. $2x^2-3x+1$, $2x-1$, $3x-2$.
8. a^2-ab+b^2, $3a-2b$, $2a+3b$.
9. $x-a$, x^2+ax+a^2, x^3+a^3.
10. $x-1$, $x+1$, x^2+1, x^4+1.
11. x^2+x+1, x^2-x+1, x^4-x^2+1.
12. a^2+ab+b^2, a^2-ab+b^2. $a^4-a^2b^2+b^4$.

CHAPTER VIII.

DIVISION.

1. *Division* has been defined as the inverse of multiplication.

Thus $a \div b$ is that which when multiplied by b gives a or $(a \div b) \times b = a$. Hence if $a = bc$, then $a \div b = c$ or $a \div c = b$.

The result $a \div b$ is often written $\frac{a}{b}$ or a/b.

In $a \div b$, a is called the dividend, b the divisor, the result is called the quotient.

It follows that quotient = dividend ÷ divisor ; dividend = divisor × quotient ; divisor = dividend ÷ quotient.

2. *To prove that* (i) $a \times (b \div c) = a \times b \div c$.

(ii) $a \div (b \times c) = a \div b \div c$.

(i) We have $(b \div c) \times c = b$ by definition.

$\therefore (b \div c) \times c \times a = b \times a$,

or $a \times (b \div c) \times c = a \times b$, commutative law for multiplication.

$\therefore a \times (b \div c) = a \times b \div c$, dividing both sides by c.

(ii) Let $b \times c = k$, then $(a \div k) \times (b \times c) = a$.

$\therefore (a \div k) \times c \times b = a$, for $xb \times c = xc \times b$.

$\therefore (a \div k) \times c = a \div b$, dividing both sides by b.

$\therefore a \div k = a \div b \div c$, dividing both sides by c.

$\therefore a \div (b \times c) = a \div b \div c$, restoring the value of k.

The above results state the **associative law** for multiplication and division.

3. *To prove that* $a \div b \div c = a \div c \div b$.

$$\left.\begin{array}{l}\text{We have } a \div b \div c = a \div (b \times c) \\ \text{and } a \div c \div b = a \div (c \times b)\end{array}\right\} \text{by (ii), art. 2.}$$

But $b \times c = c \times b$; hence $a \div b \div c = a \div c \div b$.

This states the **commutative law** for division *viz.*, *in a chain of divisions the order of operations is indifferent.*

The same law may be proved when multiplications and divisions are combined. Thus $a \div b \times c = a \times c \div b$, *i.e. in a chain of multiplications and divisions combined, operations may be conducted in any order, taking into account the sign of operation* ($\times$ *or* $\div$) *before each symbol.*

4. Index law for division. *To prove that* $a^m \div a^n = a^{m-n}$, *where m and n are positive integers and* $m > n$.

As a particular case, suppose we want $a^5 \div a^2$. Now we know that $a^5 = (a^3) \times (a \times a) = a^3 \times a^2$;

$\therefore$ by definition $a^5 \div a^2 = a^3$ or a^{5-2}.

Generally, $a^m = a \times a \times a \times$ ......to m factors.

$= (a \times a \times \ldots$to $m - n$ factors$) \times (a \times a \times \ldots$to n factors$)$.

$= a^{m-n} \times a^n$.

$\therefore$ by definition, $a^m \div a^n = a^{m-n}$, where $m > n$.

Otherwise thus :—

$$a^m \div a^n = \frac{a^m}{a^n} = \frac{a \times a \times \text{......to } m \text{ factors}}{a \times a \times \text{......to } n \text{ factors}}$$

$$= \frac{(a \times a \times \text{to } m-n \text{ factors}) \times (a \times a \times \text{...to } n \text{ factors})}{(a \times a \times \text{...to } n \text{ factors})}$$

$\therefore a^m \div a^n = a \times a \times \ldots$to $m - n$ factors, by striking out common factors

$= a^{m-n}$ by def. of a power.

The above result is called the *index law for division* and states that *in dividing one power of a base by another power of the same base we take a power of the base of which the index is the difference of the indices of the given powers. The rule is therefore to subtract the index of the divisor from that of the dividend.*

Cor. If in the above law we put $n = m$. we get $a^m \div a^m = a^{m-m} = a^0$. But $a^m \div a^m = 1$; hence $a^0 = 1$, a curious result which we shall examine afterwards.

5. Division of monomial by a monomial. To divide one monomial by another we proceed as shown below :—

Ex. 1. Divide $2abc$ by bc.

Since $2abc = 2a \times bc$, we have $2abc \div bc = 2a$.

Ex. 2. Divide $-24a^5b^4c^3$ by $6a^2b^3c$.

We have $(-24a^5b^4c^3)\div(6a^2b^3c)$.

$=(-24\times a^5\times b^4\times c^3)\div(6\times a^2\times b^3\times c)$

$= -24\times a^5\times b^4\times c^3 \div 6\div a^2\div b^3\div c$, by associative law

[see (ii) art. 2]

$= -24\div 6\times a^5\div a^2\times b^4\div b^3\times c^3\div c$, by commutative law

[see art. 3, latter part.]

$= (-24\div 6)\times(a^5\div a^2)\times(b^4\div b^3)\times(c^3\div c)$, by associative law

[see (i) art. 2]

$= -4a^3bc^2$, by index Law (art. 4).

The above examples lead to the following rule for dividing one monomial by another :—

Write down the quotient of the numerical coefficient of the dividend by that of the divisor, attending to the rule of signs (see art. 16, Chap. IV) and place after it each letter involved in the dividend, raising it to a power of which the index is equal to the index of its power in the dividend minus the index of its power in the divisor.

Ex. 3. Divide $20x^7y^9z^{10}$ by $-4x^2y^5z^4$.

By the rule, quotient$= -5x^{7-2}y^{9-5}z^{10-4} = -5x^5y^4z^6$.

EXERCISE XX.

Divide

1. $4a$ by $-a$. **2.** $-2xy$ by $-y$. **3.** abc by $-ab$.

4. $\frac{2}{5}x^3$ by $-\frac{3}{4}x$. **5.** $\frac{4}{5}a^5$ by $-\frac{3}{7}a^3$. **6.** $-\frac{1}{2}a^3b^4$ by $\frac{4}{3}ab^2$.

7. $-27a^5b^5$ by $3ab$. **8.** $21x^3y^7$ by $-7xy^4$.

9. $18x^7y^9z^2$ by $-6x^2y^5$. **10.** $-36a^9b^3x^4$ by $9a^5x^2$.

11. $-12a^5b^4c^3x^6y^7z^9$ by $-3a^3b^2x^3$, $-2c^2x^4y^3z^4$, $6a^2b^2c^2x^4y^4z^4$.

12. $la^{3p}b^{2q}c^{5r}$ by $ma^pb^qc^r$, $na^{2p}b^{2q}c^{3r}$, pb^qc^{4r}.

13. $\frac{3}{4}.(bc)^2.(ca)^3.(ab)^4$ by $\frac{3}{16}a^2b$, $-\frac{3}{8}a^3b^2c^3$, $-\frac{9}{2}(abc)^3$.

14. Prove that (i) $a\div b\times c\div d=a\times c\div b\div d$.

(ii) $a\div(b\times c\times d)=a\div b\div c\div d$.

6. Division of a polynomial by a monomial. To prove that $(a+b+c+\ldots)\div m = a\div m + b\div m + c\div m + \ldots$, where $m, a, b, c\ldots$ are any quantities.

For all values of m, A, B, C,...we have (art. 9 Chap. VII) $(A+B+C+\ldots)\,m = Am+Bm+Cm+\ldots\ldots$

$$\therefore\ (Am+Bm+Cm+\ldots)\div m = A+B+C+\ldots\ldots$$
$$= Am\div m + Bm\div m + Cm\div m + \ldots$$

Hence putting $Am=a$, $Bm=b$, $Cm=c$,...we have

$$(a+b+c+\ldots)\div m = a\div m + b\div m + c\div m + \ldots\ldots$$

Hence to divide a polynomial by a monomial, *divide each term of the polynomial and add the partial quotients.*

This is the **distributive law** for division and the dividend is said to be distributed. It will be noted that the divisor cannot be distributed here ; thus $m\div(a+b)$ is *not equal* to $m\div a + m\div b$, as can be easily seen. Hence the distributive law has limited application in the case of division.

Ex. 1. Divide $6x^2 - 10xy^2 + 2x^4y$ by $2x$.

$$\text{Quotient} = \frac{6x^2 - 10xy^2 + 2x^4y}{2x}$$
$$= \frac{6x^2}{2x} - \frac{10xy^2}{2x} + \frac{2x^4y}{2x}$$
$$= 3x - 5y^2 + x^3y.$$

Ex. 2. Divide $3a^2b^7 + 9a^3b^4 - 12a^5b^3$ by $-3a^2b^2$.

$$\text{Quotient} = \frac{3a^2b^7 + 9a^3b^4 - 12a^5b^3}{-3a^2b^2}$$
$$= \frac{3a^2b^7}{-3a^2b^2} + \frac{9a^3b^4}{-3a^2b^2} + \frac{-12a^5b^3}{-3a^2b^2}$$
$$= -b^5 - 3ab^2 + 4a^3b.$$

EXERCISE XXI.

Divide

1. $10x - 5y$ by 5.
2. $2ax - 3ay$ by a.
3. $6x^4 - 4x^3$ by $-2x^2$.
4. $6abx - 12aby$ by $4ab$.
5. $9ab^3 + 12a^2b^4 - 15a^5b^5$ by $-3ab^2$.
6. $3a^3x^3 - 2a^2x^4 + 4ax$ by $\frac{1}{3}ax$.
7. $4a^3b^4c^6 - 6a^2b^5c^4 + 10a^4b^7c^9$ by $-2b^2c^3$, $\frac{1}{2}a^2b^3c^4$, $-ac^3$.

Divide

8. $3a^3b^3c^2d - 5a^2b^2c^3d^3 + 4ab^4c^2d^2$ by $-2abcd$.

9. $pa^4l^3 - qa^5l^4 + ra^6l^9$ by $-2a^3l^3$, $3a^2l$.

10. $35x^7y^9z^6 - 15x^5y^6z^7 + 45x^6y^8z^5 - 60x^4y^7z^4$ by $-5x^3y^4z^2$.

11. $a^3b(a-b)$ by a ; $(a+b)(c+d)^2$ by $c+d$.

7. Division of one compound expression by another.

To find the quotient of one compound expression by another, we are to break up the dividend into several parts each of which contains the divisor, divide each of these parts by the divisor and take the algebraical sum of these results for the required quotient. This leads to the following practical rule :—

1. Arrange the divisor and the dividend in ascending or descending powers of some letter common to both.

2. Divide the first term of the dividend by the first term of the divisor put the result down as the first term of the quotient.

3. Multiply the whole divisor by this first term of the quotient and subtract the product from the dividend.

4. Consider the remainder (on bringing down after it terms from the dividend, if necessary) as a new dividend and repeat the above operations as often as may be necessary till all the terms from the dividend are brought down.

Ex. 1. Divide $10x^2+23x+12$ by $5x+4$.

The process stands thus ; the divisor, the dividend and the quotient being arranged as in Arithmatic :—

$$\begin{array}{r|l|l}
5x+4 & 10x^2+23x+12 & 2x+3 = \text{quotient.} \\
 & \underline{10x^2+8x\qquad\quad} & \\
 & \qquad +15x+12 & \\
 & \qquad \underline{+15x+12} & \\
\end{array}$$

Explanation :—The first term of the dividend is $10x^2$ and that of the divisor is $5x$; hence the first term of the quotient is $10x^2 \div 5x = 2x$. We multiply $5x+4$ by $2x$ and subtract the product $10x^2+8x$ from the dividend. The remainder $15x+12$ is treated as a new dividend and we get $15x \div 5x$ or 3 as the second term of the quotient. By multiplying $5x+4$ by 3 we get $15x+12$ and by subtracting it from the new dividend $15x+12$, no remainder is left.

Reason for the process :—It will be seen from the above that we have in effect subtracted $2x(5x+4)$ and $3(5x+4)$ in seccession from $10x^2+23x+12$ and the remainder is found to be zero. Hence $10x^2+23x+12=2x(5x+4)+3(5x+4)=2x.\mathrm{A}+3.\mathrm{A}$ where $\mathrm{A}=5x+4$.

$\therefore (10x^2+23x+12)\div \mathrm{A}=(2x\mathrm{A}+3\mathrm{A})\div \mathrm{A}=2x+3$, by art 6.

Hence $(10x^2+23x+12)\div(5x+4)=2x+3$.

EX. 2. Divide $2x^5+x^3+3x^2-3x+2$ by $2x^2-2x+1$.

$$
\begin{array}{l}
2x^2-2x+1\,)\,2x^5 \qquad +x^3+3x^2-3x+2\,(\,x^3+x^2+x+2 = \text{Quotient.}\\
\qquad\qquad\quad \underline{2x^5-2x^4+x^3}\\
\qquad\qquad\qquad +2x^4 \qquad +3x^2\\
\qquad\qquad\qquad \underline{\ \ 2x^4-2x^3+x^2}\\
\qquad\qquad\qquad\qquad 2x^3+2x^2-3x\\
\qquad\qquad\qquad\qquad \underline{2x^3-2x^2+x}\\
\qquad\qquad\qquad\qquad\qquad 4x^2-4x+2\\
\qquad\qquad\qquad\qquad\qquad \underline{4x^2-4x+2}
\end{array}
$$

EX. 3. Divide $\frac{4}{9}a^3-\frac{19}{48}ab^2-\frac{3}{16}b^3$ by $\frac{2}{3}a-\frac{3}{4}b$.

$$
\begin{array}{l}
\frac{2}{3}a-\frac{3}{4}b\,)\,\frac{4}{9}a^3 \qquad -\frac{19}{48}ab^2-\frac{3}{16}b^3\,(\,\frac{2}{3}a^2+\frac{3}{4}ab+\frac{1}{4}b^2 \quad = \text{Quotient.}\\
\qquad\qquad \underline{\frac{4}{9}a^3-\frac{1}{2}a^2b}\\
\qquad\qquad\quad +\frac{1}{2}a^2b-\frac{19}{48}ab^2\\
\qquad\qquad\quad \underline{\ \ \frac{1}{2}a^2b-\frac{9}{16}ab^2}\\
\qquad\qquad\qquad\qquad \frac{1}{6}ab^2-\frac{3}{16}b^3\\
\qquad\qquad\qquad\qquad \underline{\frac{1}{6}ab^2-\frac{3}{16}b^3}
\end{array}
$$

Here first term of the quotient $=\frac{4}{9}a^3\div\frac{2}{3}a=\frac{2}{3}a^2$, the second term $=\frac{1}{2}a^2b\div\frac{2}{3}a=\frac{3}{4}ab$, the third term $=\frac{1}{6}ab^2\div\frac{2}{3}a=\frac{1}{4}b^2$.

EX. 4. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

Arrange the dividend in descending powers of a : thus,

$$
\begin{array}{l}
a+b+c\,)\,a^3+ \qquad\qquad -3abc+b^3+c^3\,(\,a^2-ab-ac+b^2-bc+c^2\\
\qquad\qquad \underline{a^3+a^2b+a^2c}\\
\qquad\qquad\quad -a^2b-a^2c \qquad -3abc\\
\qquad\qquad\quad \underline{-a^2b \qquad -ab^2-abc}\\
\qquad\qquad\qquad\quad -a^2c+ab^2-2abc\\
\qquad\qquad\qquad\quad \underline{-a^2c \qquad -abc-ac^2}\\
\qquad\qquad\qquad\qquad +ab^2-abc+ac^2+b^3\\
\qquad\qquad\qquad\qquad \underline{+ab^2 \qquad\qquad +b^3+b^2c}\\
\qquad\qquad\qquad\qquad\quad -abc+ac^2 \qquad -b^2c\\
\qquad\qquad\qquad\qquad\quad \underline{-abc \qquad\qquad -b^2c-bc^2}\\
\qquad\qquad\qquad\qquad\qquad\quad ac^2 \qquad\qquad +bc^2+c^3\\
\qquad\qquad\qquad\qquad\qquad\quad ac^2 \qquad\qquad +bc^2+c^3
\end{array}
$$

Ex. 5. Divide x^4-y^4 by $x+y$.

Here the terms containing x^3, x^2, x are wanting.

$$
\begin{array}{r|l|l}
x+y & x^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots -y^4 & x^3-x^2y+xy^2-y^3. \\
 & \underline{x^4+x^3y} & \\
 & -x^3y & \\
 & \underline{-x^3y-x^2y^2} & \\
 & x^2y^2 & \\
 & \underline{x^2y^2+xy^3} & \\
 & -xy^3-y^4 & \\
 & \underline{-xy^3-y^4} &
\end{array}
$$

8. Inexact division. Sometimes, after the operation of division some remainder is left. Generally the quotient in such cases is required to a certain number of terms.

Ex. 1. Divide x^2-5x+9 by $x-3$.

$$
\begin{array}{r|l|l}
x-3 & x^2-5x+9 & x-2 \\
 & \underline{x^2-3x} & \\
 & -2x+9 & \\
 & \underline{-2x+6} & \\
 & +3 &
\end{array}
$$

Hence the quotient $=x-2$, and the remainder $=3$.

Ex. 2. Divide $1+a$ by $1-a+a^2$ to four terms.

$$
\begin{array}{r|l|l}
1-a+a^2 & 1+a & 1+2a+a^2-a^3 \\
 & \underline{1-a+a^2} & \\
 & 2a-a^2 & \\
 & \underline{2a-2a^2+2a^3} & \\
 & a^2-2a^3 & \\
 & \underline{a^2-a^3+a^4} & \\
 & -a^3-a^4 & \\
 & \underline{-a^3+a^4-a^5} & \\
 & -2a^4+a^5 &
\end{array}
$$

Thus the quotient $=1+2a+a^2-a^3$ and the remainder $=-2a^4+a^5$.

EXERCISE XXII.

Divide

1. $x^2-9x+20$ by $x-5$.
2. $6x^2+5x-4$ by $3x+4$.
3. $15x^2-2xy-24y^2$ by $3x-4y$.

Divide.

4. $3a^2 - 11ab - 20b^2$ by $3a + 4b$.

5. $12x^2 - 4xy - 21y^2$ by $6x + 7y$.

6. $6 - 7x - 98x^2$ by $3 - 14x$.

7. $162b^2 - 20 + 27b$ by $9b + 4$.

8. $6l^2 - 7lm - 245m^2$ by $l - 7m$.

9. $\frac{1}{6}a^2 + \frac{35}{72}ab + \frac{1}{4}b^2$ by $\frac{1}{2}a + \frac{1}{3}b$.

10. $\frac{1}{2}x^2 - \frac{67}{48}xy + \frac{15}{16}y^2$ by $\frac{2}{3}x - \frac{3}{4}y$.

11. $6x^3 - 17x^2 + 22x - 15$ by $2x - 3$.

12. $15x^3 + 11x^2 - 23x + 6$ by $3x - 2$.

13. $14x^4 - 33x^3 + 43x - 24$ by $2x^2 - 5x + 3$.

14. $6x^5 - x^4 - 8x^3 - 7x + 10$ by $3x^2 + x - 5$.

15. $3a^4 - 25a^3b + 54a^2b^2 - 46ab^3 + 16b^4$ by $a^2 - 7ab + 8b^2$.

16. $2a^6 - 5a^5 + 25a - 33$ by $a^3 - a^2 - 2a + 3$.

17. $10c^6 - 11c^5 - 2c^4 - 26c^3 + 5c^2 + c + 35$ by $2c^3 - 3c^2 + 2c - 5$.

18. $25a^6 - 19a^4 - a^3 + 15a^2 + 4a - 6$ by $5a^3 + 3a^2 - 2$.

19. $1 - 2x^3 + x^6$ by $1 - 2x + x^2$.

20. $3l^5 - 5l^3 + 2$ by $1 - 2l + l^2$.

21.* $28x^4 + 13x^2y^2 - xy^3 + 15y^4$ by $4x^2 + 4xy + 3y^2$.

22. $6x^5 + 13x^4y - 5x^3y^2 - 6x^2y^3 + 5xy^4 - y^5$ by $3x^2 + 2xy - y^2$.

23. $6a^5 - 4a^4b - 20a^3b^2 - 7a^2b^3 - 20ab^4 + 10b^5$ by $2a^3 - 4a^2b - 5b^3$.

24. $a^3 + b^3$ by $a + b$.

25. $a^3 - b^3$ by $a - b$.

26. $a^4 + 4b^4$ by $a^2 + 2ab + 2b^2$.

27. $a^5 + b^5$ by $a + b$.

28. $a^4 + a^2b^2 + b^4$ by $a^2 - ab + b^2$.

29. $a^6 - b^6$ by $a - b$.

30. $1 + 6x^5 + 5x^6$ by $1 + 2x + x^2$

31. $a^8 - b^8$ by $a - b$.

32. $a^6 - b^6$ by $a^2 - ab + b^2$.

33. $a^7 + b^7$ by $a + b$.

34. $8a^3 - 27b^3 + 125c^3 + 90abc$ by $2a - 3b + 5c$.

35. $1 + 8x^3 - 27y^3 + 18xy$ by $1 + 2x - 3y$.

36. $x^3 + y^3 + 3xy - 1$ by $x + y - 1$.

Divide.

37. $a^3+b^3-c^3+3abc$ by $a+b-c$.

38. $a^4+b^4+c^2-2b^2c^2-2c^4a^2-2a^2b^2$ by $a^2-2ab+b^2-c^2$.

39. $\frac{1}{3}x^4-\frac{11}{12}x^3+4\frac{1}{8}x^2-2\frac{3}{4}x+6$ by $\frac{2}{3}x^2-\frac{5}{6}x+1$.

40. $\frac{1}{6}x^4-\frac{11}{36}x^3y+\frac{17}{36}x^2y^2-\frac{7}{18}xy^3+\frac{1}{6}y^4$ by $\frac{1}{2}x^2-\frac{2}{3}xy+\frac{1}{3}y^2$.

41. $\frac{9}{8}x^4-\frac{3}{2}x^3y+\frac{1}{2}x^2y^2-\frac{2}{9}y^4$ by $\frac{3}{4}x^2-\frac{1}{2}xy+\frac{1}{3}y^2$.

42. $\frac{1}{16}x^4-\frac{16}{81}y^4$ by $\frac{1}{2}x-\frac{2}{3}y$.

43. $10x^4+x^3y+18x^2y^2-72xy^3-27y^4$ by $2x^2+3xy+9y^2$.

44. $3a^4-5a^3b+6a^2b^2+5ab^3-3b^4$ by $a^2-2ab+3b^2$.

45. $6x^6-19x^5+6x^3-3x+2$ by $3x^2-2x+1$.

46. What must multiply $1+x^2+x^3$ to give $1-x^4+2x^3+x^6$?

47. By what must x^2-ax+a^2 be multiplied to give $x^5+a^2x^3+a^3x^2+a^5$?

48. If the dividend be $6x^6-19x^5+6x^3-3x+2$ and the quotient $3x^2-2x+1$, find the divisor.

49. Divide x^3-ax^2+bx-c by $x-a$ and find the remainder.

50. Divide to four terms :—

(1) $1-2x$ by $1+5x$ (2) $1-x$ by $1-2x+2x^2$.

CHAPTER IX.

BRACKETS.

1. We have already stated that an expression is enclosed within brackets when it is intended to take it *as a whole*. When we use brackets within brackets in an expression, to avoid confusion we use different forms of them, as [ ], { }, ( ),—.

Thus $a-\{b-(c-d)\}$ means that $c-d$ as a whole is to be subtracted from b and *this* result subtracted from a.

2. Removal of brackets. We know $+(2a-3b+4c)$ indicates the operation of adding the expression $2a-3b+4c$ *as a whole* and from art. 8, Chap. V it follows that $+(2a-3b+4c)=2a-3b+4c$.

Hence, when the sign + precedes a pair of brackets, the brackets may be removed without interfering with the sign of any term within them. (Rule I).

Again, we know that $-(2a-3b+4c)$ indicates the operation of subtracting the expression $2a-3b+4c$ as *a whole*, and from art. 2, Chap. VI., it follows that $-(2a-3b+4c)=-2a+3b-4c$.

Hence, when the sign — precedes a pair of brackets, the brackets may be removed by changing the sign of every term within them. (Rule II).

Ex. 1. Simplify $x-y+z-(y-z+x)+(z+x-y)$

The expr. $=x-y+z-y+z-x+z+x-y.$

$=x-3y+3z$, collecting like terms.

Ex. 2. Find the value of $2(3x-4y+5z)-3(x+2y-z)$.

We have $2(3x-4y+5z)=6x-8y+10z$

$3(x+2y-z)=3x+6y-3z$

$\therefore$ the expr. $=6x-8y+10z-(3x+6y-3z)$

$=6x-8y+10z-3x-6y+3z$

$=3x-14y+13z$

In practice the last *two* steps only are written down.

Ex. 3. Simplify $\dfrac{12x-3}{4}-\dfrac{6x+2}{3}+\dfrac{12x-8}{24}$.

Here the line separating each numerator from the denominator serves as a vinculum.

The expr. $=\dfrac{12x}{4}-\frac{3}{4}-\left(\dfrac{6x}{3}+\frac{2}{3}\right)+\dfrac{12x}{24}-\frac{8}{24}$.

[note the brackets inserted here].

$=3x-\frac{3}{4}-2x-\frac{2}{3}+\dfrac{x}{2}-\frac{1}{3}.$

$=\dfrac{3x}{2}-\frac{7}{4}.$

Ex. 4. Simplify $3(x+8)(x+3)-5(x+4)(x+5)+2(x+3)(x+4)$.

We have $(x+8)(x+3)=x(x+3)+8(x+3)$

$=x^2+3x+8x+24=x^2+11x+24.$

Similarly $(x+4)(x+5)=x^2+9x+20$,

$(x+3)(x+4)=x^2+7x+12.$

$\therefore$ the expr. $=3(x^2+11x+24)-5(x^2+9x+20)+2(x^2+7x+12)$

$=3x^2+33x+72-5x^2-45x-100+2x^2+14x+24$

$=2x-4.$

EXERCISE XXIII.

Simplify

1. $(2+3-6)-(3-4+5)-(7-8+4)$.

2. $5x+7y-9z-(2y+5z-4x)-(5x+2y)$.

3. $x-(3y+5z)+(2z-3x)-(5x+2y)$.

4. $(x+1)(x^2-x+1)-(x-1)(x^2+x+1)$.

5. $(x+3)(x+2)-2(x-3)(x-2)+(x+1)(x+4)$.

6. $3a(2a-3b)-3a(5a-7b)-(6a-11b)(a-5b)$.

7. $(2x-3)(3x+4)-2(3x-2)(4x+3)+(5x+1)(x-6)$.

8. $\frac{x-2}{2}+\frac{x-10}{9}-\frac{x-5}{3}$.

9. $\frac{12x-9}{3}-\frac{15x+21}{9}-\frac{10x-22}{6}$.

10. $3(a+2b+c)(a+2b-c)-2(a-2b+c)(a-2b-c)$.

11. $a^2b(2a-3b+4c)-ab^2(4a+3b-5c)+ab(2a^2-2b^2+3c^2)$.

12. $a(b-c)+b(c-a)+c(a-b)$.

13. Subtract $bcd^2-(a^2-c^2)bd$ from $(a^2+bc)d^2-(a^2-c^2)bd$.

14. Simplify $24\{x-\frac{1}{2}(x-3)\}\{x-\frac{2}{3}(x+2)\}\{x-\frac{3}{4}(x-1\frac{1}{3})\}$, and subtract the result from $(x+2)(x-3)(x+4)$ (M. M. 1886).

---

3. When there are brackets within brackets in one expression, in removing them we may begin from the innermost pair or the outermost pair or we may remove them all at once.

Ex. 1. Simplify by removing brackets

$$9a-3b-\{a-8b+(5a+7b)\}.$$

(i) Beginning from the innermost pair,

the expr. $=9a-3b-\{a-8b+5a+7b\}$ by rule I.

$=9a-3b-a+8b-5a-7b$ by rule II.

$=3a-2b$, by collecting like terms.

(ii) Beginning from the outermost pair,

the expr. $=9a-3b-a+8b-(5a+7b)$ by rule II.

$=9a-3b-a+8b-5a-7b$ by rule II.

$=3a-2b$ as before.

(iii) If we want to remove the brackets at once we are to consider the effects of their removal upon each term. Thus in the above expression we have $-\{-8b\}=+8b, -\{+(5a)\}=-5a$, and so in other terms.

Hence the expr. $=9a-3b-a+8b-5a-7b$

$=3a-2b.$

Ex. 2. Simplify

$$3x-2y-[5z+3x+\{-2y-3z-(5x-2y-\overline{-3z+5x})\}]$$

The expression

$=3x-2y-[5z+3x+\{-2y-3z-(5x-2y+3z-5x)\}]$

$=3x-2y-[5z+3x+\{-2y-3z-5x+2y-3z+5x\}]$

$=3x-2y-[5z+3x-2y-3z-5x+2y-3z+5x]$

$=3x-2y-5z-3x+2y+3z+5x-2y+3z-5x$

$=-2y+z.$

We might begin by removing the brackets from the outermost pair. In many cases it will be useful to go on simplifying in the course of removing brackets, specially when there are multipliers before them.

Ex. 3. What is the defect of the sum of $a-4b+c$ and $5a+3b-7c$ from the difference of $10a+5b-10c$ and $4a-6b+3c$?

The required answer

$=\{(10a+5b-10c)-(4a-6b+3c)\}-\{(a-4b+c)+(5a+3b-7c)\}$

$=\{10a+5b-10c-4a+6b-3c\}-\{a-4b+c+5a+3b-7c\}$

$=(6a+11b-13c)-(6a-b-6c)$

$=6a+11b-13c-6a+b+6c=+12b-7c.$

Ex. 4. Simplify $7x-2[y-3\{4x-(5y+2x)+3y\}-4x]+9y.$

The expr. $=7x-2[y-3\{4x-5y-2x+3y\}-4x]+9y$

$=7x-2[y-3\{2x-2y\}-4x]+9y$

$=7x-2[y-6x+6y-4x]+9y$

$=7x-2[7y-10x]+9y$

$=7x-14y+20x+9y=27x-5y.$

4. Insertion of brackets. From art. 2 we get the following rules for the insertion of brackets :—

(1) Any number of terms of an expression may be enclosed within brackets preceded by the sign +, provided we keep the signs of the terms unchanged.

(2) Any number of terms of an expression may be enclosed within brackets preceded by the sign −, provided we change the signs of the terms.

Thus, $a-b+c-d+e-f$

$$=a+(-b+c-d+e-f)$$

$$=a-b+(c-d+e-f)$$

$$=a-b+c+(-d+e-f)=\text{etc.}$$

Also $a-b+c-d+e-f$

$$=a-(b-c+d-e+f)$$

$$=a-b-(-c+d-e+f)$$

$$=a-b+c-(d-e+f)=\text{etc.}$$

Ex. Arrange the expression $bx^2-cx+ax^3-px^3+rx-qx^2$ in descending powers of x by bracketing the powers so that the signs before the brackets may be alternately + and − from the left ; and state the co-efficients of the powers.

The expr. $=+(ax^3-px^3)-(-bx^2+qx^2)+(-cx+rx)$

$$=+(a-p)x^3-(q-b)x^2+(r-c)x.$$

Note that we write ax^3-px^3 as $(a-p)x^3$, [see note 2, art. 7, Chap. VII), for by multiplying $a-p$ by x^3 we get ax^3-px^3 ; similarly in other terms.

The co-efficients of x^3, x^2, x are evidently $a-p$, $-(q-b)$, $r-c$ respectively.

5. The rules for the removal and insertion of brackets as stated in arts. 2 and 4 give us in full the **associative law** (see art. 6, Chap. V) for addition and subtraction, *viz*, in a chain of additions and subtractions any number of terms may be associated into a group by enclosing them within brackets and conversely, any number of terms within brackets may be dissociated by removing the brackets ; the rules for the insertion and removal of brackets being as in arts. 4 and 2 respectively.

EXERCISE XXIV

Simplify by removing brackets

1. $x-y-\{3x+4y-(9x-7y)+7x\}-13y$.
2. $3a-\{4b-c-(3a+2b-\overline{3c+5a})\}$
3. $\{a+(b-\overline{c+d})\}+\{b-(\overline{c+d-a})\}-\{c-(d-\overline{a-b})\}$
4. $-x-[-x-\{-x-(-x-x)\}-x]$
5. $5a-2b-[4c-\{5d-(2a+4b-\overline{-3c+4d})\}]$
6. $-[2x-3y+\{4z-3x-(-5y+3z-\overline{x-2y})\}]-5x+2y$
7. $-7a-[-4b-\{-5c-(-2a-\overline{-4b-3c})\}+a]$
8. $5(x-2y+z)-\{3(2x+y-3z)-2(-3z+5y-4x)\}$
9. $2\{a-b+3(c-d)\}-\{2a-5b+3(c-d)-2.\overline{a-2b}\}$
10. $3\{5a-4(b-c)\}+5(a-3c)-\{2a-(b+c-\overline{a+b})\}$
11. $[2a-3\{b-(2c+d-\overline{3a+b})\}]-3(a-\overline{b+c})-2(c-d)$.
12. $2[2x-3\{4y-5(2z-x-y)\}]-3\{x-2(y-z+2x)\}$.
13. $-[-\{-(-x)\}]-(-(-y))$.
14. $5a-4[2b-3\{4a-2(b-3a)+5b\}-7a]-11b$.
15. $2x-[5y+\{7z-4x-(3x+y)\}+3x-(2y-5z)]$
16. $-[-\{-(3a-b+c)-2(a+b+c)-3a\}+5(b-c)]-9a$.

Evaluate when $x=1, y=2$;—

17. $5[2x-3\{4x-2(x+y)-y\}+5x]-4\{x-3(y-3.\ \overline{x-y})\}$.
18. $-4[-3\{-2(x-y)+5x\}+y]-2[-4\{-5(3y-2x)+y\}]$.
19. Subtract $b\{a-(b+c)\}$ from the sum of

$a\{a-(c-b)\}$ and $c\{a-(b-c)\}$. (M. M. 1885).

20. Express symbolically the excess of the sum of $3a-2(4b+c)$ and $3c-5(4a-3b)$ over the difference of $(2a-5b)+10c$ and $c-(4a-9b)$; and simplify the result.

21. Arrange the expression $ax^4-3bx^2+lx-mx^3+c-7x+4x^2-9-7x^4+5x^3$ in ascending powers of x by bracketing the powers so that all brackets may have (i) positive signs (ii) negative signs, before them.

Arrange in descending powers of x :—

22. $ax^3-3(bx^2+4)+2(5-7x+c)-x^2(ax+b+c)$.
23. $lx^3-\{2(n-mx)-x(hx^2+kx+l)-x^2(p-2q)\}$
24. $ax^4-[bx-c-\{px^2+qx-(r-lx^3)-mx+n\}]$.

25. In each of the following expressions enclose within brackets the last 6 terms, the last 5 terms, the last 4 terms as well as the last 3 terms, the signs before the brackets in each being $+$, $-$, $-$, $+$ respectively :—

(i) $a-b+c-d-e+f-g$.

(ii) $a+b-c-d+e-f+g$.

---

6. We shall now consider some questions on the first four rules, the co-efficients of the letters involved being literal. When the co efficients of a letter are compound quantities, it will be useful to retain them within brackets throughout the work.

Ex. 1. Add together

$$\begin{array}{lr}
 & (2a-b)x^3-(3b-c)x^2+(a-2b)x \\
 & (b+2c)x^3+(c+a)x^2+(2a+b)x \\
 & (c-2a)x^3-(2a+b)x^2+(b+c)x \\
\hline
\text{Reqd. sum}= & 3cx^3-(a+4b-2c)x^2+(3a+c)x.
\end{array}$$

Here the co-eff. of x^3 in the sum $=$ sum of $2a-b$, $b+2c$, $c-2a$ $=3c$; the co-eff. of x^2 in the sum $=$ sum of $-(3b-c)$, $(c+a)$, $-(2a+b)=-(a+4b-2c)$; the co-eff. of x in the sum $=$ sum of $a-2b$, $2a+b$, $b+c=3a+c$. Hence the above result.

Ex. 2.

$$\begin{array}{llll}
\text{From} & (7m-5n)x^3 & +3nx^2 & -4mx \\
\text{take} & (3m-2n)x^3 & +2mx^2 & -3nx \\
\hline
\text{Reqd. remainder}= & (4m-3n)x^3 & +(3n-2m)x^2 & +(-4m+3n)x
\end{array}$$

For the co-efficient of x^3 in the remainder is $(7m-5n)-(3m-2n)$ $=4m-3n$; so $(4m-3n)x^3$ is one term in the remainder. Similarly other terms are obtained.

Ex. 3. Find the products :—$(x+a)(x+b)$ and $(x+a)(x+b)(x+c)$.

$$\begin{array}{l}
x+a \\
x+b \\
\hline
x^2+ax \\
\qquad +bx+ab
\end{array}$$

$\therefore\ (x+a)(x+b)=x^2+(a+b)x+ab$

$$\begin{array}{l}
x+c \\
\hline
x^3+(a+b)x^2+abx \\
\quad +\qquad cx^2+(ac+bc)x+abc
\end{array}$$

$\therefore\ (x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$

Ex. 4. Multiply $a+bx+cx^2$ by $p+qx$

$$\begin{array}{l} a+bx+cx^2 \\ \underline{p+qx} \\ ap+bpx+cpx^2 \\ \underline{\quad +aqx+bqx^2+cqx^3} \end{array}$$

$\therefore$ Reqd. product $=ap+(bp+aq)x+(cp+bq)x^2+cqx^3$

Ex. 5. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

Arrange in descending powers of c, bracketing the part $a+b$ of the divisor :

$$\begin{array}{r|l|l} c+(a+b) & c^3 \qquad\qquad -3abc+a^3+b^3 & c^2-c(a+b) \\ & \underline{c^3+c^2(a+b)} & \quad +(a^2-ab+b^2). \\ & -c^2(a+b)- \qquad 3ab.c & \\ & \underline{-c^2(a+b)-(a^2+2ab+b^2)c} & \\ & +(a^2-ab+b^2)c+a^3+b^3 & \\ & \underline{+(a^2-ab+b^2)c+a^3+b^3} & \end{array}$$

$\therefore$ Quotient $=c^2-c(a+b)+a^2-ab+b^2$
$\qquad =a^2+b^2+c^2-ab-ac-bc.$

Note. In the above we have made use of the results $(a+b)^2=a^2+2ab+b^2$ and $(a^2-ab+b^2)(a+b)=a^3+b^3$ (see examples 1 and 2, Art. 11, Chap. VII). Compare the above solution with that of the same question in Chap. VIII. and observe how we can shorten work by using brackets.

Ex. 6. Divide $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $a^2(b-c)+b^2(c-a)+b^2(a-b)$

Arranging in descending powers of c we proceed thus :—

$$\begin{array}{r|l|l} c^2(a-b)-c(a^2-b^2) & c^3(a-b)-c(a^3-b^3)+ab(a^2-b^2) & \\ +ab(a-b) & \underline{c^3(a-b)-c^2(a^2-b^2)+c.ab(a-b)} & c+(a+b) \\ & c^2(a^2-b^2)-c(a^3+a^2b-ab^2-b^3)+ab(a^2-b^2) & \\ & \underline{c^2(a^2-b^2)-c(a^3+a^2b-ab^2-b^3)+ab(a^2-b^2)} & \end{array}$$

$\therefore$ the quotient $=c+(a+b)=a+b+c.$

EXERCISE XXV.

Add together

1. $px^3,\ -qx^2,\ rx,\ -s,\ -ax^3,\ bx^2,\ -cx,\ d.$
2. $(a+b)x^2+(a-b)x+2a,\ (a-b)x^2-(a+b)x+b.$

Add together

3. $(3p-2q+r)x-(5q-3r+2p)y+(2p-q)z$,
$(4r-7p+2q)x+(5r-7p+q)y+(3r-p)z$,
$(3p-4q)x-(7q-3r)y+(5q-6r)z$.

4. $(5a-2b)xy-(3b-2c)yz+(7c-a)zx+(a-2b)$,
$-(7b-2c)yz+(5a-c)zx-(9b-a)xy-(3b-5c)$,
$(5a-2b)zx-(2b-9c)xy+(7a-2b)yz+(9c-5a)$.

5. $(x+2y-z)a+(x-y-3z)b-(5z-2x)c$,
$(2x+2y-3z)b+(x-y-z)c-(2y+z-x)a$,
$(x-y+2z)c-(z-3x+y)a+(3x-2y+5z)b$,
$(x+y+z)a-(z-x+y)b+(x-y+z)c$.

Subtract

6. $(b+c-2a)xy+(c+a-2b)yz-(a+b-2c)zx$ from
$(5a-2b+1)yz-(7a-b+4)xz+(a-2b)xy$.

7. $(\frac{1}{2}x-\frac{2}{3}y+\frac{1}{4}z)a^2-(3x-4v+5z)b^2+(2x+y-3z)c^2$ from
$(7x-2z+y)a^2+(\frac{1}{3}z-\frac{1}{2}x+\frac{2}{3}y)b^2-(3z-2y+x)c^2$.

8. $(2p-3q)l+(5q-2r)m-(7r-p)n+(2p-3q+r)$ from
$(3r-5q)l-(2p-5r)m+(5q-2p)n-(3r+2p-5q)$.

9. $(m+n-r)ax+(2n-r+2m)\ by\ -(m-n+r)cz$ from
$(n+2r-m)\ by\ -(3r-2m-2l)cz+(3l-2m+7r)ax$.

10. $(x-2y+z)a+(2x+y+z)b-(2y-z-x)c$ from
$(x+y+z)a-(z-x+y)b+(x-y-z)c$.

Multiply together

11. $x-a,\ x-b$. 12. $x+a,\ x-b$. 13. $x-a,\ x+b$.

14. $ax^2+2hxy+by^2,\ lx+m$. 15. $lx^2-mx+n,\ px-q$.

16. $1+ax+bx^2+cx^3,\ 1+x+x^2$.

17. $(a+b)x^3+(a-b)x^2-(a+b)x+(a-b),\ (a+b)x+1$.

Divide

18. $x^3-px^2+qx-r^3+pr^2-qr$ by $x-r$.

19. $x^3-x(b^2+bc+c^2)+bc(b+c)$ by $x^2-x(b+c)+bc$.

20. $(a-b)^2.c^2+(a-b)c^3-(c^2-a^2)b^2+(c-a)b^3$
by $(a-b)c^2-(c-a)b^2$ (C. E 1883).

21. $a^3-b^3+c^3+3abc$ by $a-b+c$.

22. $bc(b-c)+ca(c-a)+ab(a-b)$ by $a-b$.

23. $a^2(b-c)+b^2(c-a)+c^2(a-b)$ by $b-c$.

24. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$ by $a+b$.

25. $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $a+b+c$ and $a^2-bc+ac-b^2$.

26. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$ by $a^2c-ab^2+a^2b-b^2c$.

27. $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$ by $a+b+c$.

28. $a^3(1-x)+ab(a-b)(x+y)+b^3(1+y)$
by $a(1-x)+b(1+y)$ M. M. 1898.

29. $(a+1)^2x^3+(a+1)x^2+a^2(a-1)x-a^5$
by $(a+1)x-a^2$ M. M. 1899.

30. Simplify
$(x-a)^2(x-b)(x-c)-[bc(x-a)+\{(a+b+c)x$
$-a(b+c)\}x]$ A. E. 1882.

31. Find the co-efficient of x^4 in the product of
$x^4-ax^3+bx^2-cx+d$ and x^2+px+q. (C. E. 1885).

7. In a chain of multiplication and division (not interrupted by addition or subtraction) any number of the members may be associated into a group by enclosing them within brackets and conversely any number of members within brackets may be dissociated by removing the brackets.

This is the **associative law** for multiplication and division (see art. 2, Chap. VIII) and may be stated symbolically thus :—

$$a\times(b\div c\times d\div e)=a\times b\div c\times d\div e, \ldots\ldots\ldots(1)$$
$$a\div(b\div c\times d\div e)=a\div b\times c\div d\times e \ldots\ldots\ldots(2)$$

The student will observe that if in (1) and (2) we replace $\times$ by $+$ and $\div$ by $-$, we get

$$a+(b-c+d-e)=a+b-c+d-e, \ldots\ldots\ldots(3)$$
$$a-(b-c+d-e)=a-b+c-d+e \ldots\ldots\ldots(4)$$

Hence the rules for removing brackets in a chain of multiplication and division are the same as in a chain of addition and subtraction, the signs $\times$ *and* $\div$ *taking the places of* $+$ *and* $-$ *respectively in the latter chain.*

Ex. Simplify $15\times[3\div\{5\div(24\div\overline{2\times4})\}]$

$$\begin{aligned}\text{The expr. } &= 15\times3\div\{5\div(24\div\overline{2\times4})\}\\ &= 15\times3\div5\times(24\div\overline{2\times4})\\ &= 15\times3\div5\times24\div\overline{2\times4}\\ &= 15\times3\div5\times24\div2\div4\\ &= (15\times3\times24)\div(5\times2\times4)\\ &= 27.\end{aligned}$$

EXERCISE XXVI.

Simplify

1. $\{12\times(2\times3)\}\div\{12\div(2\times3)\}$.
2. $48\div[2\times3\div\{2\div(4\times2)\times3\}\div2]$.
3. $96\div2\times3\div\{3\div2\times(4\times2)\}\div2\times3\times(4\div3)$.
4. $a\div b\times[c\div\{a\div b\div(c\div a)\times b\}\div c]$.
5. $(4x^2)^2\div[(xy)^3\div(y^2)^4\times\{(x^2y)^2\div(x^2)^2\}\times y^3]$.

CHAPTER X.

EASY EQUATIONS.

1. An **equation** is a statement of equality of two expressions

Thus $2x+3=15$......(1), $(3x+4)+(2x+3)=5x+7$......(2) are equations.

The parts of an equation connected by the sign of equality are called its **sides** or **members**; the left hand and the right-hand parts being called the *left-hand side* and the *right-hand side* respectively.

The student will observe that there is a difference in the equations (1) and (2). He can easily transform the left-hand side of (2) into the right-hand side by collecting terms on the left, and hence the relation (2) must be true for *all values* of x; but no algebraical process can transform the left-hand side of (1) into the right hand side and the relation (1) cannot be true for all values of x. In fact $x=6$ is the only value of x which makes the two sides of (1) equal.

2. An equation of which one side can be transformed into the other by algebrical processes and which is therefore true for *all values* of the letters involved is called an **identical equation** or simply, an **identity.**

An equation of which one side can be transformed into the other only by giving particular values to the letter or letters involved is called a **conditional equation** or simply, an **equation.**

Thus, $(3x+4)+(2x+3)=5x+7$ and $4(x+1)+3x=7x+4$ are identities, while $2x+3=15$ and $4x=12$ are equations.

3. The letter or letters in an equation on whose particular value or values the equality depends are called **unknown quantities,** and to **solve** the equation is to find the values of the unknown quantities, *i.e.* values which make the two sides of the equation true. These values are called **roots** or **solutions** of the equation, and they are said to **satisfy** the equation.

The unknown quantities in equations are denoted by the last letters of the alphabet, *viz.* x, y, z, w...... ; the first letters of the alphabet *viz.*, a, b, c......stand for known quantities.

4. We shall here consider some easy equations in one unknown quantity when there is the first and no higher power of the quantity in the equations. An equation which contains the first and no higher power of the unknown quantity than the first is called a **simple equation** or an **equation of the first degree.**

To solve equations generally the following axioms will be found very useful :—

I. If equals are added to equals, the sums are equal.

II. If equals are subtracted from equals, the remainders are equal.

III. If equals are multiplied by equals, the products ars equal.

IV. If equals are divided by equals, the quotients are equal.

Ex. 1. Solve the equation $5x=15$.

Dividing both sides of the equation by 5, we have

$5x \div 5 = 15 \div 5$ or $x=3$

Ex. 2. Solve the equation $\frac{3x}{4}=6$.

Multiplying both sides of the equation by 4, we have

$\frac{3x}{4} \times 4 = 6 \times 4$ or $3x=24$.

Hence dividing both sides by 3, we get $x=8$.

Ex. 3. Solve the equation $\frac{7x}{9}=\frac{5}{2}$

Multiplying both sides by 9, we have $7x=\frac{5}{2} \times 9$.

Hence dividing both sides by 7, we have

$$x=\frac{5 \times 9}{2 \times 7}=\frac{45}{14}=3\frac{3}{14}.$$

Ex. 4. Solve the equation $6x+7=37$.

Subtracting 7 from both sides, we have

$6x+7-7=37-7$ or $6x=30$.

Hence dividing both sides by 6, we get $x=5$.

Ex. 5. Solve the equation $11x-15=29$.

Adding 15 to both sides, we have

$11x-15+15=29+15$, or $11x=44$.

Hence dividing both sides by 11, we have $x=4$.

Ex. 6. Solve the equation $4x-15=2x+1$.

Adding 15 to both sides we have

$$4x-15+15=2x+1+15,$$
$$\text{or } 4x=2x+16.$$

Again, subtracting $2x$ from both sides,

$$4x-2x=2x+16-2x,$$
$$\text{or } 2x=16.$$

Hence dividing both sides by 2, we get $x=8$.

Note. In the first three of the above examples we have the terms containing x on one side and numerical quantities on the other side and we apply axioms III or IV to get their solutions. In the last three examples we apply axioms I and II to bring the terms containing x on one side and the known quantities on the other, and then solve the equations by applying axioms III and IV.

EXERCISE XXVII.

Solve the equations

1. $4x=16$ 2. $6x=42$ 3. $9x=72$ 4. $10x=-10$
5. $-15x=45$ 6. $22x=-33$ 7. $5x=-12$ 8. $-26x=-$
9. $\frac{5x}{3}=2$ 10. $\frac{3x}{7}=-3$ 11. $-\frac{x}{2}=7$ 12. $\frac{3x}{10}=-7$
13. $\frac{7x}{3}=-14$ 14. $\frac{11x}{12}=5$ 15. $\frac{9x}{7}=7$ 16. $\frac{7x}{5}=-9$
17. $\frac{x}{4}=\frac{3}{8}$ 18. $\frac{5}{2x}=\frac{25}{16}$ 19. $\frac{56}{27}=\frac{7}{9x}$ 20. $\frac{3}{2x}=-\frac{9}{8}$
21. $\frac{4x}{5}=-\frac{8}{10}$ 22. $\frac{6x}{11}=-\frac{4}{3}$ 23. $\frac{8}{5}=\frac{15x}{8}$ 24. $\frac{12x}{13}=-\frac{12}{13}$
25. $12x-5x=-35$. 26. $9x-4x=20$.
27. $11x-5x=38+4$. 28. $19x+5x=50+22$.
29. $4x+19=3$. 30. $5x-2=13$.
31. $5x+21=3x+3$. 32. $7x-9=2x+16$.
33. $3x-5=9x-41$. 34. $2x+12=7x-53$.
35. $7x-32=2x+8$. 36. $15x+15=70+4x$.

5. Transposition. The axioms I and II of art. 4 give us an important principle called **Transposition** which we already employed without naming it. The principle may be stated thus :

Any term may be transposed or brought over from one side of an equation to another by simply changing its sign.

Consider the equation $9x=5x+65$...............(1)

Subtracting $5x$ from both sides we get

$$9x-5x=65 \quad ....(2)$$

Thus $+5x$ which was on the right-hand side of (1) appears as $-5x$ on the left-hand side in (2).

Again, take the equation $63-2x=5x$..............(3)

Adding $2x$ to both sides, we get

$$63=5x+2x \quad ....(4)$$

Thus $-2x$ on the left-hand side of (3) appears as $+2x$ in the right-hand side of (4).

The above principle is very important and shortens the process of solving equations. It is evident that any number of terms may be simultaneously transposed from left to right and right to left.

6. *We may change the sign of every term on the two sides of an equation without altering the equation.*

For, consider the equation $5x-22=2x-15$..............(1).

Multiplying both sides by -1, we get

$$-1\times(5x-22)=-1\times(2x-15) \text{ or } -5x+22=-2x+15.$$

This result may be otherwise obtained :—

In (1) transpose the left-hand side terms to the right and the right-hand side terms to the left ;

$$\text{then } -2x+15=-5x+22.$$

$\therefore$ interchanging sides, $-5x+22=-2x+15$, as before.

7. Rule for solving a simple equation. The following rule for solving a simple equation may now be given :—

Transpose all the terms containing the unknown quantity on one side and the known quantities on the other. Then simplify both sides and divide them by the co-efficient of the unknown quantity.

In many cases preliminary simplification is necessary before applying the rule (see examples 4 and 5 following).

Ex. 1. Solve $7x+15=4x+45$.

Transposing $4x$ to the left and 15 to the right, we have

$$7x-4x=45-15,$$
$$\text{or } 3x=30, \therefore x=10$$

Note. The student should verify by substitution that the solution obtained satisfies a given equation. Thus here left-hand side $=7\times10+15=85$ and right-hand side $=4\times10+45=85$; hence the solution is correct.

Ex. 2. Solve $\frac{3x}{4}-\frac{5}{8}=\frac{7x}{12}+\frac{7}{6}$.

When fractional co-efficients occur as in this example, it is generally convenient to avoid fractions by multiplying both sides of the equation by the L. C. M. of the denominators. Hence here multiplying by 24.

$$24\left(\frac{3x}{4}-\frac{5}{8}\right)=24\left(\frac{7x}{12}+\frac{7}{6}\right) \text{ or } 18x-15=14x+28.$$

$\therefore$ by transposition, $18x-14x=28+15$.

$\therefore 4x=43$, whence $x=\frac{43}{4}=10\frac{3}{4}$.

Ex. 3. Solve $\frac{x-2}{4}-\frac{3x+1}{2}=10$

Multiplying both sides by 4, we have

$$(x-2)-2(3x+1)=40 \text{ or } x-2-6x-2=40.$$

$\therefore$ by transposition, $x-6x=40+2+2.$

$$\therefore\ -5x=44 \text{ or } x=-\tfrac{44}{5}=-8\tfrac{4}{5}.$$

Ex. 4. Solve $(2x-1)(3x-7)=6(x+2)^2+18.$

Multiplying out, $6x^2-17x+7=6(x^2+4x+4)+18.$

or $6x^2-17x+7=6x^2+24x+24+18.$

Hence subtracting $6x^2$ from both sides,

$$-17x+7=24x+24+18$$

$\therefore$ transposing, $-17x-24x=24+18-7$

$$\therefore\ -41x=35. \text{ hence } x=-\tfrac{35}{41}.$$

Note. The above equation seems to contain x^2 but on simplification terms containing x^2 cancel from both sides.

Ex. 5. Solve the equation

$$30x-[5x-2\{6x+7(x-8)\}]=4-[3x-2\{x-6(x-1)\}]$$

Here the left-hand side $=30x-[5x-2\{6x+7x-56\}]$

$$\begin{aligned} &=30x-[5x-2\{13x-56\}]\\ &=30x-[5x-26x+112]\\ &=30x-5x+26x-112\\ &=51x-112.\end{aligned}$$

The right-hand side $=4-[3x-2\{x-6x+6\}]$

$$\begin{aligned} &=4-[3x-2\{-5x+6\}]\\ &=4-[3x+10x-12]\\ &=4-3x-10x+12\\ &=16-13x.\end{aligned}$$

$\therefore$ the equation reduces to $51x-112=16-13x.$

Hence transposing, $51x+13x=16+112$

$$\therefore\ 64x=128 \text{ whence } x=2.$$

EXERCISE XXVIII.

Solve the following equations :—

1. $3x+5=23$. **2.** $21-5x=6$. **3.** $5=9x-4$.

4. $9x+12=12x-3$. **5.** $100-7x=4x-21$.

6. $(3x-7)-(5x+2)=2x+9$. **7.** $5+3(x+3)=17$.

8. $8(x-7)+3(x-1)=3x+1$. **9.** $2(x+5)=4(3x+1)-3(x+5)$.

10. $(2x+1)(x+5)+3(x+1)^2=5(x+1)(x+2)$.

11. $(x+5)(2x+3)=(x-1)(2x-1)+19$.

12. $4(x^2-2x+7)=(x+1)(2x-9)+2(x-1)(x-5)$.

13. $(4x+5)(3x+1)+12(x+1)^2=4(2x+3)(3x+1)$.

14. $(2x-3)(3x-2)+(x-1)(4x-1)=(5x+1)(2x-3)$.

15. $(3x+1)(4x-3)-3(2x+1)^2=12$.

16. $3x-4(x-2)=2(x+4)+3(x-6)$.

17. $5(x+2)-(3x+1)=4(2x-3)+3(4x-5)$.

18. $\dfrac{3x+2}{4}=\dfrac{5x-1}{3}$. **19.** $\dfrac{x}{3}+\dfrac{x-6}{4}=8$.

20. $\dfrac{x-1}{2}+\dfrac{x-3}{3}=5$. **21.** $\dfrac{2x+1}{6}+\dfrac{3x-5}{2}=\dfrac{x-5}{4}$.

22. $\dfrac{3(x-1)}{4}-\dfrac{2(2x-3)}{5}=3$. **23.** $\dfrac{x+3}{7}+\dfrac{x-3}{3}=1\frac{1}{3}$.

24. $\dfrac{7}{2}+\dfrac{5}{x}=3+\dfrac{x-1}{4x}$. **25.** $\dfrac{3}{x}-\dfrac{5}{12}+\dfrac{1}{3x}=\dfrac{3}{4}-\dfrac{5}{4x}$.

26. $12x-\{5-3x+2(x-2)\}=30$.

27. $7x-[5-x-\{-3x+2(x-50)+5x\}-12]=6x+5$.

CHAPTER XI.

EASY PROBLEMS.

1. We have already explained (see p. 45) how statements in words are translated by symbols into algebraical language, and we here propose to consider cases of this translation when it leads to the formation of Algebraical equations.

Ex. 1. The sum of the squares of 3 consecutive numbers is 50. Express this fact by an equation.

Consecutive numbers differ by unity. Hence if x be the middle one of 3 consecutive numbers, the one next greater than x is $x+1$ and the one next smaller than x is $x-1$. Thus the numbers are $x-1, x, x+1$. By the question,

$$(x-1)^2+x^2+(x+1)^2=50, \text{ the equation required.}$$

Ex. 2. Five times the sum of two numbers is equal to three times their product. Represent this fact by an equation.

Let x, y stand for the numbers. Then 5 times their sum is $5(x+y)$ and 3 times their product is $3xy$. Hence by the question,

$$5(x+y)=3xy, \text{ the equation required.}$$

Ex. 3. Five years ago I was twice as old as my son who is now x years old, and my present age is y years. Express the above by an equation.

5 years ago my son's age was $x-5$ years and therefore my age then was $2(x-5)$ years. Hence my present age is $2(x-5)+5$ years; but by the question my present age is also y years.

$$\therefore\ 2(x-5)+5=y \text{ is the equation required.}$$

EXERCISE XXIX.

1. If a is divided into 3 parts of which the first is x and the second is less than the first by y, what is the third part?

2. How many miles can I walk in x hours if I take a hours to walk b miles?

3. x years hence A will be m times as old as B who is of y years now. What is the present age of A?

4. Write down (i) 3 consecutive odd numbers (ii) 3 consecutive even numbers.

Express the following statements in equational forms :—

5. The sum of 4 consecutive numbers of which y is the greatest, is x.

6. The difference of the squares of two consecutive numbers is equal to their sum.

7. x is greater than y by z.

8. Five times the excess of a number over 7 is equal to twice the number increased by 12.

9. The area of the four walls of a room of length a feet, breadth b feet and height c feet, is x square feet.

10. x divided by y gives b as quotient.

11. A man now aged x years was a years old b years ago. He will be y years old after c years.

12. A man had Rs. a with him ; he purchased Rs. b worth of things, gave in charity Rs. c and lost Rs. d. He is now left with Rs. x.

13. I walked at the rate of a miles per hour and found at the end of the journey that I had walked x miles in b hours.

14. a men finish a piece of work in x days, while b men finish it in y days.

15. The interest of Rs. 500 at x per cent per annum for y years is z rupees.

16. Two trains have the same speed : one runs at the rate of $30a$ miles per hour and the other at the rate of $11b$ feet per second.

2. The most important application of equations is the **solution of problems.** A **problem** is a statement of relation between quantities of which some are known and some are unknown, and the solution of a problem is the finding of the unknown quantities in terms of the known.

We shall here consider easy problems in which there is one unknown quantity to be determined and we may give the following method for their solution,

Let the unknown quantity be represented by x. Translate the verbal statement of the problem into algebraical language in the form of an equation and by solving the equation determine the value of the unknown quantity.

Ex. 1. The sum of 4 consecutive numbers is 54, find the numbers.

Let x be the smallest of the consecutive numbers. Then because consecutive numbers differ by unity the four numbers are x, $x+1$, $x+2$ and $x+3$. Hence, since the sum of these numbers is 54, we have

$$x+(x+1)+(x+2)+(x+3)=54$$

$$\therefore 4x+6=54 \text{ ; hence transposing,}$$

$$4x=54-6 \text{ or } 4x=48.$$

$$\therefore \quad x=12.$$

Hence the numbers are 12, 13, 14, 15.

Verification. $12+13+14+15=54$.

Ex. 2. Divide 40 into 2 parts so that four times the greater may be as much more than 30 as three times the smaller is less than 115.

Let x be the greater part, then $40-x$ is the smaller.

Now 4 times the greater part is $4x$ and this is greater than 30 by $4x-30$; also 3 times the smaller part is $3\ (40-x)$ and this is less than 115 by $115-3\ (40-x)$.

$$\therefore \text{ by the question, } 4x-30=115-3(40-x).$$

$$\therefore 4x-30=115-120+3x$$

$$\text{Transposing, } 4x-3x=115-120+30$$

$$\therefore x=25 \text{ ; hence } 40-x=15.$$

Thus the parts are 25 and 15.

Ex. 3. Find two consecutive odd numbers such that one-third of the greater exceeds one-fourth of the less by $2\frac{3}{4}$.

It is evident that $2x$ represents any even number. Hence any odd number may be represented by $2x+1$ and consequently the next greater odd number (being greater than $2x+1$ by 2) is $2x+3$.

By the question $\dfrac{2x+3}{3}-\dfrac{2x+1}{4}=2\frac{3}{4}$ or $\frac{11}{4}$.

Multiplying both sides by 12, we have $4\ (2x+3)-3(2x+1)=33$.

$$\therefore 8x+12-6x-3=33,$$

$$\text{Transposing } 8x-6x=33-12+3.$$

$$\therefore 2x=24 \text{ ; hence } x=12.$$

$$\therefore 2x+1=25, \text{ and } 2x+3=27.$$

Thus the numbers are 25 and 27.

Verification. $\frac{27}{3}-\frac{25}{4}=9-6\frac{1}{4}=2\frac{3}{4}$.

Ex. 4. A and B together can do a piece of work in 20 days, which A alone can do in 25 days. In what time can B alone do it?

Suppose B alone can do the work in x days.

Now A can do $\frac{1}{25}$th of the work in one day.

B can do $\frac{1}{x}$th of the work in one day.

$\therefore$ A and B together can do $\left(\frac{1}{25}+\frac{1}{x}\right)$th part of the work in one day.

But by the question, A and B together can do $\frac{1}{20}$th of the work in one day.

$$\therefore \frac{1}{25}+\frac{1}{x}=\frac{1}{20} \text{ or } \frac{1}{x}=\frac{1}{20}-\frac{1}{25}=\frac{1}{100}.$$

$\therefore$ $x=100$ or A alone can do the work in 100 days.

Ex. 5. The sum of the ages of A and B is 60 years, and 12 years hence A will be twice as old as B. How old are they now?

Let A's age be x years, then B's age is $(60-x)$ years. Now 12 years hence A's age will be $(x+12)$ years and B's age, $(60-x+12)$ or $(72-x)$ years. Hence by the question,

$$x+12=2(72-x)$$

$$\text{or } x+12=144-2x.$$

$$\therefore \text{ transposing, } x+2x=144-12 \text{ or } 3x=132.$$

$$\therefore x=44\text{; hence } 60-x=16.$$

Thus A and B are now aged 44 and 16 years respectively.

Ex. 6. Divide £1000 among A, B, C so that B may get 3 times as much as A, and C may get £200 more than the shares of A and B together.

Let A's share $=£x$, then B's share $=£3x$,

and C's share $=£(x+3x+200)$.

$$\therefore x+3x+(x+3x+200)=1000$$

$$\therefore 8x+200=1000 \text{ or } 8x=800$$

$$\therefore x=100.$$

A gets £100, B gets £300, C gets £600.

Ex. 7. Divide 44 into 4 parts such that the first part increased by 2, the second diminished by 8, the third multiplied by 3 and that the fourth divided by 4, shall be all equal.

Let first part $+2=x$,	$\therefore$	first part $=x-2$
second part $-8=x$,		second part $=x+8$
third part $\times 3=x$,		third part $=x\div 3$
fourth part $\div 4=x$,		fourth part $=4x$.

$\therefore$ by the question, $(x-2)+(x+8)+x/3+4x=44$.

$$\therefore 3x-6+3x+24+x+12x=132$$

$$\therefore 19x=132+6-24=114$$

$$\therefore x=6.$$

Thus the parts are, 4, 14, 2, 24, respectively.

The student will note here that we do not put x for any of the quantities to be determined, but for something upon which all the quantities depend.

EXERCISE XXX.

1. Find two numbers whose sum is 30 and difference 24.

2. One number is greater than another by 3 and their sum is 27, determine the numbers.

3. The difference of 2 numbers is 15 and their snm is twice their difference ; find the numbers.

4. The sum of 5 consecutive numbers is 125, what are the numbers ?

5. The difference of the squares of two consecutive numbers is 25 ; find the numbers.

6. Of 2 numbers which differ by 18 one is as much above 37 as the other is below 45 ; find the numbers.

7. A number increased by 12 is 3 times its defect from 24, what is the number ?

8. Divide 20 into 2 parts so that 5 times the greater part is as much greater than 40 as 3 times the smaller part is less than 44.

9. What number is that which exceeds its fifth part by 12 ?

10. Find two consecutive numbers such that one-third of the the smaller exceeds one-fourth of the greater by $1\frac{1}{3}$.

11. A post has a fourth of its length in the mud, and a third of its length in the water and 10 feet above the water. What is its length?

12. Two sums of money are together equal to £54. 12*s*., and there are as many pounds in the one as shillings in the other. What are the sums? (C. E. 1885).

13. *If I subtract from the double of my present age, the treble of my age six years ago, the result is my present age. What is my present age?* (A. E. 1893).

14. A father is 30 years and his son 4 years old. After how many years will the father be 3 times as old as his son?

15. Find the number of which the excess over 13 is greater by 10 than its defect from 27.

16 Divide Rs. 100 among A, B, C, so that B may get Rs. 5 more than A, and C Rs. 15 more than B.

17. A is twice as old as B, 12 years ago he was thrice as old; find the age of each.

18. A is three times as old as B, 20 years hence he will be twice as old. What is the age of each?

19. Ten years ago A was four times as old as B, and ten years hence A will be only twice as old. What is the age of A?

20. A can do a piece of work in 12 days, B can do it in 15 days; in how many days can they together do it?

21. Two pipes can fill a cistern in 15 hours, and one alone can fill it in 20 hours; in what time can the other alone fill it?

22. The sum of 3 consecutive even numbers is 48, find the numbers.

23 A bag contains a certain number of rupees, twice as many half-rupees, three times as many four anna pieces and four times as many half anna pieces. The total sum in the bag is *Rs.* 57 8*as.* Find the number of coins of each kind.

24. A is twice as old as B and 4 years older than C. The sum of the ages of A, B and C is 96 years. Find B's age. (C. E. 1860).

25. A sum of *Rs.* 63. 4*as.* was paid in rupees and two-anna pieces. The total number of coins being 100, how many of each kind were used? (M. M. 1890).

26. A servant was asked to bring 28 one-anna and half-anna postage stamps of the value of one rupee, how many of each kind should he get?

27. What are the numbers of which the difference is 6 and the difference of their squares 156?

28. Divide 50 into 2 parts such that three times one part with five times the other may be 190.

29. Divide 78 into 4 parts such that the first increased by 10, the second diminished by 3, the third multiplied by 4 and the fourth divided by 2, shall all be equal.

30. Divide the number 127 into four such parts that the first increased by 18, the second diminished by 5, the third multiplied by 6 and the fourth divided by $2\frac{1}{2}$, shall all be equal. (B. M. 1883).

MISCELLANEOUS EXERCISE PAPERS (I).

PAPER I.

1. If $x=5$, $y=-2$, $z=7$, evaluate

(i) $\dfrac{x^3+y^3}{x^2-xy+y^2}$ (ii) $\dfrac{x^3+y^3+z^3-3xyz}{x^2+y^2+z^2-yz-zx-xy}$.

2. Plot the points :— $(3,-4)$, $(5,-1)$, $(-7, 2)$ and measure the distance between every two of them.

3. Add $3a-(2b+5c)$ to $2a-\{5b-7c+(a-b)-5c\}$ and subtract the result from $-3a-2\{5b-3(7c-2a)+b\}$.

4. Multiply $\frac{1}{2}x^2-3xy+\frac{1}{3}y^2$ by $\frac{1}{4}x-\frac{1}{5}y$.

5. Divide x^6-3x+2 by $x-1$.

6. Divide x^5+y^5 by a number which is greater than x by y.

7. Solve $2(x+1)(x+3)+11=(2x+1)(x+7)$.

8. A labourer is engaged for 30 days, on condition that he receives 2*s*. 6*d*. for each day he works, and loses 1*s*. for each day he is idle: he receives £2. 7*s* in all. How many days does he work and how many days is he idle? (C. E. 1869 & B. M. 1893).

PAPER II.

1. If $a=7$, $b=-5$, evaluate.

$$\frac{a^4-a^3b-ab^3+b^4}{a^4+3a^3b+4a^2b^2+3ab^3+b^4}.$$

2. Prove on squared paper :—

(i) $-5+2=-3$. (ii) $-7+4+5-2=0$.

3. Simplify $25-3[2-2\{3-2(5-2)-1\}-1]-1$.

4. Add together $5ax-by+7cz$, $-3by+4cz-2ax$, and $-9cz+2ax-5by$; and subtract the result from zero.

5. What must be added to the expression $3x-\{4y-3(z-2x)-y\}$ to make it equal to $x+y+z$?

6. Multiply $a^2+2b^2+9c^2-3ab+6ac-9bc$ by $a+2b-3c$, and divide the result by $a-b+3c$ (M. M. 1880).

7. Solve $4x-\frac{x-1}{2}=x+\frac{2x-2}{5}+24$ (C. E. 1880).

8. Divide 48 into 4 parts such that the first part increased by 3, the second diminished by 9, the third multiplied by 4 and the fourth divided by 3 give the same result in each case.

PAPER III.

1. If $a=6$, $b=2$, $c=3$, $d=4$, prove that

$$\frac{(a+b)^3-(b+c)^3+(c+d)^3-(d+a)^3}{(a+b)^2-(b+c)^2+(c+d)^2-(d+a)^2}=22\tfrac{1}{2}.$$

2. Simplify

$$24\left\{\frac{2x}{3}-\frac{1}{4}\left(\frac{5y}{2}-\frac{z}{3}\right)\right\}-36\left\{\frac{7x}{9}-\frac{3}{2}\left(\frac{5y}{2}-\frac{2z}{3}\right)\right\}.$$

3. I walk 8 miles to the east; then 12 miles to the north and next 13 miles to the west, find graphically how far I am from the starting point.

4. Subtract $2b\{3a-(5b+c)\}$ from the sum of $c\{3a-2(b-3c)\}$ and $3a\{2a-3(2b-c)\}$.

5. Divide $x^4+5ax^3+(25a-b-29)x^2-5(4a+b-4)x+4b$ by x^2+5x-4. (M. M. 1893).

6. Simplify $a^2-(b-c)^2+b^2-(c-a)^2+c^2-(a-b)^2$, and find its numerical value when $a=7$, $b=5$, $c=2$.

7. Multiply $x^2-\frac{1}{2}xy-3y^2$ by $\frac{1}{2}x-\frac{1}{3}y$.

8. Find two consecutive even numbers such that one-third of the smaller exceeds one-tenth of the larger by 4.

PAPER IV.

1. Find the value of

$$\frac{4y}{5}(y-x)-35\left[\frac{3x-4y}{5}-\frac{1}{10}\left\{3x-\frac{5}{7}(7x-4y)\right\}\right]$$

when $x=-\frac{1}{2}$, $y=2$.

2. If $x=2a-3b+c$, $y=a+2b-3c$, find the product of $(x+2y)$ and $(2x-y)$ in terms of a, b, c.

3. The product of two quantities is $6x^6-19x^5+6x^3-3x+2$ and one of them is $3x^2-2x+1$, find the other.

4. Find the coefficient of x^2 in the product

$$(5x^3+7x^2-3x+1)\ (2x^3-5x^2+7x+2)$$

5. Simplify $\frac{1}{2}(x-5)(2x+9)-\frac{1}{3}(3x-1)(x+2)+\frac{1}{4}(2x-7)(2x+1)$, and subtract the result from $\frac{1}{2}x^2-\frac{1}{3}x+\frac{1}{4}$.

6. Remove the brackets from

$$6\{x-2[y-4(2a+b)]\}-4\{x-3[y-4(2a-b)]\}.$$

7. Solve $\frac{7x-1}{4}-\frac{1}{3}\left(2x-\frac{1-x}{2}\right)=6\frac{1}{3}$ (C. E. 1872).

8. A person bought 166 mangoes for 10 rupees; some he bought at the rate of 18 per rupee, and the rest at 15 per rupee. How many did he buy of each sort? (M. M. 1889).

PAPER V.

1. If $a=\sqrt{2}$, $b=\sqrt{3}$, $c=4$ and $d=0$, find the value of

$$\sqrt{\{(a^2+b^2+c^2)(b^2+c^2)(b^2+d^2)\}} \qquad \text{(C. E. 1868).}$$

2. If $a=6$, $b=7$, $c=8$, evaluate

$$a\left\{\frac{a}{7}-\frac{2}{3}\left(\frac{b}{4}+\frac{c}{8}\right)\right\}-\frac{2}{7}\left\{b(a+1)+\frac{1}{2}(a^2-2b)-\frac{7a}{3}\right\}.$$

3. Multiply $2a-3\{b-(2c-a)+b\}-c$ by $a-b-2\{3c-(a-2b)+c\}$, and find the value of the product when $a=\frac{1}{2}$, $b=-\frac{1}{3}$, $c=\frac{1}{4}$.

4. Divide $2x^4-6ax^3+(4a^2+ab-2b^2)x^2+3ab^2x-a^2b^2$ by $x^2-(2a-b)x-ab$ (B. M. 1901).

5. Simplify the expression

$7(a-3b+c)-[4(2b+4c)(6c-3b)-3(a-4b)(a+3b)+$
$\{(5a-4b+3c)\times 4+a-47b+2c\}\div 7]$ (M. M. 1891).

6. From what must $3x-[5y-\{6z-(4x-11y)+z\}-x]$ be subtracted to get $6x-[15y-\{3x-(11y-5z)-7y\}+2x]$?

7. Solve $\frac{5-3x}{4}+\frac{5x}{3}=\frac{3}{2}-\frac{3-5x}{3}$ (C. E. 1866).

8. How much milk at *Rs.* 2 8*as.* per maund must be mixed with 60 maunds of milk at *Rs.* 3. 8*as.* per maund so that the mixture may be worth *Rs.* 3. 4*as.* per maund?

PAPER VI.

1. When $x=11$, find the value of

$\sqrt[3]{[(x+2)\sqrt{x-2}-2\{\sqrt[3]{11x^2}-x+2\sqrt{x-2}\}]}$ (M. M. 1880).

2. Simplify

$$42\left\{\frac{4a-3b}{6}-\tfrac{2}{7}(a-\tfrac{4}{3}b)\right\}-36\left\{\tfrac{1}{7}(3a-2b)-\tfrac{3}{8}(\tfrac{16}{3}a-b)\right\}$$

3. Arrange in descending powers of x :—

$2ax^2-3bx+cx^3-d+px-qx^2-rx^3.$

4. Divide $x^4-3(a+1)x^3+2(3a+1)x^2+3(a+1)(a^2-1)x$
$+(a^2-1)^2$ by $x^2-(a+3)x-a^2+1$. (B. M. 1900).

5. Prove that $x(x-1)(x-2)(x-3)+1$
$=x^4-2x^2(3x-1)+(3x-1)^2.$

6. Simplify $(a+b+c)(x+y+z)+(a+b-c)(x+y-z)$
$+(b+c-a)(y+z-x)+(c+a-b)(z+x-y).$

7. Solve $\frac{x+3}{8}-\frac{x-3}{10}=\frac{x+5}{6}-\frac{x-7}{3}.$ (M. M. 1880).

8. Find two consecutive odd numbers such that one-fifth of the smaller with one-third of the larger is 14.

PAPER VII.

1. If $a=5$, $b=12$, prove that

$(2a+b)^3+9a(2a+b)(a-b)+(a-b)^3=3375.$

2. Add together $(a+b-c)\,xy+(b+c-a)yz+(c+a-b)zx,$
$(2b+3c-5a)yz+(2c+3a-5b)zx+(2a+3b-5c)xy,$
$(b+c)zx+(c+a)xy+(a+b)yz.$

3. If $x=b+c-2a$, $y=c+a-2b$, $z=a+b-2c$, find the value of $2x-(3y-2x)+5z$.

4. Simplify
$\{m-n-(3x-2y)\}-[3m+2n-\{x-y+(m+2n)-(2y-x)\}]$

5. Arrange the expression
$x(p+x)\{p^2+q^2-x(p-x)\}-(p^2+qx)(2x^2-qx+q^2)$
in powers of x; and divide it by $x^2+(p-q)x-p^2$.

6. Divide $(a^2-c^2)^2-2b^2(a^2+c^2)+b^4$ by $a^2-b^2-c^2+2bc$.

7. Solve $x-\frac{3-x}{5}=3\cdot\frac{x-1}{2}+\frac{x+1}{5}-\frac{3}{10}$. (C. E. 1891).

8. A is twice as rich as B and one-fourth of A's property with one-tenth of B's amount to £150. Find the worth of each.

PAPER VIII.

1. Simplify :—
$3a-[a+b-2\{a+b+c-(a-b+c-d)\}+a]$ (C. E. 1876)

2. From $4a-[3b+\{3-5c-(1-\overline{3a-3})-b\}+c]$ take the sum of $a-(b-c)+\{2c-(3b-7a)\}$ and $2b-\{3a-(b-c)-5c\}$.

3. Multiply $\frac{2}{3}a^2-\frac{5}{6}ab+b^2$ by $\frac{1}{2}a^2-\frac{3}{4}ab+6b^2$.

4. Divide x^6-6x+5 by $(x-1)^2$.

5. Evaluate $x^2(y+z)+y^2(z+x)+z^2(x+y)+3xyz$, when $x=72$, $y=-100$, $z=28$.

6. The product of two expressions is $28x^4+13x^2y^2-xy^3+15y^4$ and one of them is $4x^2+4xy+3y^2$, find the other.

7. Solve $x-\left(3x-\frac{2x+5}{10}\right)=\frac{1}{6}(2x+57)+\frac{5}{3}$. (A. E. 1890.

8. How old is a man whose age 10 years ago was one-half of what it will be 15 years hence?

PAPER IX.

1. If $a=15$, $b=16$, $c=-31$, prove that
$$a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2).$$

2. Tabulate the values of $-5x^2+7x-9$, when $x=-3, -2, -1, 0, 1, 2, 3$.

3. Add together $36\left\{\frac{5a}{9}-\frac{1}{6}\left(\frac{2b}{3}+c\right)\right\}$,

$$60\left\{\frac{7c}{15}-\frac{9}{5}(2a-3b)\right\} \text{ and } 24\left\{\frac{3a}{8}-\frac{2b-5c}{12}\right\}.$$

4. Multiply $\frac{1}{2}x+\frac{2}{3}y$ by $\frac{3}{4}x-\frac{1}{3}y$.

5. If $a=47$, $b=48$, $c=49$, then $a^3+c^3=2b^3+6b$.

6. Divide $(x+y)^3-8y^3$ by $x-y$.

7. Solve $\frac{x+5}{6}+\frac{1}{9}\left(\frac{x}{2}+\frac{2}{5}\right)-\frac{2}{3}(3+2x)$

$$=\frac{4x-14}{3}+\frac{x+10}{10}. \qquad \text{(C. E. 1894).}$$

2. A says to B: two-fifths of my present salary is $\frac{4}{25}$th of yours and the difference between our salaries is Rs. 600. What is A's salary? (P. E. 1894).

Paper X.

1. If $a=10$, $b=11$, $c=12$, prove that

$$\frac{(2a-5b)^3+(5b-7c)^3+(7c-2a)^3}{(2a-5b)(5b-7c)(7c-2a)}=3.$$

2. If $x=78$, evaluate $x(x+1)(x+2)(x+3)+1$, and find the square root of the result.

3. Simplify $3ab-a[2a-3\{b-2c-2(a-2b)+c\}-2b]$.

4. By what must x^2+x+41 be multiplied to give

$$x^6+3x^5+46x^4+89x^3+132x^2+169x+205\,?$$

5. Add together $(3x-1)(x+2)(2x-5)$, $(1-x)(2-3x)(x+2)$, and $(3x+2)(1+x)(1-2x)$.

6. Solve $\frac{4x+3}{9}+\frac{13x}{108}=\frac{8x+19}{18}$. (C. E. 1878).

7. Solve $\frac{17-3x}{5}-\frac{4x+2}{3}=5-6x+\frac{7x+14}{3}$. (B. M. 1883).

8. What number is that whose third part exceeds its ninth part by 12?

CHAPTER XII.

FORMULÆ AND IDENTITIES.

1. Any general result expressed in symbols is called a **formula**. We shall in this Chapter consider important formulæ in multiplication and division, and the student will do well to commit them thoroughly to memory, as this will enable him to solve questions readily.

2. It is proved in art. 9 Chap. VII that

$$(\mathbf{a}+\mathbf{b}+\mathbf{c}+\ldots)\mathbf{m}=\mathbf{am}+\mathbf{bm}+\mathbf{cm}+\cdots \qquad \text{Formula I.}$$

Hence conversely we have

$$\mathbf{am}+\mathbf{bm}+\mathbf{cm}+\ldots=(\mathbf{a}+\mathbf{b}+\mathbf{c}+\ldots)\mathbf{m}\,;$$

that is an expression of the form $am+bm+cm+\ldots$ can be *resolved* into the factors $a+b+c+\ldots$and m, where m may be any quantity, simple or compound.

Thus if any factor is common to each term of an expression the quotient of the expression by that factor may be enclosed within brackets, the factor being placed outside as a multiplier.

Ex. 1. Resolve into factors $3x^2-5x$.

Here x is common to each term, hence taking it out, we have $3x^2-5x=x(3x-5)$.

Ex. 2. Resolve into factors $b^2c^2+2bc^3$.

Here bc^2 is common to each term, hence taking it out, $b^2c^2+2bc^3=bc^2(b+2c)$.

Ex. 3. Resolve into factors $4ax-6a^2x^2$.

Here $2ax$ is common to each term, hence taking it out, we have $4ax-6a^2x^2=2ax(2-3ax)$.

Ex. 4. Resolve into factors $2x^3y-3xy^2+4x^2y^3$.

Here xy is common to each term, hence taking it out, $2x^3y-3xy^2+4x^2y^3=xy(2x^2-3y+4xy^2)$.

Ex. 5. Resolve into factors $3xy(x-y)-9z^2(x-y)$.

Here $3(x-y)$ is common to each term, hence taking it out we have $3xy(x-y)-9z^2(x-y)=3(x-y)(xy-3z^2)$.

EXERCISE XXXI.

1. Prove that $a(b+c)+b(c+a)+c(a+b)=2(ab+bc+ca)$.

2. Prove that $a(b-c)+b(c-a)+c(a-b)=0$.

3. Prove that $a(a-b+c)+b(a+b-c)+c(-a+b+c)=a^2+b^2+c^2$.

4. Prove that $a^2(b+c)+b^2(c+a)+c^2(a+b)$
$$=bc(b+c)+ca(c+a)+ab(a+b)$$
$$=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$$
$$=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2$$

5. Prove that $a^2(b-c)+b^2(c-a)+c^2(a-b)$
$$=bc(b-c)+ca(c-a)+ab(a-b)$$
$$=-\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}$$
$$=a^2b-ab^2+b^2c-bc^2+c^2a-ca^2.$$

[The student is advised to remember the equivalent forms of examples 4 and 5].

Resolve the following expressions into factors :—

6. a^2+ab. 7. a^3-2a^2 8. $3a-5a^2$.

9. $7x^2+x$. 10. $5ax-10a^2x^2$. 11. $3x^2-6xy$.

12. a^2b-ab^2+abc. 13. $a^3-a^2x+ax^2$. 14. $2a^3+4a^4-6a^5$.

15. $x^2(y-z)+y^2(v-z)$. 16. $3a(4a-x^2)-b(4a-x^2)$.

17. $a(a+b+c)+b(a+b+c)+c(a+b+c)$.

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3. Square of a binomial. By actual multiplication we have
$$(\mathbf{a}+\mathbf{b})^2=\mathbf{a}^2+2\mathbf{ab}+\mathbf{b}^2\text{.........Formula II.}$$
$$(\mathbf{a}-\mathbf{b})^2=\mathbf{a}^2-2\mathbf{ab}+\mathbf{b}^2\text{.........Formula III.}$$

It may be remarked that formula III may be deduced from formula II thus : $(a-b)^2=\{a+(-b)\}^2$
$$=a^2+2a(-b)+(-b)^2 \text{ by formula II.}$$
$$=a^2-2ab+b^2.$$

The formulæ II and III may be stated in words thus :

(1) The square of the sum of two quantities is equal to the sum of their squares plus twice their product.

(2) The square of the difference of two quantities is equal to the sum of their square minus twice their product.

Obs. From formulæ I and II we have by transposition,
$$a^2+b^2=(a+b)^2-2ab,$$
$$a^2+b^2=(a-b)^2+2ab.$$

Ex. 1. Find the square of $3a+2b$.

We have $(3a+2b)^2=(3a)^2+2.3a.2b+(2b)^2$
$$=9a^2+12ab+4b^2.$$

Ex. 2. Find the square of $2x-5y$.

We have $(2x-5y)^2=(2x)^2-2.2x.5y+(5y)^2=4x^2-20xy+25y^2$.

Ex. 3. Find the square of 999.

We have $999^2=(1000-1)^2$
$=1000^2-2.1000.1+1$
$=1000000-2000+1=998001$.

Ex. 4. Prove that (i) $(a+b)^2+(a-b)^2=2(a^2+b^2)$
(ii) $(a+b)^2-(a-b)^2=4ab$.

We have $(a+b)^2=a^2+2ab+b^2$................ (1)
$(a-b)^2=a^2-2ab+b^2$..................(2)

(i) By adding (1) and (2)
$$(a+b)^2+(a-b)^2=2a^2+2b^2=2(a^2+b^2).$$

(ii) By subtracting (2) from (1),
$$(a+b)^2-(a-b)^2=4ab.$$

This result may be put in the form $ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2$, and may be stated verbally thus :—**the product of two quantities = (semi-sum)^2 − (semi-difference)^2.**

Ex. 5. Prove that $\frac{1}{2}\{(b-c)^2+(c-a)^2+(a-b)^2\}$
$=a^2+b^2+c^2-bc-ca-ab$.

We have $(b-c)^2=b^2-2bc+c^2$
$(c-a)^2=c^2-2ca+a^2$
$(a-b)^2=a^2-2ab+b^2$.

$\therefore$ adding, $(b-c)^2+(c-a)^2+(a-b)^2$
$=2a^2+2b^2+2c^2-2bc-2ca-2ab$.

Dividing both sides by 2, the required result follows.

Ex. 6. Find the value of a^2+b^2 if (i) $a+b=11$, $ab=-5$
(ii) $a-b=8$, $ab=7$.

(i) We have $a^2+b^2=(a+b)^2-2ab=11^2-2\times(-5)$
$=121+10=131$.

(ii) We have $a^2+b^2=(a-b)^2+2ab=8^2+2\times 7$
$=64+14=78$.

4. Square of a trinomial. The square of a trinomial may be obtained by taking two of the terms of the trinomial as one and thus regarding the trinomial as a binomial. The square of any polynomial will be considered afterwards.

To find the square of $a+b+c$.

We have $(a+b+c)^2=\{(a+b)+c\}^2$

$=(a+b)^2+2(a+b)c+c^2$ by formula II

$$\left.\begin{aligned} i.e.\ (a+b+c)^2 &= a^2+b^2+c^2+2ab+2ac+2bc \\ &= a^2+b^2+c^2+2(ab+ac+bc) \end{aligned}\right\}\text{...Formula IV.}$$

Hence the square of the sum of three quantities is equal to the sum of their squares together with twice the product of every two of them.

Obs. From formula IV we have by transposition

$a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+bc)$,

also $2(ab+ac+bc)=(a+b+c)^2-(a^2+b^2+c^2)$.

Ex. 1. Find the square of $3x+5y+4z$.

We have $(3x+5y+4z)^2$

$=(3x)^2+(5y)^2+(4z)^2+2.3x.5y+2.3x.4z+2.5y.4z.$

$=9x^2+25y^2+16z^2+30xy+24xz+40yz.$

Ex. 2. Find the value of $(ab+ac+bc)^2$

We have $(ab+ac+bc)^2$

$=(ab)^2+(ac)^2+(bc)^2+2.ab.ac+2.ab.bc+2ac\,bc$

$=a^2b^2+a^2c^2+b^2c^2+2a^2bc+2ab^2c+2abc^2$

$=a^2b^2+a^2c^2+b^2c^2+2abc(a+b+c)$

Ex. 3. Find the square of $2a-3b-c$.

We have $(2a-3b-c)^2=\{2a+(-3b)+(-c)\}^2$

$=(2a)^2+(-3b)^2+(-c)^2+2.2a(-3b)+2.2a(-c)+2.(-3b)(-c)$

$=4a^2+9b^2+c^2-12ab-4ac+6bc.$

Ex. 4. If $a+b+c=15$, $ab+ac+bc=47$, find the value of $a^2+b^2+c^2$.

We have $a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+bc)$

$=15^2-2\times 47$

$=225-94=131.$

EXERCISE XXXII.

Write down the squares of :—

1. $x+1$.	2. $x+3$.	3. $3x+4$.	4. $7x+6$.
5. $3x+5y$.	6. $7x-9y$.	7. $3ab-4bc$.	8. $ax+2by$.
9. $lx+my$.	10. $px-qy$.	11. a^2-bc.	12. $2lm-5mn$.
13. $\frac{2}{3}a+\frac{3}{4}b$.	14. $\frac{5}{7}x-\frac{7}{5}y$.	15. $\frac{1}{2}ab-\frac{1}{3}c$.	16. $\frac{1}{2}a^2-\frac{4}{5}b^2$.

Find the value of

17. 75^2. **18.** 98^2. **19.** 102^2. **20.** 497^2.

21. 799^2. **22.** $1{\cdot}005^2$. **23.** $8{\cdot}997^2$. **24.** $99{\cdot}9^2$.

25. Find the value of a^2+b^2 if (i) $a+b=7$, $ab=2$.
(ii) $a+b=12$, $ab=-5$.
(iii) $a-b=16$, $ab=9$.
(iv) $a-b=-4$, $ab=-21$.

Find the squares of

26. $a-b+c$. **27.** $a+b-c$. **28.** $a-b-c$.

29. $2a-b-1$. **30.** $2-3x-5x^2$. **31.** $3x^2-5x+2$.

32. $\frac{5}{2}x^2-\frac{3}{4}xy+\frac{2}{3}y^2$. **33.** ax^2+bx+c. **34.** $\frac{2}{3}l-\frac{3}{4}m+\frac{1}{2}n$.

35. Find the value of $a^2+b^2+c^2$ if
(i) $a+b+c=25$, $ab+ac+bc=72$.
(ii) $a+b+c=40$, $ab+ac+bc=109$.

36. Find the value of $ab+ac+bc$ if
(i) $a^2+b^2+c^2=75$, $a+b+c=20$.
(ii) $a^2+b^2+c^2=215$, $a+b+c=25$.

37. If $a+b+c=p$, $ab+ac+bc=q$, prove that
$$a^2+b^2+c^2=p^2-2q.$$

38. Simplify
$$(a+b-c)^2+(a-b+c)^2+(-a+b+c)^2+(a+b+c)^2.$$

39. Simplify $(a+b)^2+(b+c)^2+(c+a)^2-(a+b+c)^2$.

40. Prove that $n^3=\left\{\dfrac{n(n+1)}{2}\right\}^2-\left\{\dfrac{n(n-1)}{2}\right\}^2$; hence express the cubes of the following as the difference of two squares :— 15, 21, 32.

5. Product of sum and difference of two quantities.

By actual multiplication we have

$$(a+b)(a-b)=a^2-b^2 \text{......Formula V.}$$

Hence the product of the sum and difference of any two quantities is equal to the difference of their squares.

Ex. 1. $(3x+2y)(3x-2y)=(3x)^2-(2y)^2=9x^2-4y^2$.

Ex. 2. $(5ab-7cd)(5ab+7cd)=(5ab)^2-(7cd)^2=25a^2b^2-49c^2d^2$.

Ex. 3. $(\sqrt{3}x+1)(\sqrt{3}x-1)=(\sqrt{3}x)^2-1=3x^2-1$.

Ex. 4. Multiply 101 by 99.

We have $101\times 99=(100+1)(100-1)=100^2-1=9999$.

Ex. 5. Find the continued product of $a-b$, $a+b$, a^2+b^2.

$$\text{Product}=(a-b)(a+b)(a^2+b^2)$$
$$=(a^2-b^2)(a^2+b^2)=a^4-b^4.$$

EXERCISE XXXIII.

Write down the products :—

1. $(x+1)(x-1)$. **2.** $(x+4)(x-4)$. **3.** $(x+a)(x-a)$.

4. $(2x+9)(2x-9)$. **5.** $(3a+5b)(3a-5b)$. **6.** $(x-6y)(x+6y)$.

7. $(x+2y)(x-2y)$. **8.** $(\frac{2}{3}a-\frac{3}{4}b)(\frac{2}{3}a+\frac{3}{4}b)$.

9. $(\frac{1}{2}ab-\frac{1}{3}bc)(\frac{1}{2}ab+\frac{1}{3}bc)$. **10.** $(\frac{3}{5}xy+1)(\frac{3}{5}xy-1)$.

11. $(a+\sqrt{2}b)(a-\sqrt{2}b)$. **12.** $(8+7a)(7a-8)$.

13. $(-2a-3b)(3b-2a)$. **14.** $(x^2+a^2)(x^2-a^2)$.

15. $(ax+by)(ax-by)$. **16.** $(pa+qb)(pa-qb)$.

17. $(3x+5y)(x-\frac{5}{3}y)$. **18.** $(4x+y)(x-\frac{1}{4}y)$.

19. $(2x+3y)^2(2x-3y)^2$. **20.** $(5ab+3cd)^2(5ab-3cd)^2$.

Find the value of (by formula V) :—

21. 28×22. **22.** $\times 43$. **23.** 92×68.

24. 125×75. **25.** $\times 54$. **26.** 73×47.

Write down the products :—

27. $(1-x)(1+x)(1+x^2)$. **28.** $(2x+5y)(2x-5y)(4x^2+25y^2)$.

29. $(mx-n)(mx+n)(m^2x^2+n^2)$.

30. $(x-2)(x+2)(x^2+4)(x^4+16)$.

31. $(a-b)(a+b)(a^2+b^2)(a^4+b^4)$.

32. $(a-b)^2(a+b)^2(a^2+b^2)^2$. **33.** $(2a+3b)^2(2a-3b)^2(4a^2+9b^2)^2$.

34. Find the value of $(b+c)(b-c)+(c+a)(c-a)+(a+b)(a-b)$.

35. Find the value of

$$(2x+3y)(2x-3y)+(3y+5z)(3y-5z)+(5z+2x)(5z-2x).$$

6. In the result of art. 5 we can put for a and b any two expressions; hence **the product of the sum and difference of any two expressions is equal to the difference of their squares.**

Ex. 1. $(a+b+c)(a+b-c)$
$=\{(a+b)+c\}\{(a+b)-c\}$, regarding $a+b$ as one term
$=(a+b)^2-c^2$, by formula V
$=a^2+2ab+b^2-c^2$.

Ex. 2. $(a^2+\sqrt{2ab}+b^2)(a^2-\sqrt{2ab}+b^2)$
$=\{(a^2+b^2)+\sqrt{2ab}\}\{(a^2+b^2)-\sqrt{2ab}\}=(a^2+b^2)^2-(\sqrt{2ab})^2$
$=a^4+2a^2b^2+b^4-2a^2b^2=a^4+b^4$.

Ex. 3. $(a^2+ab+b^2)(a^2-ab+b^2)$
$=\{(a^2+b^2)+ab\}\{(a^2+b^2)-ab\}$
$=(a^2+b^2)^2-(ab)^2$
$=a^4+2a^2b^2+b^4-a^2b^2$
$=a^4+a^2b^2+b^4$

This result is important and should be remembered as a formula.

Ex. 4. $(a+b+c+a)(a-b+c-d)$
$=\{(a+c)+(b+d)\}\{(a+c)-(b+d)\}$
$=(a+c)^2-(b+d)^2$
$=(a^2+2ac+c^2)-(b^2+2bd+d^2)$
$=a^2-b^2+c^2-d^2+2ac-2bd$.

Ex. 5. Find the product

$$(\mathbf{a}+\mathbf{b}+\mathbf{c})(\mathbf{a}-\mathbf{b}+\mathbf{c})(\mathbf{a}+\mathbf{b}-\mathbf{c})(-\mathbf{a}+\mathbf{b}+\mathbf{c})$$

Product $=\{(a+c)+b\}\{(a+c)-b\}\times\{b+(a-c)\}\{b-(a-c)\}$
$=\{(a+c)^2-b^2\}\times\{b^2-(a-c)^2\}$
$=(a^2+2ac+c^2-b^2)(b^2-a^2+2ac-c^2)$
$=\{2ac+(a^2-b^2+c^2)\}\{2ac-(a^2-b^2+c^2)\}$
$=(2ac)^2-(a^2-b^2+c^2)^2$
$=4a^2c^2-(a^4+b^4+c^4-2a^2b^2+2a^2c^2-2b^2c^2)$
$=\mathbf{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}$.

This result is important and should be remembered as a formula.

EXERCISE XXXIV.

Find the following products

1. $(a-b+c)(a+b-c)$.
2. $(a+b-c)(b+c-a)$.
3. $(2a-3b+5c)(2a+3b-5c)$.
4. $(x^2+2x+3)(x^2-2x+3)$.
5. $(3a^2-4b^2+7c^2)(7c^2+4b^2-3a^2)$.
6. $(4ab+6bc-9ca)(4ab-6bc+9ca)$.
7. $(3ax-by+2cz)(3ax-by-2cz)$.
8. $(la+mb+nc)(la-mb+nc)$.
9. $(px^2+qy^2+rz^2)(px^2-qy^2+rz^2)$.
10. $(a+b-c-d)(a-b+c-d)$
11. $(b-c+d-a)(c-d-a-b)$.
12. $(2x-3v+4z-5w)(2x+3y+4z+5w)$
13. $(3a^2-5b^2+7c^2-9d^2)(3a^2-5b^2-7c^2+9d^2)$.
14. $(px+qy+rz+sw)(-px+qv-rz+sw)$.
15. $(ab+bc+cd+da)(ab+bc-cd+da)$.
16. $(1+x+x^2+x^3+x^4)(1-x+x^2-x^3+x^4)$.

Find the continued product of

17. x^2-xy+y^2, x^2+xy+y^2, $x^4-x^2y^2+y^4$.
18. x^2-x+1, x^2+x+1, x^4-x^2+1, x^8-x^4+1.

Simplify

19. $(a+b+c)(a+b-c)+(a-b+c)(a-b-c)$
$+(-a+b+c)(a-b+c)$.
20. $(a+b+c+d)(a+b-c-d)+(a-b+c-d)(-a+b+c-d)$.
21. Add together $x^2-(x-y+z)(x+y-z)$, $y^2-(y-x+z)\times$ $(y+x-z)$, and $z^2-(z-x+y)(z+x-y)$. (C. E. 1864.)

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7. Product $(x+a)(x+b)$.

By actual multiplication we have

$$(\mathbf{x}+\mathbf{a})(\mathbf{x}+\mathbf{b})=\mathbf{x}^2+(\mathbf{a}+\mathbf{b})\mathbf{x}+\mathbf{ab}\text{......Formula VI.}$$

i.e., $(\mathbf{x}+\mathbf{a})(\mathbf{x}+\mathbf{b})=\mathbf{x}^2+$(sum of $\mathbf{a}$ and $\mathbf{b}$) $\mathbf{x}+$ product of $\mathbf{a}$ and $\mathbf{b}$.

Observe that the product involves x^2, x and a term not containing x, and that a, b in the formula may stand for any positive or negative quantities.

Obs. We can also write the formula thus:

$$(a+x)(b+x)=ab+(a+b)x+x^2$$

Ex. 1. $(x+a)(x-b)$
$=x^2+(\text{sum of } a \text{ and } -b)x+\text{product of } a \text{ and } -b$
$=x^2+(a-b)x-ab.$

Ex. 2. $(x-a)(x-b)$
$=x^2+(\text{sum of } -a \text{ and } -b)x+ \text{ product of } -a \text{ and } -b$
$=x^2-(a+b)x+ab.$

Ex. 3. $(x-4)(x+5)$
$=x^2+(\text{sum of } -4 \text{ and } 5)x+\text{product of } -4 \text{ and } 5$
$=x^2+x-20.$

Ex. 4. $(x+2)(x-9)$
$=x^2+(\text{sum of } 2 \text{ and} -9)x+\text{product of } 2 \text{ and } -9$
$=x^2-7x-18.$

Ex. 5. $(x-3)(x-8)$
$=x^2+(\text{sum of } -3 \text{ and } -8)x+\text{product of } -3 \text{ and } -8$
$=x^2-11x+24.$

Ex. 6. $(x-7y)(x+5y)$
$=x^2+(\text{sum of } -7y \text{ and } 5y)\ x+\text{product of } -7y \text{ and } 5y$
$=x^2+(-2y)x-35y^2$
$=x^2-2xy-35y^2.$

Ex. 7. Find the continued product of $(x+a)(x+b)(x-a)(x-b)$.

$$\begin{aligned}\text{Product}&=(x+a)(x-a)\times(x+b)(x-b)\\&=(x^2-a^2)(x^2-b^2)\\&=x^4-(a^2+b^2)x^2+a^2b^2.\end{aligned}$$

Ex. 8. Multiply $x+a+b$ by $x+c+d$, arranging the product in descending powers of x.

$$\begin{aligned}\text{Product}&=\{x+(a+b)\}\{x+(c+d)\}\\&=x^2+(a+b+c+d)x+(a+b)(c+d).\end{aligned}$$

EXERCISE XXXV.

Write down the products :—

1. $(x+2)(x+3)$. **2.** $(x+1)(x+5)$. **3.** $(x+7)(x-4)$.
4. $(a+4)(a-12)$. **5.** $(y-9)(y+7)$. **6.** $(5-c)(c-13)$.
7. $(ab+2)(ab-3)$. **8.** $(5x+2)(5x-12)$. **9.** $(x^2+y^2)(x^2+3y^2)$.
10. $(x-4y)(x-6y)$ **11.** $(x-7y)(x+5y)$.
12. $(a+bx)(a+cx)$. **13.** $(a+x)(a-5x)$.

Write down the products :—

14. $(3-lm)(3+5lm)$.
15. $(1-bx)(1-cx)$.
16. $(\frac{2}{3}x+\frac{3}{4}y)(\frac{2}{3}x+\frac{5}{4}y)$.
17. $(3y+z)(3y-2z)$.
18. $(6x-5y)(6x+y)$.
19. $(3x^2+7)(3x^2-1)$.
20. $(ax+by)(ax+cy)$.

Find the value of

21. $(x+2)(x+3)(x-2)(x-3)$.
22. $(x+a)(x+24)(x-a)(x-24)$.
23. $(ax+by)(ax+cy)(ax-by)(ax-cy)$.

---

8. Product $(ax+b)(cx+d)$.

We have by actual multiplication,

$$(\mathbf{ax}+\mathbf{b})(\mathbf{cx}+\mathbf{d})=\mathbf{acx}^2+(\mathbf{bc}+\mathbf{ad})\mathbf{x}+\mathbf{bd} \text{......Formula VII.}$$

Hence the product of the binomials $ax+b$ and $cx+d$ consists of three terms of which the first contains x^2, the second contains x and the third does not contain (*i.e.* is independent of) x.

To get the first term of the product multiply together the first terms of the binomials ; to get the second term of the product multiply the second term of *each* binomial by the first term of the other and add the results ; and to get the last term of the product multiply together the last terms of the binomials.

It should be noted that a, b, c, d in formula VII may be any quantities, positive or negative.

Ex. 1. Multiply $ax+b$ by $cx-d$.

Here first term of the product $=ax\times cx=acx^2$;

second term " " $=b\times cx+(-d)ax$

$=bcx-adx=(bc-ad)x$;

third term " " $=b\times(-d)=-bd$.

$$\therefore\ (ax+b)(cx-d)=acx^2+(bc-ad)x-bd.$$

Similarly, $(ax-b)(cx-d)=acx^2-(bc+ad)x+bd$.

Ex. 2. Multiply $5x+3$ by $4x+7$.

$$(5x+3)(4x+7)=5x.4x+(3.4x+7.5x)+3.7$$
$$=20x^2+47x+21.$$

Ex. 3. Multiply $3x+2$ by $5x-9$.

$$(3x+2)(5x-9)=3x.5x+(2.5x-9.3x)+2(-9)$$
$$=15x^2-17x-18.$$

Ex. 4. Multiply $5x-2$ by $3x-11$.

$(5x-2)(3x-11)=5x.3x-(2.3x+11.5x)+(-2)(-11)$.

$=15x^2-61x+22$.

Ex. 5. Multiply $3x+4y$ by $2x+5y$.

$(3x+4y)(2x+5y)=3x.2x+(4y.2x+5y.3x)+4y.5y$.

$=6x^2+23xy+20y^2$.

EXERCISE XXXVI.

Write down the product:—

1. $(2x+1)(3x+2)$	**2.** $(3x+5)(2x+7)$	**3.** $(4x+3)(3x-8)$
4. $(6x-5)(4x+3)$	**5.** $(4x-5)(2x+7)$	**6.** $(2x+3)(3x-5)$
7. $(9x-5)(7x-2)$	**8.** $(3x-11)(2x+1)$	**9.** $(4x-12)(2x-9)$
10. $(4x+7)(6x-5)$	**11.** $(8x-3)(6x-11)$	**12.** $(7x+4)(2x-3)$
13. $(3a-4)(2a-5)$	**14.** $(11a-2)(3a-7)$	**15.** $(7a-3)(5a-9)$
16. $(5x-9y)(2x+3y)$		**17.** $(4a-5b)(6a-b)$.

---

9. Product $(x+a)(x+b)(x+c)$.

By actual multiplication we have

$\mathbf{(x+a)(x+b)(x+c)}$

$\mathbf{=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc}$...Formula VIII.

Thus $(x+a)(x+b)(x+c)=x^3+(\text{sum of } a, b, c)\ x^2$

$+(\text{sum of products of every two of } a, b, c)\ x+\text{product of } a, b, c$.

Note that a, b, c may be any quantities, positive or negative.

Ex. 1. Find the product $(x+a)(x-b)(x+c)$.

Product$=x^3+(\text{sum of } a, -b, c)x^2$

$+(\text{sum of products of every two of } a, -b, c)\ x+\text{product of } a, -b, c$.

$=x^3+(a-b+c)x^2+(-ab+ac-bc)x-abc$.

Similarly, $(x-a)(x-b)(x-c)$

$=x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$.

Ex. 2. Find the product $(x-2)(x+3)(x-4)$.

Here sum of $-2, +3, -4=-3$;

sum of products of every two of $-2, +3, -4$

$=-2\times 3+(-2)(-4)+3(-4)$

$=-6+8-12=-10$;

product of $-2, +3, -4=24$.

$\therefore$ Required product$=x^3-3x^2-10x+24$.

Ex. 3. Find the product $(x-4)(x-5)(x-1)$.

Here sum of $-4, -5, -1 = -10$;

sum of products of every two of $-4, -5, -1$

$$=(-4)(-5)+(-4)(-1)+(-5)(-1)$$
$$=20+4+5=29.$$

product of $-4, -5, -1 = -20$.

$\therefore$ Required product $=x^3-10x^2+29x-20$.

EXERCISE XXXVII.

Obtained the following products

1. $(x+2)(x+3)(x+4)$
2. $(x+3)(x+5)(x+6)$
3. $(x-1)(x+7)(x-8)$
4. $(x-3)(x-5)(x-7)$
5. $(x+y)(x+2y)(x+3y)$
6. $(x+2y)(x+3y)(x-4y)$
7. $(a-3b)(a-5b)(a-7b)$
8. $(a+3b)(a-4b)(a-2b)$
9. $(2x-1)(2x+5)(2x-4)$
10. $(3a-4b)(3a-2b)(3a-b)$

---

10. By actual multiplication we have

$(\mathbf{a}+\mathbf{b})(\mathbf{a}^2-\mathbf{ab}+\mathbf{b}^2)=\mathbf{a}^3+\mathbf{b}^3$............Formula IX.

$(\mathbf{a}-\mathbf{b})(\mathbf{a}^2+\mathbf{ab}+\mathbf{b}^2)=\mathbf{a}^3-\mathbf{b}^3$............Formula X.

It may be noted that formula X may be deduced from formula IX by changing b into $-b$. Also we observe that there is a^3 *plus* b^3 or a^3 *minus* b^3 on the right according as we have a *plus* b or a *minus* b on the left.

Ex. 1. $(2x+3y)(4x^2-6xy+9y^2)=(2x)^3+(3y)^3=8x^3+27y^3$.

Ex. 2. $(3x-5y)(9x^2+15xy+25y^2)=(3x)^3-(5y)^3=27x^3-125y^3$.

Ex. 3. $(a+b)(a-b)(a^2+ab+b^2)(a^2-ab+b^2)$

$$=\{(a+b)(a^2-ab+b^2)\}\times\{(a-b)(a^2+ab+b^2)\}$$
$$=(a^3+b^3)(a^3-b^3)=a^6-b^6.$$

EXERCISE XXXVIII.

Write down the products :—

1. $(x+1)(x^2-x+1)$
2. $(x-1)(x^2+x+1)$
3. $(2x+5y)(4x^2-10xy+25y^2)$
4. $(3x-4y)(9x^2+12xy+16y^2)$.
5. $(2x-y)(4x^2+2xy+y^2)$
6. $(5x-2y)(25x^2+10xy+4y^2)$
7. $(a+6b)(a^2-6ab+36b^2)$
8. $(ab+2)(a^2b^2-2ab+4)$

Write down the products :—

9. $(5-xy)(25+5xy+x^2y^2)$ **10.** $\left(\frac{a}{2}-\frac{2b}{3}\right)\left(\frac{a^2}{4}+\frac{ab}{3}+\frac{4b^2}{9}\right)$

11. $(x-3)(x+3)(x^2-3x+9)(x^2+3x+9)$.

12. $(2x+y)(2x+y)(4x^2+2xy+y^2)(4x^2-2xy+y^2)$.

---

11. **Cube of a binomial.**

By actual multiplication we have

$$\left.\begin{aligned}(\mathbf{a}+\mathbf{b})^3&=\mathbf{a}^3+3\mathbf{a}^2\mathbf{b}+3\mathbf{a}\mathbf{b}^2+\mathbf{b}^3\\ \text{or}&=\mathbf{a}^3+\mathbf{b}^3+3\mathbf{a}\mathbf{b}(\mathbf{a}+\mathbf{b})\end{aligned}\right\}\text{......Formula XI,}$$

$$\left.\begin{aligned}(\mathbf{a}-\mathbf{b})^3&=\mathbf{a}^3-3\mathbf{a}^2\mathbf{b}+3\mathbf{a}\mathbf{b}^2-\mathbf{b}^3\\ \text{or}&=\mathbf{a}^3-\mathbf{b}^3-3\mathbf{a}\mathbf{b}(\mathbf{a}-\mathbf{b})\end{aligned}\right\}\text{......Formuia XII.}$$

Obs. It is to be noted that formula XII can be deduced from formula XI by changing b into $-b$. From these formulæ we have by trantposition

$$a^3+b^3=(a+b)^3-3ab(a+b)$$
$$a^3-b^3=(a-b)^3+3ab(a-b).$$

Ex. 1. Find the cube of $2x+3y$.

$$(2x+3y)^3=(2x)^3+3(2x)^2 3y+3.2x.(3y)^2+(3y)^3$$
$$=8x^3+36x^2y+54xy^2+27y^3$$

Ex. 2. Find the cube of $2a-5b$.

$$(2a-5b)^3=(2a)^3-3.(2a)^2.5b+3.2a.(5b)^2-(5b)^3$$
$$=8a^3-60a^2b+150ab^2-125b^3.$$

Ex. 3. If $a+b=7$, $ab=2$, find the value of a^3+b^3.

We have $a^3+b^3=(a+b)^3-3ab(a+b)$

$$=7^3-3.2\ 7=443-42=401.$$

Ex. 4. If $a-b=5$, $ab=6$, find the value of a^3-b^3.

We have $a^3-b^3=(a-b)^3+3ab(a-b)$

$$=5^3+3.6.5=215.$$

12. **Cube of a trinomial.** To find the cube of $a+b+c$.

$(a+b+c)^3=\{(a+b)+c\}^3$, taking $a+b$ as one term

$=(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$......by formula XI.

$=\{a^3+b^3+3ab(a+b)\}+3(a+b)^2c+3(a+b)c^2+c^3$......(1)

$=a^3+b^3+c^3+3(a+b)\{ab+(a+b)c+c^2\}$

$=a^3+b^3+c^3+3(a+b)(a+c)(b+c)$, by formula VI, Obs.

$\therefore (\mathbf{a}+\mathbf{b}+\mathbf{c})^3=\mathbf{a}^3+\mathbf{b}^3+\mathbf{c}^3+3(\mathbf{a}+\mathbf{b})(\mathbf{b}+\mathbf{c})(\mathbf{c}+\mathbf{a})$...formula XIII.

If in the above we multiply out the right-hand side at the stage (1) we have

$$(a+b+c)^3=a^3+b^3+3ab(a+b)+3(a^2+2ab+b^2)c+3(a+b)c^2+c^3.$$
$$=a^3+b^3+3a^2b+3ab^2+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3.$$
$$\therefore (\mathbf{a}+\mathbf{b}+\mathbf{c})^3=\mathbf{a}^3+\mathbf{b}^3+\mathbf{c}^3+3(\mathbf{a}^2\mathbf{b}+\mathbf{a}\mathbf{b}^2+\mathbf{b}^2\mathbf{c}+\mathbf{b}\mathbf{c}^2+\mathbf{c}^2\mathbf{a}+\mathbf{c}\mathbf{a}^2+2\mathbf{abc}) \text{.........formula XIV.}$$

Ex. 1. Expand $(a-b+c)^3$.

Changing b into $-b$ in formula XIV, we have $(a-b+c)^3$

$$=a^3-b^3+c^3+3(-a^2b+ab^2+b^2c-bc^2+c^2a+ca^2-2abc).$$

Ex. 2. Expand $(x+2y-3z)^3$.

Putting $a=x$, $b=2y$, $c=-3z$ in formula XIV,

$$(x+2y-3z)^3=x^3+8y^3-27z^3+3\{x^2.2y+x.(2y)^2+(2y)^2(-3z)+2y(-3z)^2+(-3z)^2x+(-3z)x^2+2.x.2y(-3z)\}$$
$$=x^3+8y^3-27z^3+3\{2x^2y+4xy^2-12y^2z+18yz^2+9z^2x-3zx^2-12xyz\}.$$

EXERCISE XXXIX.

Find the cubes of

1. $x+2y$.
2. $2x+y$.
3. $3x-4$.
4. $a+\frac{1}{2}b$.
5. $3x^2-4x^3$.
6. a^2-2b^2.
7. $a-b-c$.
8. $2a+b-3c$.
9. $ab+bc+ca$.
10. $a^2+b^2-c^2$.
11. $2l-5m+n$.
12. $3a-2b-5c$.
13. Find the value of a^3+b^3 if (i) $a+b=9$, $ab=11$.
    (ii) $a+b=7$, $ab=4$.
    and of a^3-b^3 if (i) $a-b=3$, $ab=5$.
    (ii) $a-b=4$, $ab=8$.
14. If $a+b=6$, prove that $a^3+b^3+18ab=216$.
15. If $a-b=2$, prove that $a^3-b^3=6ab+8$.
16. If $a+b=p$, $ab=q$, prove that $a^3+b^3=p^3-3pq$.
17. If $a-b=p$, $ab=q$, prove that $a^3-b^3=p^3+3pq$.
18. If $2x+3y=4$, $xy=1$, prove that $8x^3+27y^3+8=0$.
19. If $3a-5b=1$, $ab=2$, prove that $27a^3-125b^3=91$.

Simplyfy

20. $(a+b)^3+(a-b)^3$
21. $(a+b)^3-(a-b)^3$.
22. $(x+y)^3-(x+y-3)^3-27$.

13. We have by actual multiplication

$$(\mathbf{a}+\mathbf{b}+\mathbf{c})(\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2-\mathbf{bc}-\mathbf{ca}-\mathbf{ab})$$
$$=\mathbf{a}^3+\mathbf{b}^3+\mathbf{c}^3-3\mathbf{abc}\text{......formula XV.}$$

We observe that on the left hand side one factor is the sum of 3 quantities and the other factor is the sum of their squares *minus* sum of products of every two of them; while on the right hand side we have the sum of their cubes *minus* 3 times their product.

Ex. 1. Find the product $(a-b+c)(a^2+b^2+c^2+bc-ca+ab)$.

Changing b into $-b$ in formula XV, we have

$$(a-b+c)(a^2+b^2+c^2+bc-ca+ab)=a^3+(-b)^3+c^3-3a(-b)c$$
$$=a^3-b^3+c^3+3abc.$$

Ex. 2. Find the product

$$(2x-3y-4z)\ (4x^2+9y^2+16z^2-12yz+8zx+6xy).$$

Put $a=2x$, $b=-3y$, $c=-4z$ in formula XV, then

$$(2x-3y-4z)(4x^2+9y^2+16z^2-12yz+8zx+6xy)$$
$$=(2x)^3+(-3y)^3+(-4z)^3-3.2x(-3y)(-4z)$$
$$=8x^3-27y^3-64z^3-72xyz.$$

Ex. 3. If $a+b+c=6$, $bc+ca+ab=3$, find the value of $a^3+b^3+c^3-3abc$.

We have $a^2+b^2+c^2=(a+b+c)^2-2(bc+ca+ab)$, art. 4, obs.

$$=6^2-2\times 3=30.$$

Now $a^3+b^3+c^3-3abc$

$$=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$
$$=6\times(30-3)=6\times 27=162.$$

EXERCISE XL.

Find the following products

1. $(a-b-c)(a^2+b^2+c^2-bc+ca+ab)$.
2. $(x-y+1)(x^2+xy+y^2-x+y+1)$.
3. $(x-y-1)(x^2+xy+y^2+x-y+1)$.
4. $(2x-3y+4z)(4x^2+9y^2+16z^2+12yz-8zx+6xy)$.
5. $(1+a-ab)(a^2b^2+a^2b+ab+a^2-a+1)$.
6. $(bc+ca+ab)(b^2c^2+c^2a^2+a^2b^2-bc^2a-ca^2b-ab^2c)$.
7. $(bc-ca+2)(b^2c^2+c^2a^2+bc^2a+2ca-2bc+4)$.
8. $(x^2+3x-2)(x^4-3x^3+11x^2+6x+4)$.

9. If $a+b+c=12$, $bc+ca+ab=5$, find the value of $a^3+b^3+c^3-3abc$.

10. If $a+b+c=16$, $bc+ca+ab=9$, $abc=4$, find the value of $a^3+b^3+c^3$.

11. If $a+b+c=p$, $bc+ca+ab=q$, $abc=r$, prove that $a^3+b^3+c^3=p^3-3pq+3r$.

12. If $a+b+c=11$, $a^2+b^2+c^2=50$, $abc=27$, find the value of $a^3+b^3+c^3$.

14. The following results in division are important.

I. By actual division we get,

$$\frac{a^2-b^2}{a-b}=a+b,$$

$$\frac{a^3-b^3}{a-b}=a^2+ab+b^2,$$

$$\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3,$$

$$\frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4 \text{ ; and so on}$$

Thus it appears that a^n-b^n is divisible by $a-b$ when n is a positive whole number.

II. By actual division we get,

$$\frac{a^2-b^2}{a+b}=a-b,$$

$$\frac{a^3-b^3}{a+b}=a^2-ab+b^2-\frac{2b^3}{a+b},$$

$$\frac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3,$$

$$\frac{a^5-b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4-\frac{2b^5}{a+b},$$

$$\frac{a^6-b^6}{a+b}=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5 \text{ ; and so on.}$$

Thus it appears that a^n-b^n is divisible by $a+b$ only when n is an even whole number but not when n is odd.

III. By actual division we get

$$\frac{a^2+b^2}{a+b}=a-b+\frac{2b^2}{a+b},$$

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2,$$

$$\frac{a^4+b^4}{a+b}=a^3-a^2b+ab^2-b^3+\frac{2b^4}{a+b},$$

$$\frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4,$$

$$\frac{a^6+b^6}{a+b}=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5+\frac{2b^6}{a+b},$$

$$\frac{a^7+b^7}{a+b}=a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6, \text{ and so on.}$$

Thus it appears that a^n+b^n is divisible by $a+b$ only when n is an odd whole number but not when n is an even integer.

IV. By actual division similarly it will be found that a^n+b^n is never divisible by $a-b$ where n is any positive integer.

Thus when n is a positive integer :—

(1) **a^n-b^n is always divisible by $a-b$.**

(2) **a^n-b^n is divisible by $a+b$ only when n is even but not when n is odd.**

(3) **a^n+b^n is divisible by $a+b$ only when n is odd but not when n is even.**

(4) **a^n+b^n is never divisible by $a-b$.**

(For proof see Remainder Theorem, Chap. XXIII.)

15. We shall here consider some identities of an easy type. To establish an identity we can use, any one of the following methods :—

(i) to reduce *one* side and prove that it leads to the other side ;

(ii) to reduce *both* sides and to prove that they lead to the same result.

Cyclic order. In some cases we shall find it convenient to use the letters occuring in an expression in cyclic order. If we arrange any number of letters along the circumference of a circle and if starting from any one of them proceed round the circumference in the *same direction* we meet the letters in cyclic order. Thus a, b, c, d or b, c, d, a or $c, d, a\ b$ or d, a, b, c will be in cyclic order.

6. Difference of two squares. We have from formula V

$$a^2-b^2=(a+b)(a-b).$$

To factorize directly, we have $a^2-b^2=(a^2+ab)-(ab+b^2)$

$$=a(a+b)-b(a+b)=(a+b)(a-b)$$

This result is very important and may be stated thus :—*The difference of the squares of two quantities is equal to the product of their sum and difference.*

Ex. 1. $1-9a^2=1^2-(3a)^2=(1+3a)(1-3a)$.

Ex. 2. $4x^2-25y^2=(2x)^2-(5y)^2=(2x+5y)(2x-5y)$.

Ex. 3. $a^4-b^4=(a^2)^2-(b^2)^2=(a^2+b^2)(a^2-b^2)$
$=(a^2+b^2)(a+b)(a-b)$.

Ex. 4. $5x^2-20a^2=5(x^2-4a^2)$
$=5(x+2a)(x-2a)$.

Ex. 5. $xy^2-xz^2=x(y^2-z^2)=x(y+z)(y-z)$.

Ex. 6. Find the value of $(7958)^2-(7953)^2$

The expr. $=(7958+7953)(7958-7953)$
$=15911\times5=79555$.

Ex. 7. Factorize $a^2-(b-c)^2$.

$a^2-(b-c)^2=\{a+(b-c)\}\{a-(b-c)\}$
$=(a+b-c)(a-b+c)$.

Ex. 8. Factorize $(3x-4y)^2-(2x-y)^2$.

The expr. $=\{(3x-4y)+(2x-y)\}\{(3x-4y)-(2x-y)\}$
$=(5x-5y)(x-3y)=5(x-y)(x-3y)$.

Ex. 9. Factorize $a^2-b^2+2bc-2ca$.

The expr. $=(a^2-b^2)-(2ca-2bc)$
$=(a+b)(a-b)-2c(a-b)=(a-b)(a+b-2c)$.

EXERCISE XLVI.

Find the factors of (mentally) :—

1. x^2-1.	2. x^2-4.	3. $4x^2-1$.
4. $25a^2-16$.	5. x^2y^2-49.	6. $a^2b^4-c^4$.
7. $a^2x^4-a^2y^2$.	8. a^6-36.	9. x^4-121.
10. $36a^4-169$.	11. $25x^2-16y^2$.	12. $49x^2-64y^2$.
13. $x^2y^2-100x^2y^2z^2$.	14. $16a^4b^2-36a^2b^4$.	15. $16a^5-100a$.
16. $l^2m^2-9a^4$.	17. $121a^2-81x^2$.	18. $3a^2-27b^2$.
19. $4xy^2-9xz^2$.	20. $12ab^2-75ac^2$.	21. $a^2p^4-b^2q^2$

Find the value of

22. $5932^2 - 5929^2$. **23.** $7129^2 - 7128^2$ **24.** $98397^2 - 98297^2$.

Factorize

25. $x^2y^4 - x^6$. **26.** $81x - x^9$. **27.** $a^8 - b^8$

28. $(a+b)^2 - c^2$. **29.** $(a-b)^2 - c^2$. **30.** $a^2 - (b+c)^2$.

31. $(a+b)^2 - (c+d)^2$. **32.** $1 - (x+y+z)^2$.

33. $(3x-4y)^2 - (2a+b)^2$ **34.** $(7x+9)^2 - (4x+5)^2$.

35. $(x+a)^2 - (y+a)^2$. **36.** $16(a+b)^2 - 9(a-b)^2$.

37. $(x^2+y^2)^2 - 4x^2y^2$. **38.** $3(3x-5y)^2 - 12(x-7y)^2$.

39. $4x^2 - 9y^2 - 2x - 3y$. **40.** $a^2 - 4b^2 + 2ac + 4bc$.

41. $9a^2 - b^2 - 12a - 4b$. **42.** $16x^2 - 25y^2 + 4xz + 5yz$.

Simplify

43. $(a+b)^2 - (a-b)^2$. **44.** $(3x+4y-7z)^2 - (3x-4y+7z)^2$

45. $(5a-9b+11c)^2 - (5a+9b-11c)^2$.

46. $9(2a+3b+2c)^2 - 4(a+b+3c)^2$

47. $(2a-3b+4c-5d)^2 - (3b+4c+5d+2a)^2$.

48. $(a+b+c)^2 - (b+c-a)^2 + (c+a-b)^2 - (a+b-c)^2$.

---

7. Expression reducible to the difference of two squares.

Sometimes an expression may be expressed (by adding and subtracting the same quantities, if necessary) as the difference of two squares, and then we can factorize it as in art. 6.

Ex. 1. Factorize $a^2 - 2ab + b^2 - c^2$.

The expr. $= (a^2 - 2ab + b^2) - c^2$

$= (a-b)^2 - c^2$

$= (a-b+c)(a-b-c)$.

Ex. 2. Factorize $a^2 - b^2 - c^2 + d^2 + 2ad + 2bc$.

The expr. $= (a^2 + 2ad + d^2) - (b^2 - 2bc + c^2)$

$= (a+d)^2 - (b-c)^2$

$= (a+d+b-c)(a+d-b+c)$.

Note Here $2ad$ suggests that we are to take it with $a^2 + d^2$; so we take $2bc$ with $-b^2 - c^2$.

Ex. 3. Factorize $\mathbf{a^4+4b^4}$.

We have $a^4+b^4=(a^2)^2+(2b^2)^2$; hence to get a perfect square we add $2a^2.2b^2$ or $4a^2b^2$, subtracting the same to keep the value of the expression unchanged.

Thus a^4+4b^4

$$=a^4+4b^4+4a^2b^2-4a^2b^2$$
$$=(a^2+2b^2)^2-(2ab)^2$$
$$=(\mathbf{a^2+2ab+2b^2})(\mathbf{a^2-2ab+2b^2}).$$

It appears that the sum of the squares of two quantities can be factorized if twice their product is a square quantity.

Ex. 4. Factorize $\mathbf{a^4+a^2b^2+b^4}$.

Here we add a^2b^2 to get a perfect square and subtract the same to keep the value of the expression unchanged.

Thus $a^4+a^2b^2+b^4$

$$=(a^4+2a^2b^2+b^4)-a^2b^2$$
$$=(a^2+b^2)^2-(ab)^2$$
$$=(\mathbf{a^2+ab+b^2})(\mathbf{a^2-ab+b^2}).$$

This example is an important one and the method of it may be applied to any expression which, when a square quantity is added to it, becomes a perfect square.

Ex. 5. Divide $(x-1)^4+4(x-1)^2y^2+16y^4$ by

$$x^2+4y^2+2xy-2x-2y+1.$$

Putting $x-1=a$, $2y=b$, the dividend

$=a^4+a^2b^2+b^4$

$=(a^2+ab+b^2)(a^2-ab+b^2)$ by ex. 4

$=\{(x-1)^2+(x-1)2y+4y^2\}\{(x-1)^2-(x-1)2y+4y^2\}$

$=(x^2+4y^2+2xy-2x-2y+1)(x^2+4y^2-2xy-2x+2y+1)$

$\therefore$ quotient $=x^2+4y^2-2xy-2x+2y+1$.

EXERCISE XLVII.

Factorize

1. x^2-y^2-2y-1.
2. $x^2+2xy+y^2-9z^2$.
3. $x^2-4xy+4y^2-z^2$.
4. $x^2-6xy+9y^2-25$.
5. $4a^2-12ab+9b^2-4c^2$.
6. $9a^2+30ab+25b^2-100c^2$.
7. $x^2-y^2-z^2+2yz$.
8. $a^2+b^2-c^2-d^2+2ab-2cd$.
9. $4x^2-49z^2+y^2-4xy$.
10. $4a^2-16b^2+8b-1$.

Factorize

11. $9a^2-16b^2+25x^2-y^2+8by-30ax$.

12. $4a^2+4b^2-9c^2-9d^2-8ab+18cd$.

13. $25a^2+16b^2-c^2-40ab-14c-49$.

14. a^4+4. **15.** $64+x^4$. **16.** $81a^4+64b^4$.

17. $4x^4+81y^4$. **18.** $9x^4+36y^4$. **19.** $1024+81a^4$.

20. a^4+a^2+1. **21.** x^4+9x^2+81.

22. $a^4+4a^2b^2+16b^4$. **23.** $16x^4+36x^2y^2+81y^4$.

24. x^8+x^4+1. **25.** $a^8+a^4b^4+b^8$.

Divide (using factors)

26. $(2x+3y)^4+4(x-y)^4$ by $(x+y)^2+x^2$.

27. $x^4+9x^2y^2+81y^4$ by $x^2+3xy+9y^2$.

28. $(x-y)^4+(x-y)^2z^2+z^4$ by $x^2+y^2+z^2-yz+zx-2xy$.

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8. Sum or difference of two cubes. We have from formulæ IX and X

(i) $\mathbf{a^3+b^3=(a+b)(a^2-ab+b^2)}$.

(ii) $\mathbf{a^3-b^3=(a-b)(a^2+ab+b^2)}$.

To factorize directly we proceed thus :

$$\text{(i)}\quad \begin{aligned} a^3+b^3 &= a^3-ab^2+ab^2+b^3\\ &= a(a^2-b^2)+b^2(a+b).\\ &= (a+b)\{a(a-b)+b^2\}\\ &= (a+b)(a^2-ab+b^2). \end{aligned}$$

$$\text{(ii)}\quad \begin{aligned} a^3-b^3 &= a^3-ab^2+ab^2-b^3\\ &= a(a^2-b^2)+b^2(a-b)\\ &= (a-b)\{a(a+b)+b^2\}\\ &= (a-b)(a^2+ab+b^2). \end{aligned}$$

The above formulæ enable us to factorize any expression which is the *sum or difference of two cubes.*

Ex. 1. Factorize $8x^3+27y^3$.

$$\begin{aligned} \text{The expr.} &= (2x)^3+(3y)^3\\ &= (2x+3y)\{(2x)^2-2x.3y+(3y)^2\}\\ &= (2x+3y)(4x^2-6xy+9y^2). \end{aligned}$$

Ex. 2. Factorize $64a^3-125x^3$.

The expr. $=(4a)^3-(5x)^3$

$=(4a-5x)\{(4a)^2+4a.5x+(5x)^2\}$

$=(4a-5x)(16a^2+20ax+25x^2)$.

Ex. 3. Factorize a^6-b^6.

$a^6-b^6=(a^3)^2-(b^3)^2$

$=(a^3+b^3)(a^3-b^3)$, formula V.

$=(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.

Here we might begin by puttting $a^6-b^6=(a^2)^3-(b^2)^3$, but then the process would have been tedious.

Ex. 4. Factorize $a^3+b^3+2a+2b$

The expr. $=(a^3+b^3)+(2a+2b)$

$=(a+b)(a^2-ab+b^2)+2(a+b)$

$=(a+b)(a^2-ab+b^2+2)$.

EXERCISE XLVIII.

Find the factors of (mentally) :—

1. x^3+1. **2.** x^3-1. **3.** $27+8a^3$. **4.** $27-8a^3$.
5. $8x^3-y^3$. **6.** $27x^3+64$. **7.** a^3+27b^3. **8.** $8a^3-b^6$.

Factorize

9. $x^3y^3-27z^3$. **10.** l^3-216m^3. **11.** $8x^4-xy^3$. **12.** $x^{12}-y^{12}$.
13. $(2x+y)^3-8x^3$. **14.** $(a^2+b^2)^3-8x^6y^6$.
15. $(3a+5b)^3+(4b-3a)^3$. **16.** $27(a+2b)^3+8(3b-2a)^3$.

Factorize

17. $a^3-b^3+ma-mb$. **18.** $27x^3+a^3+9x+3a$.
19. $x^3+y^3+2(x+y)^2$. **20.** $x^3-8y^3+5x-10y$.

Divide

21. $(x+y)^3-8z^3$ by $x+y-2z$.
22. $(5x+7y)^3+(3x+5y)^3$ by $2x+3y$.
23. $(3x^2-5x+6)^3-(2x^2+x-1)^3$ by x^2-6x+7.
24. Divide $(ax+by)^3+(ax-by)^3-(ay-bx)^3+(ay+bx)^3$
by $(a+b)^2x^2-3ab(x^2-y^2)$ (C. E. 1888).

9. Trinomials of the form x^2+px+q.

First method. One method of factorizing an expession of the form x^2+px+q is obtained from formula VI. Suppose by trial we determine two numbers a and b such that $a+b=p$, $ab=q$. Then

$$\begin{aligned} x^2+px++q &= x^2+(a+b)x+ab \\ &=(x^2+ax)+(bx+ab) \\ &=x(x+a)+b(x+a) \\ &=(x+a)(x+b). \end{aligned}$$

Thus we get the following rule :—

To factorize x^2+px+q we find by trial (if possible) two numbers a and b such that $a+b=p$ and $ab=q$; then by putting for p and q these values and grouping terms we get the factors to be $x+a$ and $x+b$.

The success of this method of inspection depends upon practice and the student will do well to attend to the following examples.

Ex. 1. Factorize $x^2+13x+36$.

Here we are to find two numbers whose product is 36 and sum is 13. We may observe that since the product is positive, the numbers may be both positive or both negative ; but since the sum is positive, the numbers can only be both positive. Now pairs of *positive* factors of 36 are the following :—

$$1, 36 \; ; \; 2, 18 \; ; \; 3, 12 \; ; \; 4, 9 \; ; \; 6, 6.$$

Of these pairs 4 and 9 have their sum equal to 13. Hence the following process :—

$$\begin{aligned} x^2+13x+36 &= x^2+4x+9x+36 \\ &=x(x+4)+9(x+4) \\ &=(x+4)(x+9). \end{aligned}$$

Ex. 2. Factorize $x^2-20x+64$.

Here we want two numbers whose product $=64$ and sum $=-20$. Now since the product is positive, the numbers may be both positive or both negative ; but because the sum is negative, the numbers can only be both negative. Now pairs of *negative* factors of 64 are

$$-1, -64 \; ; \; -2, -32 ; \; -4, -16 \; ; \; -8, -8.$$

Of these pairs the sum of -4 and -16 is -20. Hence the process stands thus : –

$$\begin{aligned} x^2-20x+64 &= x^2-4x-16x+64 \\ &=x(x-4)-16(x-4) \\ &=(x-4)(x-16). \end{aligned}$$

Ex. 3. Factorize $x^2-7x-30$.

Here we want two numbers whose product $=-30$ and sum $=-7$. Since the product is negative, the factors are of opposite signs, and since the sum is negative, the numerically greater one is negative.

Now the pairs of factors of -30 which are of *opposite signs* (the numerically greater factor being negative) are

$$1, -30;\ 2, -15;\ 3, -10;\ 5, -6.$$

Of these 3 and -10 have their sum $=-7$. Thus we have

$$\begin{aligned} x^2-7x-30 &= x^2+3x-10x-30 \\ &= x(x+3)-10(x+3) \\ &= (x+3)(x-10). \end{aligned}$$

Ex. 4. Factorize $x^2+5x-24$.

Here we want two numbers whose product $=-24$ and sum $=5$. Since the product is negative, the numbers are of opposite signs and since the sum is positive, the greater one is positive. Now pairs of factors of -24 which are of *opposite signs* (the greater one being positive) are the following :—

$$-1, 24;\ -2, 12;\ -3, 8;\ -4, 6.$$

Of these pairs -3 and 8 have their sum equal to 5. Thus we have $x^2+5x-24=x^2-3x+8x-24$.

$$\begin{aligned} &= x(x-3)+8(x-3) \\ &= (x-3)(x+8) \end{aligned}$$

Ex. 5. Factorize $x^2+2xy-48y^2$.

Here we want two quantities whose sum $=2y$, and product $=-48y^2$. By trial we find them to be $8y$ and $-6y$. Hence $x^2+2xy-48y^2=x^2+8xy-6xy-48y^2$.

$$\begin{aligned} &= x(x+8y)-6y(x+8y) \\ &= (x+8y)(x-6y). \end{aligned}$$

Ex. 6. Factorize $a^2-11ab+30b^2$.

Here we want two quantities whose sum $=-11b$ and product $=30b^2$, and they are found to be $-5b$ and $-6b$.

Thus
$$\begin{aligned} a^2-11ab+30b^2 &= a^2-5ab-6ab+30b^2 \\ &= a(a-5b)-6b(a-5b) \\ &= (a-5b)(a-6b). \end{aligned}$$

Ex. 7. Factorize x^4-10x^2+9.

Putting $x^2=a$, the given expr.

$$=a^2-10a+9=a^2-a-9a+9$$
$$=a(a-1)-9(a-1)=(a-1)(a-9)$$
$$=(x^2-1)(x^2-9), \text{ restoring } x^2$$
$$=(x+1)(x-1)(x+3)(x-3).$$

Ex. 8. Factorize $(x+y)^2-11(x+y)-42$.

Putting $x+y=a$, the expr.

$$=a^2-11a-42$$
$$=a^2-14a+3a-42$$
$$=a(a-14)+3(a-14)$$
$$=(a-14)(a+3)$$
$$=(x+y-14)(x+y+3), \text{ substituting for } a.$$

Ex. 9. Factorize $(2a-3b)^2+18(2a-3b)(a-2b)-63(a-2b)^2$.

Putting $2a-3b=x$, $a-2b=y$, the expr.

$$=x^2+18xy-63y^2$$
$$=x^2+21xy-3xy-63y^2$$
$$=x(x+21y)-3y(x+21y)$$
$$=(x+21y)(x-3y)$$
$$=\{(2a-3b)+21(a-2b)\}\{2a-3b-3(a-2b)\}$$
$$=(23a-45b)(-a+3b).$$

Ex. 10. Factorize $x^2+2(3p+2)x+5p(p+4)$.

Putting $5p=a$, $p+4=b$, we find $a+b=5p+p+4=2(3p+2)$

Hence the given expr.

$$=x^2+(a+b)x+ab$$
$$=(x+a)(x+b)$$
$$=(x+5p)(x+p+4).$$

EXERCISE XLIX.

Find the factors of (mentally) :—

1. x^2+4x+3.
2. x^2+6x+5.
3. $x^2+9x+20$.
4. $x^2-8x+15$.
5. $x^2+17x+60$.
6. $x^2-11x+28$.
7. x^2-5x+6.
8. x^2-9x+8.
9. $x^2+10x+24$.
10. x^2-5x+4.
11. x^2-x-12.
12. $x^2+17x-18$.
13. x^2-x-72.
14. $x^2+20x+96$.
15. $x^2+5x-84$.

Find the factors of :—

16. $x^2+3x-70$. 17. $x^2-13x+12$. 18. $x^2-x-380$.
19. $x^2-10x-75$. 20. $x^2+x-182$. 21. $3x-40+x^2$.
22. $a^2-6a-40$. 23. $30+13x+x^2$. 24. $x^2+8x-240$.
25. $l^2+19l+48$. 26. c^2-c-56. 27. $a^2-23a-78$.
28. $x^2+4xy-5y^2$. 29. $x^2-2xy-48y^2$. 30. $x^2-5xy-150y^2$.
31. $x^2+8xy-240y^2$. 32. $x^2+ax-30a^2$. 33. $a^2+11ab-60b^2$.
34. $a^2b^2+10ab+16$. 35. $p^2+10pq-24q^2$. 36. $l^2+2lm-63m^2$.
37. $c^2-12cd-45d^2$. 38. $h^2+hk-110k^2$. 39. $a^2-29ab+54b^2$.
40. $m^2-8mn-84n^2$. 41. $a^2-34ax-72x^2$. 42. $b^2-11bk-80k^2$.
43. $x^2-xy-240y^2$. 44. x^4+17x^2+66. 45. m^4+16m^2+63.
46. c^4-41c^2+400. 47. $a^6-9a^3b^3+8b^6$. 48. x^6-15x^3+54.
49. $x^8-x^4y^4-42y^8$. 50. $a^{10}-17a^5b^5+16b^{10}$.

Factorize

51. $(2x+3)^2+3(2x+3)+2$. 52. $(5x-9y)^2+16(5x-9y)-80$.
53. $(x^2+3x)^2-22(x^2+3x)-48$.
54. $(3a+5b)^2-14(3a+5b)(2a-3b)+48(2a-3b)^2$.
55. $(5a+7b)^2-11(5a+7b)(a+2b)+30(a+2b)^2$.
56. $(4x-3b)^2+7(4x-3b)(b-x)+12(b-x)^2$.
57. $(l^2+m^2)^2+(l^2+m^2)(m^2+n^2)-90(m^2+n^2)^2$.
58. $(3p+q)^2-11(3p+q)(2q+p)-26(2q+p)^2$.
59. $a^2+(2b+1)a+b(b+1)$.
60. $x^2-x(a-7b)-6b(a-b)$.
61. $x^2+x-a(a-1)$. 62. $x^2+2x(4a-1)+3a(5a-2)$.
63. $x^2-(2c-d)x-(5c-9d)(7c-10d)$.
64. $(2x^2-8x+9)^2-2(2x^2-8x+9)(2x-5)-3(2x-5)^2$.
65. $(5x^2+13x+14)^2-2(5x^2+13x+14)(3x+4)-24(3x+4)^2$.

---

10. Trinomials of the form x^2+px+q (*continued*).

Second method. Another method of factorizing x^2+px+q is by expressing it as the difference of two squares. We know x^2+px becomes a perfect square by the addition of $(\frac{1}{2}p)^2$ or *the square of one-half of the co-efficient of x*. The method of procedure will be understood from the following example :—

Ex. 1. Factorize x^2+6x+8.

We have $x^2+6x+8=x^2+6x+9-9+8$.

[Adding 9, the square of half the coeff. of x, to get a perfect square and subtracting the same to keep the value unchanged]

$$=(x+3)^2-1$$
$$=(x+3+1)(x+3-1)$$
$$=(x+4)(x+2).$$

Ex. 2. Factorize $x^2-5x-14$

We have $x^2-5x-14=x^2-5x+(\frac{5}{2})^2-(\frac{5}{2})^2-14$

[adding and subtracting $(\frac{5}{2})^2$]

$$=x^2-5x+(\tfrac{5}{2})^2-\tfrac{81}{4}$$
$$=(x-\tfrac{5}{2})^2-(\tfrac{9}{2})^2$$
$$=(x-\tfrac{5}{2}+\tfrac{9}{2})(x-\tfrac{5}{2}-\tfrac{9}{2})$$
$$=(x+2)(x-7).$$

Ex. 3. Factorize $a^2-15ab+50b^2$

The expr. $=a^2-15ab+\left(\frac{15b}{2}\right)^2-\left(\frac{15b}{2}\right)^2+50b^2.$

$$=a^2-15ab+\left(\frac{15b}{2}\right)^2-\frac{25b^2}{4}$$
$$=\left(a-\frac{15b}{2}\right)^2-\left(\frac{5b}{2}\right)^2$$
$$=\left(a-\frac{15b}{2}+\frac{5b}{2}\right)\left(a-\frac{15b}{2}-\frac{5b}{2}\right)$$
$$=(a-5b)(a-10b).$$

EXERCISE L.

Factorize by the method of art. 10.

1. $x^2+5x-14$.	2. x^2-x-2.	3. x^2+x-12.
4. $x^2+9x-36$.	5. $a^2-8a-48$.	6. $x^2-7x-30$.
7. $x^2-4x-45$.	8. $a^2-12a+27$.	9. $a^2+12a+27$.
10. $a^2-9ab+20b^2$.	11. $c^2+cd-2d^2$.	12. $p^2-17p+72$.
13. $a^2+11ab+30b^2$.	14. $x^2+3xy-28y^2$.	15. $x^2-4xy-45y^2$.

11. Trinomials of the form ax^2+bx+c.

First method. We have (see formula VII.).

$(lx+m)(px+q)=lpx^2+(mp+lq)x+mq=ax^2+bx+c$, if we put a for lp, b for $mp+lq$ and c for mq.

Now we observe that $ac = lp \times mq = mp \times lq$, *i.e.*, the product of mp and lq; and $b = mp + lq$, *i.e.*, the sum of mp and lq. Hence **to factorize ax^2+bx+c we find by trial (if possible) two quantities whose product is ac and sum is b, put b equal to this sum and proceed by grouping terms.**

The following examples illustrate the method.

Ex. 1. Factorize $6x^2+7x+2$.

Here we want two factors of 6×2 or 12, such that their sum $=7$. Evidently the factors are 3 and 4. Hence the following process:

$$\begin{aligned}6x^2+7x+2 &= 6x^2+3x+4x+2\\ &= 3x(2x+1)+2(2x+1)\\ &= (2x+1)(3x+2)\end{aligned}$$

Ex. 2. Factorize $3x^2-5x-8$.

Here we want to resolve $3 \times (-8)$ or -24 into two factors whose sum is -5. Evidently the factors are -8 and 3. Hence we have.

$$\begin{aligned}3x^2-5x-8 &= 3x^2-8x+3x-8\\ &= x(3x-8)+(3x-8)\\ &= (3x-8)(x+1).\end{aligned}$$

Ex. 3. Factorize $5x^2-29xy-6y^2$.

Here we want to resolve $5 \times (-6y^2)$ or $-30y^2$ into two factors whose sum $-29y$. The factors are $-30y$ and y. Hence we have

$$\begin{aligned}5x^2-29xy-6y^2 &= 5x^2-30xy+xy-6y^2\\ &= 5x(x-6y)+y(x-6y)\\ &= (x-6y)(5x+y)\end{aligned}$$

Ex. 4. Factorize $48x^2+150x-63$.

Here we want to resolve $48 \times (-63)$ into two factors whose sum $=150$. Now $48 \times (-63) = 8 \times 6 \times (-3 \times 21) = (8 \times 21) \times (-3 \times 6) = 168 \times (-18)$; and the sum of 168 and -18 is 150. Hence we have

$$\begin{aligned}&48x^2+150x-63\\ &= 48x^2+168x-18x-63\\ &= 24x(2x+7)-9(2x+7)\\ &= (2x+7)(24x-9) = 3(2x+7)(8x-3)\end{aligned}$$

Ex. 5. Factorize $ax^2+(a^2+1)x+a$

Here we want to break up $a\times a$ or a^2 into two factors whose sum is a^2+1. The factors are evidently a^2 and 1. Hence we have

$$ax^2+(a^2+1)x+a$$
$$=ax^2+a^2x+x+a$$
$$=ax(x+a)+(x+a)=(x+a)(ax+1).$$

Ex. 6. Factorize $6(a+b)^2-7(a+b)-20$

Putting $a+b=x$, the expr. $=6x^2-7x-20$

$$=6x^2-15x+8x-20$$
$$=3x(2x-5)+4(2x-5)$$
$$=(2x-5)(3x+4)$$
$$=\{2(a+b)-5\}\{3(a+b)+4\}$$
$$=(2a+2b-5)(3a+3b+4).$$

Ex. 7. Factorize $5(2x-3y)^2+4(2x-3y)(x+4y)-12(x+4y)^2$

Putting $2x-3y=a$, $x+4y=b$.

the expr. $=5a^2+4ab-12b^2$

$$=5a^2+10ab-6ab-12b^2$$
$$=5a(a+2b)-6b(a+2b)$$
$$=(a+2b)(5a-6b)$$
$$=\{(2x-3y)+2(x+4y)\}\{5(2x-3y)-6(x+4y)\}$$
$$=(4x+5y)(4x-39y).$$

EXERCISE LI.

Factorize

1. $3x^2-xy-4y^2$. **2.** $3x^2-11x-20$.

3. $4x^2-37x-30$. **4.** $12x^2-5x-3$.

5. $10x^2-29xy-21y^2$. **6.** $7ab-10b^2+6a^2$.

7. $60a^2-77a+24$. **8.** $6x^2+17x+12$.

9. $10p^2+37p-21$. **10.** $21q^2-17q+2$.

11. $30+49a+20a^2$. **12.** $98a^2b^2+7ab-6$.

13. $14a^2-65a+56$. **14.** $16x^2+145x+9$.

15. $12x^2+47x+45$. **16.** $63x^2+x-64$.

17. $28x^2-69x+36$. **18** $15x^2-57x-72$

19. $56x^2+26x-55$. **20.** $45x^2+39x-28$. **21.** $2x^4+31x^2+75$.

22. $3x^6-7x^3+2$. **23.** $2a^8-7a^4+6$. **24.** $3(a+b)^2+a+b-2$

25. $5(3a-5b)^2+13(3a-5b)+6$.

Factorize.

26. $4(5x+7y)^2-12(5x+7y)(2x-3y)+5(2x-3y)^2$

27. $6(2a-5b)^2-7(2a-5b)(a+3b)-20(a+3b)^2$

28. $28(3a+5b)^2-41(3a+5b)(4a+7b)+15(4a+7b)^2$

29. $3(3x^2+9x+2)^2+10(3x^2+9x+2)(x^2-4x-11)$
$-8(x^2-4x-11)^2$.

30. $2(2x^2+11x+14)^2+7(2x^2+11x+14)(x^2-7x-18)$
$+3(x^2-7x-18)^2$.

---

12. Trinomials of the form ax^2+bx+c *(continued)*. **Second method.** We can reduce ax^2+bx+c to the form x^2+px+q as in the following examples and then proceed according to art. 9 or 10.

Ex. 1. Factorize $2x^2+x-3$.

$2x^2+x-3=\frac{1}{2}(4x^2+2x-6)$, multiplying and dividing by 2

$=\frac{1}{2}(z^2+z-6)$, putting z for $2x$

$=\frac{1}{2}(z+3)(z-2)$, by art. 9

$=\frac{1}{2}(2x+3)(2x-2)$, restoring x............(1)

$=(2x+3)(x-1)$.

Note that we multiply the expression by 2, so as to make the co-eff. of x^2 a perfect square.

Otherwise thus:—

$$2x^2+x-3=2(x^2+\frac{x}{2}-\frac{3}{2}).$$

$=2\{x^2+\frac{x}{2}+(\frac{1}{4})^2-(\frac{1}{4})^2-\frac{3}{2}\}$, as in art. 10

$=2\{(x+\frac{1}{4})^2-(\frac{5}{4})^2\}$

$=2(x+\frac{1}{4}+\frac{5}{4})(x+\frac{1}{4}-\frac{5}{4})$

$=2(x+\frac{3}{2})(x-1)$

$=(2x+3)(x-1)$.

Ex. 2. Factorize $5x^2+11x-12$.

$5x^2+11x-12=\frac{1}{5}(25x^2+11\times 5x-60)$

$=\frac{1}{5}(z^2+11z-60)$, putting z for $5x$

$=\frac{1}{5}(z+15)(z-4)$, by art. 9.

$=\frac{1}{5}(5x+15)(5x-4)$..............(1)

$=(x+3)(5x-4)$.

Obs. It is useful to observe from steps (1) of the preceding examples that we can *at once* factorize a trinomial of the form ax^2+bx+c.

For example, $8x^2+6x-5$

$$=\tfrac{1}{8}(8x+10)(8x-4)\ldots\ldots\ldots\ldots\ldots(1)$$
$$=(4x+5)(2x-1)$$

Here the numbers $10, -4$, in line (i) are such that their sum is 6 and their product is $8\times(-5)$.

EXERCISE LII.

Factorize (by the method of art. 12)

1. $15x^2-14x+3$.	**2.** $4x^2+11x-3$.	**3.** $3x^2+10x+3$.
4. $8x^2+6x-9$	**5.** $4x^2+8x-5$.	**6.** $6x^2+5x-21$.
7. $9x^2-3xy-2y^2$.	**8.** $2x^2+5ax-25a^2$	**9.** $2x^2-3x-9$.
10. $8c^2+22c+15$.	**11.** $24a^2-29a-4$.	**12.** $9x^2-31x+12$.
13. $45x^2+39x-28$.	**14.** $56x^2+26x-55$.	**15.** $108x^2-204x+91$.

Factorize immediately

16. $15x^2+11x+2$.	**17.** $14x^2-41x+15$.	**18.** $3x^2+5x+2$.
19. $2x^2+3x-2$.	**20.** $5x^2+11x+2$.	**21.** $3x^2+14x-17$.
22. $4a^2+8ab-5b^2$	**23.** $5x^2-29xy-6y^2$.	
24. $15x^2+7xy-2y^2$.	**25.** $6a^2+a-35$.	

13. Perfect cubes. From formulæ XI, XII and XIII,

(i) $\mathbf{a^3+3a^2b+3ab^2+b^3=a^3+b^3+3ab(a+b)=(a+b)^3}$.

(ii) $\mathbf{a^3-3a^2b+3ab^2-b^3=a^3-b^3-3ab(a-b)=(a-b)^3}$.

(iii) $\mathbf{a^3+b^3+c^3+3(b+c)(c+a)(a+b)=(a+b+c)^3}$.

From (i) and (iii) we get by transposition,

$$(a+b)^3-(a^3+b^3)=3ab(a+b)\ldots\ldots\ldots\ldots\ldots(1)$$
$$(a+b+c)^3-(a^3+b^3+c^3)=3(b+c)(c+a)(a+b)\ldots\ldots\ldots(2).$$

These results are important and show that we can factorize *the cube of the sum of two (or three) quantities diminished by the sum of the cubes of those two (or three) quantities.*

To prove (2) directly we proceed thus : –

$$(a+b+c)^3-(a^3+b^3+c^3)$$
$$=\{a+(b+c)\}^3-a^3-b^3-c^3$$
$$=a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3-a^3-b^3-c^3, \text{ formula XI.}$$
$$=3a^2(b+c)+3a(b+c)^2+3bc(b+c), \text{ by (1) above}$$
$$=3(b+c)\{a^2+a(b+c)+bc\}$$
$$=3(b+c)(a+b)(a+c), \text{ formula VI.}$$
$$=3(b+c)(c+a)(a+b)$$

Ex. 1. $8x^3+36x^2+54x+27.$
$$=(2x)^3+3.(2x)^2.3+3.2x.3^2+3^3=(2x+3)^3.$$

Ex. 2. $27x^3-27x^2y+9xy^2-y^3$
$$=(3x)^3-3.(3x)^2.y+3.3x.y^2-y^3=(3x-y)^3.$$

Ex. 3. If $2s=a+b+c$, prove that
$$(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3.$$

We have $(s-a)+(s-b)+(s-c)=3s-(a+b+c)=s$,
$$\because\ a+b+c=2s.$$

$\therefore$ cubing, by formula XIII,

$$(s-a)^3+(s-b)^3+(s-c)^3+3(2s-b-c)(2s-c-a)(2s-a-b)=s^3,$$
or $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3$,

for, $2s-b-c=a,\ 2s-c-a=b,\ 2s-a-b=c.$

EXERCISE LIII.

Factorize :—

1. $8x^3+12x^2+6x+1.$
2. $x^3-12x^2+48x-64.$
3. $x^3+15x^2+75x+125.$
4. $64a^3-144a^2b+108ab^2-27b^3.$
5. $(2x+3y)^3-8x^3-27y^3.$
6. $(5a-2b)^3-125a^3+8b^3.$
7. $(x^2+x-1)^3-x^6-x^3+1.$
8. $(2x-3y+z)^3-8x^3+27y^3-z^3.$
9. Prove that $8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3$
$$=3(b+c+2a)(c+a+2b)(a+b+2c). \qquad \text{(M.M. 1881)}$$
[Observe that $2(a+b+c)=(b+c)+(c+a)+(a+b)$]
10. Prove that $27(a+b+c)^3-(a+2b)^3-(b+2c)^3-(c+2a)^3$
$$=3(a+3b+2c)(b+3c+2a)(c+3a+2b).$$
[Observe that $3(a+b+c)=(a+2b)+(b+2c)+(c+2a)$]

14. Expressions of the form $a^3+b^3+c^3-3abc$.

We have from formula XV

$$\mathbf{a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab).}$$

This shows that we can factorize the *sum of the cubes of any three quantities diminished by three times their product.* We may prove it directly thus :—

$$(a+b)^3=a^3+b^3+3ab(a+b)$$
$$\therefore\ a^3+b^3=(a+b)^3-3ab(a+b)$$

Adding c^3-3abc to both sides,

$$a^3+b^3+c^3-3abc$$
$$=(a+b)^3-3ab(a+b)+c^3-3abc$$
$$=\{(a+b)^3+c^3\}-3ab(a+b+c)$$
$$=(a+b+c)\{(a+b)^2-(a+b)c+c^2\}-3ab(a+b+c),\text{ formula IX.}$$
$$=(a+b+c)\{(a+b)^2-(a+b)c+c^2-3ab\}$$
$$=(a+b+c)(a^2+b^2+c^2-bc-ca-ab),\text{ on simplification.}$$

Again, we have (see ex. 5, art. 3 chap. XII).

$$a^2+b^2+c^2-bc-ca-ab=\tfrac{1}{2}\{(b-c)^2+(c-a)^2+(a-b)^2\}$$
$$\therefore\ a^3+b^3+c^3-3abc$$
$$=\tfrac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}\ \ldots\ldots\ldots\text{(A)}$$

This form (A) of the formula is sometimes important.

Obs. If we put $a+b+c=0$ in either of the above results, we get $a^3+b^3+c^3-3abc=0$, for $0\times$any quantity$=0$; hence $a^3+b^3+c^3=3abc$, a result already obtained independently. [Ex. 3, art. 16, Chap. XII.]

Ex. 1. Factorize $x^3-y^3+3xy+1$

The expr. $=x^3+(-y)^3+(1)^3-3x(-y)\cdot 1$

$$=\{x+(-y)+1\}\{x^2+(-y)^2+1^2-(-y).1-1.x-x(-y)\}$$

[Putting x, $-y$, 1 for a, b, c respectively in the formnla]

$$=(x-y+1)(x^2+y^2+xy-x+y+1)$$

Ex. 2. Factorize $a^3-b^3-c^3-3abc$.

The expr. $=a^3+(-b)^3+(-c)^3-3a.(-b)(-c)$

$$=\{a+(-b)+(-c)\}$$
$$\times\{a^2+(-b)^2+(-c)^2-(-b)(-c)-(-c)a-a(-b)\}$$
$$=(a-b-c)(a^2+b^2+c^2-bc+ca+ab).$$

Ex. 3. Factorize $8x^3-27y^3+64z^3+72xyz$.

The expr. $=(2x)^3+(-3y)^3+(4z)^3-3.(2x)(-3y).4z,$

$$=\{2x+(-3y)+4z\}$$
$$\times\{(2x)^2+(-3y)^2+(4z)^2-(-3y)4z-4z.2x-(2x)(-3y)\}$$
$$=(2x-3y+4z)(4x^2+9y^2+16z^2+12yz-8zx+6xy).$$

Ex. 4. Prove that $(y+z)^3+(z+x)^3+(x+y)^3-3(y+z)(z+x)(x+y)$ $=2(x^3+y^3+z^3-3xyz)$.

Putting $y+z=a,\ z+x=b,\ x+y=c,$

left-side $=a^3+b^3+c^3-3abc$

$$=\tfrac{1}{2}(a+b+c)\times\{(b-c)^2+(c-a)^2+(a-b)^2\},\text{ by (A)}$$

Now $a+b+c=y+z+z+x+x+y=2(x+y+z)$,

also $(b-c)=(z+x)-(x+y)=-(y-z)$,

Similarly $(c-a)=-(z-x)$, $a-b=-(x-y)$.

$\therefore$ left side $=\frac{1}{2}\times 2(x+y+z)\{(y-z)^2+(z-x)^2+(x-y)^2\}$

$$=2(x^3+y^3+z^3-3xyz)\text{ by (A)}$$

Ex. 5. Prove that $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$

$$=3abc(b-c)(c-a)(a-b)$$

Put $a(b-c)=x$, $b(c-a)=y$, $c(a-b)=z$,

then $x+y+z=a(b-c)+b(c-a)+c(a-b)=0$ evidently.

$\therefore x^3+y^3+z^3=3xyz$, see obs. above.

Hence substituting for x, y, z, we get the required result.

Ex. 6. Find the value of $(3{\cdot}75)^3+(1{\cdot}25)^3+15\times 3{\cdot}75\times 1{\cdot}25$

We have $3{\cdot}75+1{\cdot}25=5$,

or $3{\cdot}75+1{\cdot}25+(-5)=0$

$\therefore\ (3{\cdot}75)^3+(1{\cdot}25)^3+(-5)^3=3\times 3{\cdot}75\times 1{\cdot}25\times(-5)$

$$=-15\times 3{\cdot}75\times 1{\cdot}25$$

$\therefore$ by transposition, $(3{\cdot}75)^3+(1{\cdot}25)^3+15\times 3{\cdot}75\times 1{\cdot}25$

$$=-(-5)^3=125.$$

EXERCISE LIV.

Resolve into factors

1. $x^3+y^3+3xy-1$.
2. $x^3-y^3-3xy-1$.
3. $x^3-y^3+z^3+3xyz$.
4. $x^3+y^3-z^3+3xyz$.
5. $8a^3-b^3-c^3-6abc$.
6. $27a^3-8b^3-18ab-1$.
7. $a^6+b^6+c^6-3a^2b^2c^2$.
8. $x^3y^3+y^3z^3+z^3x^3-3x^2y^2z^2$.
9. $(b-c)^3+(c-a)^3+(a-b)^3$.
10. $(x-2y)^3+(2y-z)^3+(z-x)^3$.

Prove that

11. $a^3+b^3+3abc=c^3$, if $a+b=c$

12. $8x^3-18xy=1+27y^3$, if $2x-3y=1$.

13. $(3a-2b)^3+(2b-5c)^3+(5c-3a)^3=3(3a-2b)(2b-5c)(5c-3a)$.

14. $(b+c)^3-(c-a)^3-(a+b)^3=3(b+c)(c-a)(a+b)$.

15. $(a+b-2c)^3+(b+c-2a)^3+(c+a-2b)^3$
$$=3(a+b-2c)(b+c-2a)(c+a-2b).$$

16. $(3x+5y-8z)^3+(3y+5z-8x)^3+(3z+5x-8y)^3$
$$=3(3x+5y-8z)(3y+5z-8x)(3z+5x-8y).$$

17. If $2s=a+b+c$, prove that
$$(s-a)^3+(s-b)^3+3(s-a)(s-b)c=c^3.$$

18. If $3s=a+b+c$, prove that
$$(s-a)^3+(s-b)^3+(s-c)^3=3(s-a)(s-b)(s-c).$$

19. Divide $x^3+8y^3-27z^3+18xyz$ by $x+2y-3z$.

20. Divide $a^3+8b^3+27c^3-18abc$ by
$$a^2+4b^2+9c^2-6bc-3ca-2ab.$$

21. Divide $(a+b)^3+(b+c)^3+(c+a)^3-3(b+c)(c+a)(a+b)$
by $a^2+b^2+c^2-bc-ca-ab$.

22. Find the value of
$$(4567)^3-(6759)^3+(2192)^3+3\times4567\times6759\times2192.$$

23. Find the value of $(\cdot729)^3+(\cdot271)^3+3\times\cdot729\times\cdot271$.

N.B. At the option of the teacher the learner may take up only selected portions of the remainder of this chapter on a first reading.

15. We have from formula XVII

$$\left.\begin{array}{l} \mathbf{a^2(b-c)+b^2(c-a)+c^2(a-b)} \\ \text{or } \mathbf{bc(b-c)+ca(c-a)+ab(a-b)} \\ \text{or } -\{\mathbf{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)}\} \end{array}\right\} = -\mathbf{(b-c)(c-a)(a-b)}$$

We can directly factorize any one of the above expressions. Thus taking the first we have

$$a^2(b-c)+b^2(c-a)+c^2(a-b)$$
$=a^2(b-c)+b^2c-ab^2+ac^2-bc^2$, multiplying out partly
$=a^2(b-c)-a(b^2-c^2)+bc(b-c)$, in descending powers of a
$=(b-c)\{a^2-a(b+c)+bc\}$
$=(b-c)(a-b)(a-c)$ formula VI
$=-(b-c)(c-a)(a-b)$, $\because$ $a-c=-(c-a)$.

Similarly the other two cases may be dealt with.

Ex. Prove that $(x+a)^2(b-c)+(x+b)^2(c-a)+(x+c)^2(a-b)$

$$=-(b-c)(c-a)(a-b).$$

Put $x+a=A$, $x+b=B$, $x+c=C$;

then $B-C=b-c$, $C-A=c-a$, $A-B=a-b$.

$$\begin{aligned}\therefore \text{ left side } &= A^2(B-C)+B^2(C-A)+C^2(A-B)\\ &= -(B-C)(C-A)(A-B)\\ &= -(b-c)(c-a)(a-b)\end{aligned}$$

16. We have from formula XVIII

$$\left.\begin{aligned} &\mathbf{a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc}\\ \text{or }&\mathbf{bc(b+c)+ca(c+a)+ab(a+b)+2abc}\\ \text{or }&\mathbf{a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc}\end{aligned}\right\}=\mathbf{(b+c)(c+a)(a+b)}.$$

We can directly factorize any one of the above expressions. Thus taking the first we have

$$\begin{aligned}&a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc\\ &=a^2(b+c)+b^2c+ab^2+ac^2+bc^2+2abc, \text{ multiplying out partly}\\ &=a^2(b+c)+a(b+c)^2+bc(b+c), \text{ in descending powers of } a\\ &=(b+c)\{a^2+a(b+c)+bc\}\\ &=(b+c)(a+b)(a+c), \text{ formula VI.}\\ &=(b+c)(c+a)(a+b)\end{aligned}$$

Similarly the other two cases may be dealt with.

17. The following are some miscellaneous examples on factorization.

Ex. 1. Factorize $8x^3-8x+3$.

$$\begin{aligned}8x^3-8x+3&=(8x^3-1)-(8x-4)\\ &=(2x-1)(4x^2+2x+1)-4(2x-1)\\ &=(2x-1)(4x^2+2x+1-4)\\ &=(2x-1)(4x^2+2x-3)\end{aligned}$$

Ex. 2. Factorize $a^2-5b^2-c^2-4ab+6bc$.

$$\begin{aligned}\text{The expr. }&=(a^2-4ab+4b^2)-(9b^2-6bc+c^2)\\ &=(a-2b)^2-(3b-c)^2\\ &=\{(a-2b)+(3b-c)\}\{(a-2b)-(3b-c)\}\\ &=(a+b-c)(a-5b+c)\end{aligned}$$

Ex. 3. Prove that $x^4+6x^3+x^2-24x-20$

$$=(x+1)(x+2)(x-2)(x+5).$$

We observe that $x^4+6x^3+9x^2=(x^2+3x)^2$; hence

left-side $=(x^2+3x)^2-8(x^2+3x)-20$

$$=a^2-8a-20, \text{ putting } a \text{ for } x^2+3x$$
$$=(a+2)(a-10)$$
$$=(x^2+3x+2)(x^2+3x-10), \text{ restoring } x$$
$$=(x+1)(x+2)(x-2)(x+5), \text{ formula VI.}$$

Note. It follows from the above that the expression $x^4+6x^3+x^2-24x-20$ is equal to 0 if $x+1=0$, $x+2=0$, $x-2=0$ or $x+5=0$, *i.e.* if $x=-1$, $x=-2$, $x=+2$ or $x=-5$. Hence we infer that if an expression containing x vanishes when x is put equal to, say, -1 it contains $x+1$ as a factors or if it vanishes when $x=+2$, it contains $x-2$ as a factor.

For example, $x-1$ will be a factor of $3x^3-5x+2$, for $x=1$ makes $3x^3-5x+2$ equal to 0.

$$\text{Thus } 3x^3-5x+2=3x^2(x-1)+3x(x-1)-2(x-1)$$
$$=(x-1)(3x^2+3x-2).$$

Ex. 4. Resolve into factors $(x+1)(x+2)(x+3)(x+4)-8$.

$$\text{The expr. } =\{(x+1)(x+4)\}\times\{x+2)(x+3)\}-8$$
$$=(x^2+5x+4)(x^2+5x+6)-8$$
$$=(a+4)(a+6)-8, \text{ putting } a \text{ for } x^2+5x$$
$$=a^2+10a+24-8$$
$$=a^2+10a+16$$
$$=(a+2)(a+8)$$
$$=(x^2+5x+2)(x^2+5x+8), \text{ restoring } x.$$

Note. The factors of the first term of the given expression are so grouped into pairs that the products of the two pairs may differ only in their terms independeut of x. The success of the method depends upon this fact.

Ex. 5. Divide $(2a-b)^3+8a^3-b^3$ by $4a^2-ab+b^2$.

$$\text{Dividend } =(2a-b)^3+(2a)^3-b^3$$
$$=(2a-b)^3+(2a-b)(4a^2+2ab+b^2)$$
$$=(2a-b)(4a^2-4ab+b^2+4a^2+2ab+b^2)$$
$$=(2a-b)(8a^2-2ab+2b^2)$$
$$=2(2a-b)(4a^2-ab+b^2)$$

$\therefore$ quotient $=2(2a-b)$.

Ex. 6. Prove that $(ay-bx)^2+(bz-cy)^2+(cx-az)^2+(ax+by+cz)^2$
$$=(a^2+b^2+c^2)(x^2+y^2+z^2)$$

Squaring and cancelling, left side

$$=a^2y^2+b^2x^2+b^2z^2+c^2y^2+c^2x^2+a^2z^2+a^2x^2+b^2y^2+c^2z^2$$
$$=x^2(a^2+b^2+c^2)+y^2(a^2+b^2+c^2)+z^2(a^2+b^2+c^2)$$
$$=(a+b^2+c^2)(x^2+y^2+z^2).$$

Ex 7. Resolve into factors $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$; and hence find when the expression vanishes.

The expr. $=4b^2c^2-(a^4+b^4+c^4+2b^2c^2-2c^2a^2-2a^2b^2)$

$=(2bc)^2-(a^2-b^2-c^2)^2$, formula V

$=(2bc+a^2-b^2-c^2)(2bc-a^2+b^2+c^2)$, formula V

$=\{a^2-(b-c)^2\}\{(b+c)^2-a^2\}$

$=\{a+(b-c)\}\{a-(b-c)\}\{(b+c)+a\}\{(b+c)-a\}$

$=(a+b+c)(a-b+c)(a+b-c)(-a+b+c)$.

Hence the expression vanishes when any one of its factors is zero *i.e.* when of the three quantities a, b, c the sum is zero, or, the sum of any two is equal to the third.

Putting $a+b+c=2s$, so that $a-b+c=a+b+c-2b=2(s-b)$, $a+b-c=2(s-c)$, $-a+b+c=2(s-a)$, we can write the result in the form $16s(s-a)(s-b)(s-c)$.

Ex. 8. Factorize $(a+b+c)(bc+ca+ab)-abc$.

The expr. $=\{(a+(b+c)\}\{a(b+c)+bc\}-abc$

$=a^2(b+c)+abc+a(b+c)^2+bc(b+c)-abc$

$=a^2(b+c)+a(b+c)^2+bc(b+c)$

$=(b+c)\{a^2+a(b+c)+bc\}$

$=(b+c)(a+b)(a+c)$, formula VI.

$=(b+c)(c+a)(a+b)$.

Ex. 9. Factorize $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$.

The expr. $=\{a^2(b+c)+abc\}+\{b^2(c+a)+abc\}+\{c^2(a+b)+abc\}$

$=a\{a(b+c)+bc\}+b\{b(c+a)+ac\}+c\{c(a+b)+ab\}$

$=a(bc+ca+ab)+b(bc+ca+ab)+c(bc+ca+ab)$

$=(bc+ca+ab)(a+b+c)$.

Ex. 10. Resolve into factors $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

The expr. $=a^3(b-c)+b^3c-ab^3+ac^3-bc^3$, multiplying out partly.

$=a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)$, arranging in descending powers of a

$=(b-c)\{a^3-a(b^2+bc+c^2)+bc(b+c)\}$

$=(b-c)(a^3-ab^2-abc-ac^2+b^2c+bc^2)$

$=(b-c)\{b^2(c-a)+bc(c-a)-a(c^2-a^2)\}$, arranging in descending powers of b

$=(b-c)(c-a)\{b^2+bc-a(c+a)\}$

$=(b-c)(c-a)(b^2+bc-ac-a^2)$

$=(b-c)(c-a)\{-c(a-b))-(a^2-b^2)\}$ arranging in descending powers of c

$=(b-c)(c-a)(a-b)(-c-a-b)$

$=-(b-c)(c-a)(a-b)(a+b+c)$.

Ex. 11. Resolve into factors $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$

The expr. $=a^3(b^2-c^2)+b^3c^2-a^2b^3+a^2c^3-b^2c^3$, multiplying out partly

$=a^3(b^2-c^2)-a^2(b^3-c^3)+b^2c^2(b-c)$, arranging in descending powers of a

$=(b-c)\{a^3(b+c)-a^2(b^2+bc+c^2)+b^2c^2\}$

$=(b-c)(a^3b+a^3c-a^2b^2-a^2bc-a^2c^2+b^2c^2)$

$=(b-c)\{b^2(c^2-a^2)-ba^2(c-a)-a^2c(c-a)\}$, arranging in descending powers of b

$=(b-c)(c-a)\{b^2(c+a)-ba^2-a^2c\}$

$=(b-c)(c-a)\{b^2c+ab^2-ba^2-a^2c\}$

$=(b-c)(c-a)\{-c(a^2-b^2)-ab(a-b)\}$, arranging in descending powers of c

$=(b-c)(c-a)(a-b)\{-c(a+b)-ab\}$

$=-(b-c)(c-a)(a-b)(bc+ca+ab)$.

Examples 10 and 11 worked out above, illustrate how to factorise expressions of similar nature in which the letters involved occur in cyclic order, by arranging them in powers of the different letters in succession in different steps.

In general $a^n(b^m-c^m)+b^n(c^m-a^m)+c^n(a^m-b^m)$

$=-(a-b)(b-c)(c-a)$

$\times$(a fourth homogenous factor of $m+n-3$ dimensions), where n and m are unequal positive integers.

Thus it will be found on factorisation that

$a^4(b-c)+(b^4(c-a)+c^4(a-b)$

$=-(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$.

Ex. 12. Resolve into factors $2x^2+7xy+3xz+3y^2-yz-2z^2$.

The expression arranged in descending powers of x

$=2x^2+x(7y+3z)+3y^2-yz-2z^2$

$=2x^2+x(7y+3z)+\frac{1}{3}(3y-3z)(3y+2z)$ [see obs. Ex. 2, art. 12].

$=2x^2+x(7y+3z)+(y-z)(3y+2z)$

Now we are to find two expressions whose product $=2(y-z)(3y+2z)$ and sum $=7y+3z$

They are evidently $2(3y+2z)$ and $y-z$.

Hence the expression when factorised

$=\frac{1}{2}\{2x+2(3y+2z)\}(2x+y-z)$ [See obs. Ex. 2, art. 12].

$=(x+3y+2z)(2x+y-z)$

(For a second method of factorising this, see chapter XXIII. —Factors).

EXERCISE LV.

Resolve into factors :—

1. $x^2+2xy+y^2-3x-3y-18$.
2. $9a^2-24ab+16b^2-12a+16b-60$.
3. $(a+b)^2+(a+c)^2-(b+d)^2-(c+d)^2$.
4. $8a^3+6a-4$.
5. $(a^2-b^2)(c^2-d^2)+4abcd$.
6. $(a+b-3c)^2-a-b+3c$.
7. $(a-2b)^3+a^3-8b^3$.
8. $(a^2+8)^2+36$.
9. $4a^2+9b^2-9c^2-16d^2-12(ab-2cd)$.
10. $x^4+14x^3+53x^2+28x-96$.
11. $x^4+2x^3-3x^2-4x-1$.
12. x^4-2x^3-6x-9.
13. $x^4-2x^3+3x^2-2x-8$.
14. $x^4+6x^3+12x^2+9x-4$.
15. $a^4+a^2c^2-b^2c^2-b^4$.
16. $a^2+4ab+3b^2+2bc-c^2$.
17. $9a^3+8b^3+16a^2b$.
18. $x^3-8x^2+16x-8$.
19. $x^2+7y^2-z^2-8xy-6yz$.
20. Divide.
    (i) $x^3-125y^3+30xy+8$ by $x-5y+2$.
    (ii) $(a^2-bc)^3+8b^3c^3$ by a^2+bc.
    (iii) $(x+y)^3-8z^3$ by $x+y-2z$.
    (iv) $(ax+by)^3+(bx+ay)^3$ by $(a+b)(x+y)$.
21. Divide. $(ax+by+cz)^3+(bx+cy+az)^3$
    by $(a+b)x+(b+c)y+(c+a)z$.
22. Divide the product of $x^2+(a-b)x-ab$ and $x^2-(a-b)x-ab$
    by $x^2+(a+b)x+ab$.
23. Prove that the sum of the cubes of $11x^2-9x-3$ and $4x^2-5x-5$ is divisible by the product of $5x+2$ and $3x-4$.
24. Divide $(x+a)^3+(x+b)^3+(x+c)^3-3(x+b)(x+c)(x+a)$ by
    $a^2+b^2+c^2-bc-ca-ab$.
25. Prove that $(ac-bd)^2+(ad+bc)^2=(a^2+b^2)(c^2+d^2)$
26. If $s=a+b+c$, prove that
    $(as+bc)(bs+ca)(cs+ab)=(b+c)^2(c+a)^2(a+b)^2$.
27. If $x^2=bc+ca+ab$, prove that $(x^2+a^2)(x^2+b^2)(x^2+c^2)$ is a perfect square.

Resolve into factors

28. [(i) $(x+1)(x+2)(x+3)(x+4)-3$.
    (ii) $(x+4)(x+2)(x-3)(x-5)+40$.
    (iii) $(x+1)(x+4)(x-3)(x-6)+80$.
    (iv) $(x+2)(3x+1)(x-1)(3x+2)+11$.]

Prove the following

29. $(2y+3z)^2+(3y-5z)^2+2(4y-z)(5y-9z)$
$=(4y-z)^2+(5y-9z)^2+2(2y+3z)(3y-5z).$

30. $(3a+5b)^2-(b+c)^2$
$=(5a+4b)(5a+6b+2c)-(2a+c)(8a+10b+c).$

31. $(3a+4b)^3-(a+b)^3-(2a+3b)^3=3(3a+4b)(a+b)(2a+3b).$

32. $(x-a)^3(b-c)^3+(x-b)^3(c-a)^3+(x-c)^3(a-b)^3$
$=3(b-c)(c-a)(a-b)(x-a)(x-b)(x-c).$

33. $(a-b-c)^3-a^3+b^3+c^3=3(b+c)(c-a)(a-b).$

34. $125(x+y)^3-(x+3y)^3-(2y+z)^3+(z-4x)^3$
$=3(4x+2y)(5x+3y-z)(x+5y+z).$

35. (i) $a^2(b-c)+b^2(c-a)+c^2(a-b)=-(b-c)(c-a)(a-b).$

(ii) $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=(b-c)(c-a)(a-b).$

(iii) $bc(b-c)+ca(c-a)+ab(a-b)=-(b-c)(c-a)(a-b)$

36. (i) $a^3(b-c)+b^3(c-a)+c^3(a-b)$
$=-(b-c)(c-a)(a-b)(a+b+c).$

(ii) $bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$
$=-(b-c)(c-a)(a-b)(a+b+c).$

(iii) $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)$
$=(b-c)(c-a)(a-b)(a+b+c).$

37. (i) $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$
$=-(b-c)(c-a)(a-b)(bc+ca+ab).$

(ii) $b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$
$=-(b-c)(c-a)(a-b)(bc+ca+ab).$

(iii) $a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3)$
$=(b-c)(c-a)(a-b)(bc+ca+ab).$

38. (i) $b^2c^2(b^2-c^2)+c^2a^2(c^2-a^2)+a^2b^2(a^2-b^2)$
$=-(b+c)(b-c)(c+a)(c-a)(a+b)(a-b).$

(ii) $a^2(b^4-c^4)+b^2(c^4-a^4)+c^2(a^4-b^4)$
$=(b+c)(b-c)(c+a)(c-a)(a+b)(a-b).$

(iii) $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$
$=-(b+c)(b-c)(c+a)(c-a)(a+b)(a-b).$

39. (i) $a^4(b-c)+b^4(c-a)+c^4(a-b)$
$=-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab).$

(ii) $bc(b^3-c^3)+ca(c^3-a^3)+ab(a^3-b^3)$
$=-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab).$

(iii) $a(b^4-c^4)+b(c^4-a^4)+c(a^4-b^4)$
$=(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab).$

Divide

40. $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by

$$a^2(b-c)+b^2(c-a)+c^2(a-b).$$

41. $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$ by

$$a^2(b-c)+b^2(c-a)+c^2(a-b).$$

42. $a(b-c)^3+b(c-a)^3+c(a-b)^3$ by $a+b+c$.

43. $a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3$ by $bc+ca+ab$.

Factorize

44. $(b-c)(b+c)^2+(c-a)(c+a)^2+(a-b)(a+b)^2$.

45. $(x+a)^3(b-c)+(x+b)^3(c-a)+(x+c)^3(a-b)$.

46. $a(b-c)^2+b(c+a)^2+c(a-b)^2+4abc$.

47. $a(b-c)^2+b(c-a)^2+c(a-b)^2+9abc$.

Resolve into factors

48. $x^2+6y^2+z^2-5xy-5yz+2xz$.

49. $4x^2-9y^2-16z^2-24yz+2x-3y-4z$.

50. $6a^2+9ab+9ac+3b^2+6bc+3c^2$.

The results of the examples 35 to 39 above, should be remembered as formulæ.

CHAPTER XIV.

HIGHEST COMMON FACTOR.

1. Definitions. Degree and Dimensions.

Each of the letters occurring in a term is called a dimension of the term. The total number of letters in a term is called its **degree** and gives the number of its **dimensions**. Thus, the term xyz is said to be of *three dimensions* or of *third degree*; and x^6y^4 is of 10 dimensions or of the tenth degree ($x^6=x\times x\times x\times x\times x\times x$ and $y^4=y\times y\times y\times y$). Thus the sum of the indices of the powers of the several letters in a term gives its degree The numerical co-efficient is not taken into consideration. Hence the degree of $7a^2b^3c$ or $7a^2b^3c^1$ is $2+3+1=6$.

The degree of an expression with respect to any particular letter is given by the index of the highest power of the letter in the expression. Thus the expression $ax^4+bx^3+cx^2+d$ is of 4th

degree in x. When all the terms of an expression are of the same degree, it is said to be **homogeneous.** Thus $x^5+x^3y^2+y^5$ is a *homogeneous* expression of the fifth degree.

2. If an expression divides two or more expressions without any remainder, it is called a **common factor** of those expressions. Thus $x-a$ is a common factor of x^2-a^2 and x^3-a^3, for it divides each of the expressions x^2-a^2 and x^3-a^3 without any remainder.

The expression of the highest degree in a letter which divides without remainder each of two or more expressions in that letter, is called their **Highest Common Factor (H.C.F.)**

H. C. F. of Monomials.

3. Let us first illustrate the method to be followed by an example and thence draw the general rule.

Ex. 1. Find the H. C. F. of $24x^4y^2z$, $18x^6y^2z^2$ and $12x^3y^3z^2$.

The G. C. M. of 24, 18 and 12 is 6; the H. C. F. of the literal parts is x^3y^2z, being the expression of the highest dimensions which divides the monomials without any remainder (omitting the numerical co-efficients). Hence the H. C. F. required is $6x^3y^2z$.

Here we do not take in the H. C. F. a power of x higher than 3, for a higher power will not divide $x^3y^3z^2$ and we do not take a lower power of x, for that will not give us the *Highest* Common Factor, since there is a higher power of x which will divide all the monomials.

From the above we deduce the general rule for finding the H. C. F. of monomials.

Rule : Multiply together

(1) The G. C. M. of the numerical co-efficients.

(2) The lowest power of each letter common to the monomials.

Ex. 2. Find the H. C. F. of

$$8ab^2x^3y^4z^2,\ 16a^2b^4x^2y^3z^3 \text{ and } 24\,a^4b^3y^2z^4.$$

The G. C. M. of 8, 16 and 24 is 8.

The lowest power of a is a, of b is b^2, of x is x^2, y is y^2 and of z is z^2

$\therefore$ the reqd. H. C. F. $=8\,ab^2x^2y^2z^2$.

4. The H. C. F. of compound expressions which can be easily factorised.

The rule is as follows :—

Resolve each expression into its prime factors. The product of all the common factors, literal and numerical, of the expressions is the H.C.F.

The rule is really the same as in the case of monomials, each factor being considered as one symbol.

Ex. 1. Find the H. C. F. of

$6(x-a)^3(x+a)^2$, $8(x+a)^3(x-a)^4$ and $12(x-a)^2(x+a)(x+b)$.

The H. C. F. = product of all the common factors

$$=2(x-a)^2(x+a).$$

Ex. 2. Find the H. C. F. of a^3-a^2b and $a^2-2ab+b^2$.

Since $a^3-a^2b=a^2(a-b)$, and $a^2-2ab+b^2=(a-b)^2$,

$\therefore$ the H. C. F. reqd. $=a-b$.

Ex. 3. Find the H. C. F. of

$12x^3y-24x^2y^2+6xy^3$ and $8a^2x^2+8axy+8a^3x$.

We have $12x^3y-24x^2y^2+6xy^3=6xy(2x^2-4xy+y^2)$,

$$8a^2x^2+8axy+8a^3x=8ax(ax+y+a^2)$$

$\therefore$ the required H. C. F. $=2x$.

Ex. 4. Find the H. C. F. of $6x^2-xy-y^2$ and $4x^2-8xy+3y^2$.

Here
$$\begin{aligned}6x^2-xy-y^2&=6x^2-3xy+2xy-y^2\\&=3x(2x-y)+y(2x-y)\\&=(2x-y)(3x+y).\end{aligned}$$

$$\begin{aligned}4x^2-8xy+3y^2&=4x^2-2xy-6xy+3y^2\\&=2x(2x-y)-3y(2x-y)\\&=(2x-y)(2x-3y)\end{aligned}$$

$\therefore$ the required H. C. F. $=2x-y$.

Ex. 5. Find the H.C.F. of $6x^4-6x^3a$, $18(a^2-x^2)$, $12(a^6x^2-a^2x^6)$.

Here $6x^4-6x^3a=6x^3(x-a)=-6x^3(a-x)$,

$$18(a^2-x^2)=18(a+x)(a-x)$$

$$\begin{aligned}12(a^6x^2-a^2x^6)&=12a^2x^2(a^4-x^4)\\&=12a^2x^2(a+x)(a-x)(a^2+x^2).\end{aligned}$$

$\therefore$ the required H. C. F. $=6(a-x)$.

EXERCISE LVI.

Give the dimensions of the following :—

1. (i) $4x^3y$; $9x^2y^2z^6$. (ii) $4x^3y^2z^3$. (iii) $2ab^2c^3d^4e^5$. (iv) $7x$.

Find the H. C. F. of :—

2. x^2y^2, xy^3; x^4y^3, x^2y^5.

3. $16x^4y^2c$, $-20x^3ya^2$; $9x^4y^2z^6$, $-27x^2y^3z^9$.

4. $-30ab^2c^3d^4e^5$, $12a^2b^3c^4d^5e^6$ and $-20a^3b^4c^5d^6e^7$;

$7abcx$, $21axyz$ and $42x^2byz$.

Find the H. C. F. of :—

5. $4p^3qrst^3$, $12p^2q^2r^2s^2t^2$ and $16p^4q^4r$.

6. $-16a^4x^5$, $-12a^3x^6y$ and $8a^2y^2x^3$.

7. a^4-a^3b and a^3b-b^4 ; x^3-xy^2 and x^3-y^3.

8. a^6-4a^4 and a^2-5a+6.

9. $9x^2+12xy+4y^2+3x+2y$, $9x^2-4y^2$.

10. $x^2y^2-5xy+6$ and x^3y^3-27.

11. $a^2-b^2-c^2+2bc$ and $b^2+c^2-a^2-2bc$.

12. y^6-64x^6, $6x^2-xy-y^2$ and $4x^2-y^2$.

13. $35-2x-x^2$, x^2-6x+5 and $x^2-8x+15$.

14. x^3-8x+7, $7x^3-8x^2+1$.

15. $4x^3+32$, x^3-x^2-6x and $x^3+6x^2+11x+6$.

16. x^9+x^5+x, $2x^4-2x$ and $5x^3+5x^2+5x$.

17. $4x^2+7xy+6xz+3y^2+5yz+2z^2$ and $x^2+3xy+2y^2+4xz+5yz+3z^2$.

18. $2a^2-ab-6b^2$ and $3a^2-8ab+4b^2$.

19. $16(x^2-y^2)^4$ and $8(x-y)^2(x+y)^3$.

20. y^3-x^3, x^3-y^3, and $x^2-7xy+6y^2$.

21. $1+x^2+x^4$ and $x^5+x^4+x^3-x^2-x-1$.

22. $2x^2+xy-xz-3y^2-4yz-z^2$ and $9xz+2x^2-5xy+4z^2+8yz-12y^2$.

23 $abx^2+(b^2+ac)xy+bcy^2$, $abx^2+(b^2-ac)xy-bcy^2$.

24. $72y^3x+84y^2x^2-36yx^3$ and $24y^2x+44yx^2+12x^3$.

25. $abx^2+(ac+bc)x+c^2$, $abx^2+(bc-ac)x-c^2$.

26. $3x^3-15x^2+30x-24$ and $2x^3-8x^2+14x-12$.

27. $a^3+b^3+c^3-3abc$ and $(a+b)(b+c)(c+a)+abc$.

---

5. The following Theorems will be useful for finding the H. C. F. of any two or more compound expressions.

(1) **Every factor of A is also a factor of M.A.**

Let K be a factor of A and suppose A = K.P ; then M.A = M.K.P which shows that K is also a factor of M.A.

(2) **Every common factor of A and B is also a factor M.A ± N.B.**

For suppose K is a common factor of A and B, and suppose A = K. P. and B = K. R. ; then M. A. ± N. B. = M. K. P. ± N. K. R. = K. (M.P. ± N. R.) which shows that K is a factor M. A. ± N. B.

6. H. C. F. of any two compound expressions :

Suppose A and B are two algebraic expressions of which B is not of higher degree than A. Divide A by B and let Q_1 be the quotient and R_1 the remainder. Then divide B by R_1 and let Q_2 be the quotient and R_2 the remainder. Next divide R_1 by R_2 and let Q_3 be the quotient and R_3 the remainder and so on.

$$
\begin{array}{llll}
B)\,A\,(Q_1 & & & \\
\;\;\;\underline{BQ_1} & & & \\
\;\;\;R_1)\,B\,(Q_2 & & & \\
 & \;\;\underline{R_1Q_2} & & \\
 & \;\;R_2)\,R_1\,(Q_3 & & \\
 & & \;\;\underline{R_2Q_3} & \\
 & & \;\;R_3)\,R_2\,(Q_4 & \\
 & & & \;\;\underline{R_3Q_4}
\end{array}
$$

Now the degree of each remainder is less than that of the corresponding divisor at least by unity. Thus R_1, R_2, R_3, etc. gradually diminish in degree. If we proceed in this way, (*a*) either the division will be exact at a certain stage *i.e.* terminate with remainder *zero* or (*b*) the remainder will be a constant. If the process terminates with a remainder *zero*, the last divisor is the H.C.F. : if it terminates with a constant remainder, there is no H. C. F.

Proof : Suppose the division terminates with remainder *zero*, when the remainder R_2 is divided by the remainder R_3 as shown above. We shall prove that R_3 is the H. C. F. of A and B.

From the theory of division $A - BQ_1 = R_1$; $B - R_1Q_2 = R_2$;

$R_1 - R_2Q_3 = R_3$ and $R_2 - R_3Q_4 = 0$.

It is evident from the theorems in Art. 5 that every common factor of A and B is a factor of $A - BQ_1$ *i. e.* of R_1, hence also of $B - R_1Q_2$ *i.e.* of R_2, and so, also of $R_1 - R_2Q_3$ *i.e.* of R_3.

Thus every common factor of A *and* B *is also a factor of* R_3.

Again from the results of the divisions.

$R_3Q_4 = R_2$; $R_2Q_3 + R_3 = R_1$; $R_1Q_2 + R_2 = B$; and $R_1 + BQ_1 = A$

Now R_3 is a factor of R_3Q_4, *i.e.* R_3 is a factor of R_2 ;

$\therefore$ R_3 is a factor of $R_2Q_3 + R_3$ *i.e.* R_3 is a factar of R_1 ;

$\therefore$ R_3 is a factor of $R_1Q_2 + R_2$ *i.e.* R_3 is a factor of B ;

$\therefore$ R_3 is a factor of $R_1 + BQ_1$ *i.e.* R_3 is a factor of A.

Thus R_3 is a common factor of A and B. We have proved also that every common factor of A and B is a factor of R_3 ; that is there is no common factor of A and B, which is of higher degree than R_3. Hence R_3 must be the H. C. F. of A and B.

7. From the last article we deduce the following rules to be followed in finding the H. C. F. of two expressions which cannot be easily factorized :—

1. Arrange the two expressions in ascending or descending powers of some common letter, after taking out any monomial factors which may occur in one or both of them.

2. Divide the expression of higher degree in that letter by the other ; if they be both of the same degree, then divide the one whose highest term has the greater co-efficient by the other.

3. Regard the remainder as a new divisor and the preceding divisor as a new dividend, and go on with the process until there is no remainder. The last divisor, multiplied by the H. C. F. of the monomials (if any) taken out from the expressions, is the H. C. F. required.

4. At any stage in the above process we may multiply or divide the dividend or the divisor by any quantity which is not a factor of the other.

8. The following worked out examples will illustrate the 'rules' of art 7,

Ex. 1. Find the H. C. F. of $x^5-4x^4+6x^3-3x^2-x+1$ and x^4-4x+3.

$$
\begin{array}{r|l|l}
x^4-4x+3 & x^5-4x^4+6x^3-3x^2-x+1 & x \\
 & x^5 \qquad\qquad -4x^2+3x & \\ \hline
 & -4x^4+6x^3+x^2-4x+1 & -4 \\
 & -4x^4 \qquad\qquad +16x-12 & \\ \hline
 & 6x^3+x^2-20x+13 &
\end{array}
$$

$$
\begin{array}{r|l|l}
6x^3+x^2-20x+13 & x^4 \qquad\qquad -4x+3 & \\
 & 6 & \\ \hline
 & 6x^4 \qquad\qquad -24x+18 & x \\
 & 6x^4+x^3-20x^2+13x & \\ \hline
 & -x^3+20x^2-37x+18 & \\
 & 6 & \\ \hline
 & -6x^3+120x^2-222x+108 & -1 \\
 & -6x^3- \quad x^2+\ 20x-\ 13 & \\ \hline
121 & 121x^2-242x+121 & \\ \hline
 & x^2-\quad 2x+1 &
\end{array}
$$

Here we multiply the dividend by 6 (Rule 4, Art 7) so that its first term may be divisible by $6x^3$, the 1st term of the divisor.

Here we multiply by 6 to make the 1st term $-x^3$ divisible by $6x^3$.

Here we remove the simple factor 121 common to all *the terms. (Rule 4, art 7).*

$$
\begin{array}{r|l|l}
x^2-2x+1 & 6x^3+\quad x^2-20x+13 & 6x \\
 & 6x^3-12x^2+6x & \\ \hline
 & 13x^2-26x+13 & 13 \\
 & 13x^2-26x+13 & \\ \hline
\end{array}
$$

Hence x^2-2x+1 is the H. C. F.

Ex. 2. Find the H. C. F. of

$3x^6y - 6x^5y^2 - 6x^4y^3 + 24x^3y^4 - 21x^2y^5 + 6xy^6$ and $6x^6y^2 - 24x^3y^5 + 18x^2y^6$.

We have $3x^6y - 6x^5y^2 - 6x^4y^3 + 24x^3y^4 - 21x^2y^5 + 6xy^6$

$= 3xy(x^5 - 2x^4y - 2x^3y^2 + 8x^2y^3 - 7xy^4 + 2y^5)$
and $6x^6y^2 - 24x^3y^5 + 18x^2y^6$
$= 6x^2y^2\ (x^4 - 4xy^3 + 3y^4)$.

Here we take out monomial factors (Rule 1, art 7).

The H. C. F. of the factors $3xy$ and $6x^2y^2$ is $3xy$.

In finding the H. C. F. of other two factors we proceed thus :

$$\begin{array}{l|l}
x^5 - 2x^4y - 2x^3y^2 + 8x^2y^3 - 7xy^4 + 2y^5 & x^4 - 4xy^3 + 3y^4 \\
x^5 \qquad\qquad\qquad - 4x^2y^3 + 3xy^4 & x \\
\hline
-2y)\ -2x^4y - 2x^3y^2 + 12x^2y^3 - 10xy^4 + 2y^5 & \\
\hline
x^4 + x^3y - 6x^2y^2 + 5xy^3 - y^4 & 1 \\
x^4 \qquad\qquad\qquad - 4xy^3 + 3y^4 & \\
\hline
\end{array}$$

$$\begin{array}{l}
y)\ x^3y - 6x^2y^2 + 9xy^3 - 4y^4 \\
\hline
x^3 - 6x^2y + 9xy^2 - 4y^3
\end{array}$$

$$\begin{array}{l|l}
x^4 \qquad\qquad\qquad - 4xy^3 + 3y^4 & x^3 - 6x^2y + 9xy^2 - 4y^3 \\
x^4 - 6x^3y + 9x^2y^2 - 4xy^3 & x \\
\hline
3y)\ 6x^3y - 9x^2y^2 + 3y^4 & \\
\hline
\end{array}$$

$$\begin{array}{l|l}
2x^3 - 3x^2y \qquad\qquad + y^3 & 2 \\
2x^3 - 12x^2y + 18xy^2 - 8y^3 & \\
\hline
9y)\ 9x^2y - 18xy^2 + 9y^3 & \\
\hline
x^2 - 2xy + y^2 &
\end{array}$$

$$\begin{array}{l|l}
x^3 - 6x^2y + 9xy^2 - 4y^3 & x^2 - 2xy + y^2 \\
x^3 - 2x^2y + xy^2 & x \\
\hline
-4y)\ -4x^2y + 8xy^2 - 4y^3 & \\
\hline
\end{array}$$

$$\begin{array}{l|l}
x^2 - 2xy + y^2 & 1 \\
x^2 - 2xy + y^2 & \\
\hline
\end{array}$$

Hence $x^2 - 2xy + y^2$ is the H. C. F. of the two factors other than the monomials.

$\therefore$ the required H. C. F. $= 3xy(x^2 - 2xy + y^2)$.

Ex. 3. Find the H. C. F. of $12y^3a^3 + 5y^2xa^2 + 2ayx^2 - x^3$ and $4a^3y^3 - 13a^2y^2x + 7ayx^2 - x^3$.

Reverse the order of the expressions and change the signs.

$$\begin{array}{l|l}
x^3 - 2ax^2y - 5a^2xy^2 - 12a^3y^3 & x^3 - 7ax^2y + 13a^2xy^2 - 4a^3y^3 \\
x^3 - 7ax^2y + 13a^2xy^2 - 4a^3y^3 & 1 \\
\hline
\end{array}$$

$$\begin{array}{l}
ay)\,5ax^2y - 18a^2xy^2 - 8a^3y^3 \\
\quad 5x^2 - 18axy - 8a^2y^2
\end{array}$$

$$\begin{array}{l|l}
x^3 - 7ax^2y + 13a^2xy^2 - 4a^3y^3 & 5x^2 - 18axy - 8a^2y^2 \\
5 & \\
\hline
5x^3 - 35ax^2y + 65a^2xy^2 - 20a^3y^3 & x \\
5x^3 - 18ax^2y - 8a^2xy^2 & \\
\hline
-17ax^2y + 73a^2xy^2 - 20a^3y^3 & \\
5 & \\
\hline
-85ax^2y + 365a^2xy^2 - 100a^3y^3 & -17ay \\
-85ax^2y + 306a^2xy^2 + 136a^3y^3 & \\
\hline
\end{array}$$

$$59a^2y^2 \;\Big|\; \frac{59a^2xy^2 - 236a^3y^3}{x - 4ay}$$

$$\begin{array}{l|l}
5x^2 - 18axy - 8a^2y^2 & x - 4ay \\
5x^2 - 20axy & 5x \\
\hline
2axy - 8a^2y^2 & 2ay \\
2axy - 8a^2y^2 & \\
\hline
\end{array}$$

$\therefore$ The required H. C. F. $= x - 4ay$.

The above work may be shortly put thus :—

$$\begin{array}{r|l|l|l}
 & x^3 - 7ax^2y + 13a^2xy^2 - 4a^3y^3 & x^3 - 2ax^2y - 5a^2xy^2 - 12a^3y^3 & 1 \\
 & 5 & x^3 - 7ax^2y + 13a^2xy^2 - 4a^3y^3 & \\
\hline
 & & 5ax^2y - 18a^2xy^2 - 8a^3y^3 & ay \\
 & 5x^3 - 35ax^2y + 65a^2xy^2 - 20a^3y^3 & 5x^2 - 18axy - 8a^2y^2 & 5x \\
 & 5x^3 - 18ax^2y - 8a^2xy^2 & 5x^2 - 20axy & \\
\hline
 & -17ax^2y + 73a^2xy^2 - 20a^3y^3 & 2axy - 8a^2y^2 & 2ay \\
 & 5 & 2axy - 8a^2y^2 & \\
\hline
 & -85ax^2y + 365a^2xy^2 - 100a^3y^3 & & \\
-17ay & -85ax^2y + 306a^2xy^2 + 136a^3y^3 & & \\
\hline
59a^2y^2 & 59a^2xy^2 - 236a^3y^3 & & \\
 & x - 4ay & &
\end{array}$$

$\therefore$ H. C. F. $= x - 4ay$.

The students will do well to use this short work in finding H.C.F.

9. In finding the H. C. F. of more than two expressions the rule is as follows :—

Rule. Find the H. C. F. of two of the given expressions ; then find the H. C. F. of this H. C. F. and a third expression and so on. The final H. C. F. is the H. C. F. required.

Ex. Find the H. C. F. of $x^6 + x^5 - 6x^2 - x + 5$,

$$x^5 - 2x^4 - 3x^3 + 2x + 2, \text{ and } x^5 + x^3 + x^2 - 6x + 3.$$

The H. C. F. $x^6 + x^5 - 6x^2 - x + 5$ and $x^5 - 2x^4 - 3x^2 + 2x + 2$ is, by the process of division, found to be $x^2 - 1$.

The H. C. F. of $x^2 - 1$ and $x^5 + x^3 + x^2 - 6x + 3$ is again found to be $x - 1$.

Hence $x - 1$ is the required H. C. F.

EXERCISE LVII.

Find the H. C. F. (if any) of

1. $x^3 - 6x^2 + 11x - 6$ and $x^3 + 2x^2 - x - 2$.

2. $x^4 - 6x^3 + 11x^2 - 8x + 4$ and $2x^4 - 7x^3 + 10x^2 - 10x + 4$.

3. $x^3 - 15x^2 + 71x - 105$ and $x^4 - 5x^3 + 4x^2 + 7x - 3$.

4. $x^4 - 5x^3 + 5x^2 + 5x - 6$ and $x^4 + 6x^3 - x^2 - 54x - 72$.

5. $4x^5 - 28x^4 + 22x^3 + 98x^2 + 28x - 16$ and

$$3x^5 - 14x^4 + 19x^3 - 50x^2 + 30x - 24.$$

6. $2x^4 + 11x^3 + 16x^2 + x - 6$ and $2x^4 - x^3 - 26x^2 + 37x - 12$.

7. $x^6 + 3x^4 + 3x^2 + 2$ and $x^5 + x^4 + x^3 + x^2 + x + 1$.

8. $18x^6 - 12x^5 + 18x^4 - 12x^3 + 18x - 12$ and

$$24x^5 + 32x^4 + 40x^3 - 48x^2.$$

9. $x^5 + 2x^4 + 4x^3 - 3x^2 + 6x$ and $x^4 - 6x^3 + 12x^2 - 11x + 6$.

10. $x^4 - 2x^3y - x^2y^2 - 2xy^3 + y^4$ and $x^4 + 7x^3y + 8x^2y^2 + 7xy^3 + y^4$.

11. $4x^5y - x^4y^2 + 4x^2y^4 - xy^5$ and $8x^6y^2 + 14x^5y^3 + 12x^4y^4 + 4x^3y^5 - 2x^2y^6$.

12. $6x^4 - x^3y + 4x^2y^2 + y^4$ and $6x^4 - 5x^3y + 6x^2y^2 - 2xy^3 + y^4$.

13. $7x^3 - 19x^2 + 17x - 5$ and $2x^4 - x^3 - 9x^2 + 13x - 5$.

14. $12x^4 + 14x^3 - 18x^2 + 4x$ and $32x^6 + 24x^5 - 60x^4 + 36x^3 - 8x^2$.

15. $4x^4 + 7x^3y - 14x^2y^2 - 31xy^3 - 14y^4$ and

$$16x^4 - 48x^3y + 19x^2y^2 + 34xy^3 - 21y^4.$$

16. $x^5 + 11x - 12$ and $x^5 + 11x^3 + 54$. (P. E. 1895).

17. $4x^5 - 209x^2 + 15$ and $15x^5 - 209x^3 + 4$.

18. $6x^3 + 7x^2 - 9x + 2$ and $8x^4 + 6x^3 - 15x^2 + 9x - 2$. (M.M. 1890).

Find the H. C. F. (if any) of

19. $6x^4+x^3-6x^2-5x-2$ and $2x^4+3x^3+2x^2-7x-6$.

20. $7x^4-10ax^3+3a^2x^2-4a^3x+4a^4$ and
$8x^4-13ax^3+5a^2x^2-3a^3x+3a^4$. (B. M. 1884).

21. $6x^6-8x^5+3x^4-x^3+4x^2+x+1$, and
$6x^6+x^5-3x^4+10x^2+7x+3$.

22. $a^2x^3+(ca-b^2)x+bc$ and $a^2x^3+2abx^2+(ca+b^2)x+bc$.

23. $apx^4+(aq+bp)x^2y+(ar+cp)x^2+bqy^2+(br+cq)y+cr$
and $alx^4+(bl+am)x^2y+(an+cl)x^2+bmy^2+(bn+cm)y+cn$.

24. $a^3x^3+b^3y^3+c^3z^3-3abcxyz$ and
$alx^2+(bl+am)xy+(an+lc)xz+bmy^2+(bn+cm)yz+cnz^2$.

25. $2x^3+x^2y-xy^2-2y^3$, $3x^3-2x^2y+xy^2-2y^3$ and
$x^3-6x^2y+11xy^2-6y^3$.

26. $6x^6+5x^5+5x^4+16x^3-12x^2+16x-16$ and
$9x^6+6x^5+10x^4+3x^3+32x^2-24x+24$.

27. $x^5+3x^4+46x^3+89x^2+127x+164$ and
$x^6+3x^5+46x^4+89x^3+132x^2+169x+205$. (C. E. 1895).

28. $x^5+x^4-4x^3+2x^2+6x-9$, x^4-x^2+6x-9 and
$x^4+2x^3-5x^2-6x+9$. (B. M. 1886).

29. $8x^4-6x^3+5x^2-8x+3$, $4x^4-9x^2+6x-1$, and $6x^3-7x^2+1$.

30. $x^4-8x^3+28x^2-53x+42$, $x^4+6x^3-42x^2+129x-154$ and
$3x^4+5x^3-7x^2-9x-42$.

31. Find the H.C.D. of $x^5-4x^3-x^2+2x+2$ and x^3-x^2-2x+2 and find such a value of x as will make both the expressions vanish. (A. E. 1899).

---

9. Method of alternate Destruction of Highest and Lowest terms.

If m, n, m′ and n′ be constant quantities positive or negative and if

$$P=mA+nB \ldots\ldots\ldots\ldots\ldots(\alpha),$$
$$Q=m'A+n'B \ldots\ldots\ldots\ldots\ldots(\beta),$$

then the H. C. F. of A and B and the H. C. F. of P and Q must be the same, provided $mn'-m'n$ is not zero.

Proof. From the equations (α) and (β) it follows that every common factor of A and B is also a common factor of P and Q......(1)

Multiplying (α) by m' and (β) by m and subtracting

we get $(m'P - mQ) = (m'n - mn')B$

$= -(mn' - m'n)B$........ (γ)

Again multiplying (α) by n' and (β) by n and subtracting we get $n'P - nQ = (mn' - m'n)A$.........(δ)

Now from (γ) and (δ) it follows that every common factor of P and Q is also a common factor of A and B....................(2)

From the statements (1) and (2) it follows that the H.C.F. of A and B and that of P and Q are the same

Ex. 1. Let $A = 6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$,

$B = 4x^5 + 6x^4 - 16x^3 - 15x^2 - 2x - 5$.

Then $2A - 3B = -26x^4 + 26x^3 + 39x^2 + 13 = -13(2x^4 - 2x^3 - 3x^2 - 1)$,

and $5A - B = 26x^5 - 26x^4 - 39x^3 - 13x = 13x(2x^4 - 2x^3 - 3x^2 - 1)$.

The factors 13 and x which form no part of the H.C.F. may be rejected.

Hence the required H.C.F. is the H.C.F. of

$A' = 2x^4 - 2x^3 - 3x^2 - 1$,

and $B' = 2x^4 - 2x^3 - 3x^2 - 1$.

$\therefore$ $2x^4 - 2x^3 - 3x^2 - 1$ is the H.C.F. required.

Ex. 2. Let $A = 4x^4 + 26x^3y + 41x^2y^2 - 2xy^3 - 24y^4$,

$B = 3x^4 + 20x^3y + 32x^2y^2 - 8xy^3 - 32y^4$.

Then $4B - 3A = 2x^3y + 5x^2y^2 - 26xy^3 - 56y^4$

$= y(2x^3 + 5x^2y - 26xy^2 - 56y^3)$.

Also $4A - 3B = 7x^4 + 44x^3y + 68x^2y^2 + 16xy^3$

$= x(7x^3 + 44x^2y + 68xy^2 + 16y^3)$.

Rejecting the factors y and x which do not clearly form a part of the H.C.F., we have to find the H.C.F. of

$A' = 7x^3 + 44x^2y + 68xy^2 + 16y^3$,

$B' = 2x^3 + 5x^2y - 26xy^2 - 56y^3$

Again $2A' - 7B' = 53x^2y + 318xy^2 + 424y^3$.........(1)

$7A' + 2B' = 53x^3 + 318x^2y + 424xy^2$.........(2)

Hence rejecting $53y$, the monomial factor of (1) and $53x$, the monomial factor of (2) which clearly form no part of the H.C.F., the required H.C.F. is $x^2 + 6xy + 8y^2$.

EXERCISE LVIII.

Find by the method of alternate destruction of Highest and Lowest terms the H.C.F. (if any) of :—

1. $x^4-3x^3+6x^2-5x+3$ and $x^4-3x^3+3x^2+x-6$.
2. $2x^4-5x^3+3x^2+2x-8$ and $2x^4-8x^3+15x^2-19x-10$.
3. $2x^3-5x^2-7x+12$ and $3x^3-13x^2+8x+12$.
4. $3x^6-11x^5+22x^4-25x^3+37x^2-38x+24$ and $2x^6-3x^5-6x^4+34x^3-49x^2+52x-32$.
5. x^3-5x+4, x^3-7x+6 and $2x^3-x^2+3x-4$.
6. $2x^3+7x^2+12x+9$, $6x^3+13x^2-4x-15$, $8x^3+6x^2-5x+6$.
7. $2x^4-13x^3+31x^2-38x+24$, x^3-8 and x^3-3x-2.
8. x^4-x^2-6x-9, $x^5+x^4+3x^3+2x^2+2x+6$ and $x^4+3x^3+8x^2+9x+9$.

---

10. The term "Greatest Common Measure" is used by some writers for Highest Common Factor, but the term ought to be restricted to numerical quantities only. The highest common factor of two algebraical expressions is not necessarily the same as their greatest common measure for all numerical values of the symbols involved.

Thus the H.C.F. of x^2+4x+3 and x^2+6x+5 is $x+1$; but if we put $x=1$, they become respectively 8 and 12 of which the greatest common measure is 4, whereas the numerical value of the H.C.F. $x+1$ becomes 2.

CHAPTER XV.

LOWEST COMMON MULTIPLE.

1. Common Multiple. *An algebraical expression which is divisible by each of two or more algebraical expressions without any remainder is a common multiple of those expressions.*

Least Common Multiple (L.C.M.) *of two or more algebraical expressions is the expression of the lowest degree which is divisible by each of them.*

2. L. C. M. of monomials.

In the case of monomials the L. C. M. may be easily written down by inspection. Thus, to find the L. C. M. of $4a^4b^3$, $6a^6b^2$

we observe that the L. C. M. of 4 and 6 is 12, that of a^4 and a^5 is a^5 and that of b^3 and b^2, is b^3.

Hence the L.C.M. required is $12a^5b^3$. From this we deduce the following general rule for finding the L.C.M. of monomials :—

Rule : Multiply together

(1) **The L. C. M. of the numerical co-efficients.**

(2) **The highest power of each letter in the monomials.**

Thus the L. C. M. of $3x^2yz^2$, $6x^3y^3z^4$ and $8xyzu$ is $24x^3y^3z^4u$.

3. L. C. M. of **compound expressions which can be easily factorised :—**

Resolve each expression into its prime factors. The product of all the factors, each being raised to the highest power in which it occurs in any of the expressions, is the L. C. M.

Of course the L.C.M. of the numerical co-efficients, if any, must be included.

The rule is really the same as in the case of monomials, each factor being considered as one symbol.

Ex. 1. Find the L. C. M. of

$$6(x-1)^2(x^2+3)^3,\ 8(x-2)^2(x-3),\ 12(x-1)^5(x-2)^4(x-3)^4.$$

By the rule given above the L. C. M. is

$$24\,(x-1)^5(x-2)^4(x-3)^4(x^2+3)^3.$$

Ex. 2. Find the L. C. M. of

$$x^5-xy^4,\ y^3-x^3,\ x^6+y^6+x^2y^2(x^2+y^2).$$

Here $x^5-xy^4=x(x^4-y^4)=x(x^2-y^2)(x^2+y^2)$

$$=x(x+y)(x-y)(x^2+y^2),$$

$y^3-x^3=-(x^3-y^3)=-(x-y)(x^2+xy+y^2)$,

$x^6+y^6+x^2y^2(x^2+y^2)=(x^2+y^2)(x^4-x^2y^2+y^4)+x^2y^2(x^2+y^2)$

$$=(x^2+y^2)(x^4+y^4).$$

$$\therefore \text{ L. C. M.}=x(x+y)(x-y)(x^2+y^2)(x^2+xy+y^2)(x^4+y^4).$$

EXERCISE LIX.

Find the L. C. M. of :—

1. $4ab^2$, $6a^2b$. **2.** $9x^2yz$, $24x^3y^2z^3$.

3. $4a^4b$, $8a^2b^2c^3$, $12ab^3$ and $16a^3b^2c^5$.

4. $18a^2bc$, $12b^3$, $16a^3c$, $20c^3b$ and $45b^3a$.

5. a^3+ax, b^2a-b^2x.

6. $2(x-2)^2$, $2x^2-8$, x^3+2x^2 and $2x^2-4x$.

Find the L. C. M. of :—

7. $2xy-3y^2$, $2x^2+3xy$ and $4x^3y-9xy^3$.

8. x^2-7x+6 and x^2-5x+4.

9. $3x^3-4x^2y-7xy^2$ and $3x^4-8x^3y-11x^2y^2$.

10. $x^2+7x+12$, x^4+64x. **11.** a^4-8ab^3, $a^2b-5ab^2+6b^3$.

12. $x^3+x^2y+xy^2+y^3$ and $x^3-x^2y+xy^2-y^3$.

13. x^3+a^3, x^3-a^3, $x^4+a^2x^2+a^4$ and x^2-ax+a^2 (M.M. 1896).

14. $4x^2-6yz-(9y^2+z^2)$, $9y^2+4xz-(4x^2+z^2)$,
$z^2-12xy-(4x^2+9y^2)$. (B.M. 1896).

15. $x^3-x^2-14x+24$, x^3-2x^2-5x+6, x^2-4x+3. (C. E. 1902).

16. $1-x+x^2$, $1+x+x^2$ and $1+x^2+x^4$.

17. $2x^2-6ax+9a^2$, $2x^2+6ax+9a^2$, $4x^4-81a^4$.

18. $12(x^2-1)$, $16(x^2+1)$, $20(x-1)^2$, $4(x+1)^2$, $3(x^3-1)$, x^3+1.

19. $9x^4-28x^2+3$, $27x^4-12x^2+1$, $27x^4+6x^2-1$ and x^4-6x^2+9.
(C. E. 1886).

20. x^3-3x^2+3x-1, x^3-x^2-x+1, and x^4-2x^3+2x-1.
(B. M. 1890).

---

4. L. C. M. of two compound expressions which can not be easily factorised.

Let A and B be the two expressions and H their highest common factor. Suppose a and b to be the respective quotients of A and B when divided by H; then $A=Ha$ and $B=Hb$, (a and b having no common factor). Then L. C. M. of A and $B=ab\,H$. Denote the L. C. M. by L. Then $L=ab.H=\frac{A}{H}.\ bH=\frac{A}{H}.B=\frac{AB}{H}=$ *the product of A and B divided by their* H. C. F.

5. An important relation exists between the H. C. F. and L. C. M. of two given expressions. The relation is that *The product of two expressions is equal to the product of their H. C. F. and L. C. M.*

Let H be the H. C. F. and L be the L. C. M. of the two expressions A and B.

Let $A=Ha$, $B=Hb$ as in the preceding article; then $L=Hab$.

Hence $L.H=H.Hab=Ha.\ Hb=A.B$ which proves the relation.

6. The L. C. M. of three or more expressions.

Let A, B, C be three expressions. Find L the *L. C. M.* of A and B. Next find L' the L. C. M. of L and C; then L′ is the L. C. M. of A, B and C.

Proof. Since L is the expression of lowest degree divisible by A and B and L' is the expression of lowest degree divisible by L and C, therefore L' is the expression of lowest degree divisible by A, B and C *i.e.* is their L. C. M.

The same reasoning evidently is applicable, whatever be the number of expressions.

Hence the Rule for finding the L. C. M. of any number of expressions :—*Find the L. C. M. of any two of the expressions and then the L. C. M. of this L. C. M. and the third and so on. The last result so obtained will be the L. C. M. required.*

Ex. 1. Find the L. C. M. of

$x^3-7xy^2-6y^3$ and $x^3+8x^2y+17xy^2+10y^3$.

The H. C. F. of the expressions $=x^2+3xy+2y^2$.

$$\text{The L. C. M.}=\frac{(x^3-7xy^2-6y^3)\times(x^3+8x^2y+17xy^2+10y^3)}{x^2+3xy+2y^2}$$

$$=\frac{x^3-7xy^2-6y^3}{x^2+3xy+2y^2}\times(x^3+8x^2y+17xy^2+10y^3)$$

$$=(x-3y)\times(x^3+8x^2y+17xy^2+10y^3).$$

Ex. 2. Find the L. C. M. of

$8x^3+8x^2+4x+1$, $16x^4-4x^2-4x-1$, and $1+2x-8x^3-16x^4$.

The H. C. F. of first and second expressions $=4x^2+2x+1$.

Their L. C. M. $=(8x^3+8x^2+4x+1)\div(4x^2+2x+1)$
$\times(16x^4-4x^2-4x-1)$

$$=(2x+1)(16x^4-4x^2-4x-1).$$

$$=32x^5+16x^4-8x^3-12x^2-6x-1\text{.........(L)}$$

The L. C. M. required = the L. C. M. of the 3rd expression and the expression (L), the L. C. M. of the first two expressions.

But the H. C. F. of the 3rd expression and the expression (L)

$=8x^3+8x^2+4x+1$.

$\therefore$ The L. C. M. required

$$=(1+2x-8x^3-16x^4)\div(8x^3+8x^2+4x+1)$$
$$\times(32x^5+16x^4-8x^3-12x^2-6x-1).$$
$$=-(2x-1)(32x^5+16x^4-8x^3-12x^2-6x-1).$$

EXERCISE LX.

Find the L. C. M. of :—

1. $2x^3-x-1$ and $3x^3-2x-1$.
2. $3x^3+4x^2-2x-1$ and $3x^3-8x^2-33x-10$
3. x^5-x^3-x-1 and $x^7-x^6-x^4+1$.

Find the L. C. M. of :—

4. $21x^3-23x^2y+13xy^2-3y^3$ and $6y^3+y^2x-44yx^2+21x^3$.

5. x^3+3x^2-9x+5 and $x^3-19x+30$.

6. $9x^5+11x^3y^2-2y^5$ and $4y^5+11y^4x+81x^5$.

7. x^5+x^4+1 and $x^6-2x^4+x^2-1$.

8. $x^3-x^2-14x+24$, and x^3-2x^2-5x+6.

9. $6x^2-13x+6$, $6x^2+5x-6$, and $9x^2-4$.

10. $7x^4-2x^2-9x-2$, and $5x^3-6x^2-6x-11$.

11. $9x^4-6x^3+x^2-1$, $9x^4-7x^2+1$ and $9x^4-6x^3-5x^2+2x+1$.

12. $x^4-12x^3+42x^2-52x+21$, $x^4-10x^3+33x^2-44x+20$ and x^4-7x+6.

13. $4x^3-20x^2+17x-4$, $2x^3-15x^2+31x-12$ and $4x^3-16x^2+13x-3$ (B. M. 1902).

14. $x^3-12xy^2+16y^3$, $x^4-4x^3y-x^2y^2+20xy^3-20y^4$ and $x^4+3x^3y-11x^2y^2-3xy^3+10y^4$.

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7. We shall conclude this Chapter with a few illustrative examples depending on the theory of H.C.F. and L.C.M.

Ex. 1. If H.C.E. of two expressions is $x-2$, and L.C.M. is $x^3-2x^2-9x+18$, and one of the expressions is x^2-5x+6, find the other.

Since the product of the two expressions = their H.C.F. × their L.C.M. and one expression is x^2-5x+6,

$$\therefore \text{ the other expression} = \frac{(x^3-2x^2-9x+18)(x-2)}{x^2-5x+6}$$
$$= \frac{(x-2)(x^2-9)(x-2)}{(x-3)(x-2)}$$
$$=(x+3)(x-2)=x^2+x-6.$$

Ex. 2. Find the H.C.F. of $A=x^3+9x^2+23x+15$ and $B=x^3+7x^2+14x+8$.

The H.C.F. of A and B is a factor of $A-B$ *i.e.* $2x^2+9x+7$ *i.e.* of $(x+1)(2x+7)$. Now evidently $2x+7$ cannot be a factor of the expressions, while $x+1$ is found to be one. Hence $x+1$ = reqd. H.C.F.

Ex. 3. If $x-a$ be the H.C.F. of x^3-px^2+qx-r and x^3-mx^2+nx-r, then $a=\dfrac{n-q}{m-p}$.

Since $x-a$ is the H.C.F. of x^3-px^2+qx-r and x^3-mx^2+nx-r, it divides each of them exactly *i. e.* without any remainder.

Hence $x-a$ divides $(x^3-mx^2+nx-r)-(x^3-px^2+qx-r)$

$$i.e.\ x^2(p-m)+x(n-q);$$

$$i.e.\ x(p-m)\left(x+\frac{n-q}{p-m}\right) \text{ without any remainder.}$$

$$\therefore\ x+\frac{n-q}{p-m} \text{ must be identical with } x-a$$

$$i.e.\ -a=\frac{n-q}{p-m},\ \text{or, } a=\frac{n-q}{m-p}.$$

Ex. 4. If $x+p$ be the H. C. F. of x^2+bx+c and $x^2+b'x+c'$, then their L.C.M. $=x^3+(b+b'-p)x^2+(bb'-p^2)x+(b-p)(b'-p)p$.

Dividing x^2+bx+c by $x+p$ the quotient is $x+b-p$ and the remainder is p^2-bp+c.

Now since the expression is exactly divisible by $x+p$,

$$\therefore\ p^2-bp+c=0.$$

Similarly $p^2-b'p+c'=0$.

$$\text{Now L.C.M.}=\frac{(x^2+bx+c)(x^2+b'x+c')}{x+p}$$

$$=(x+b-p)(x^2+b'x+c')$$

$$=x^3+(b+b'-p)x^2+(bb'-b'p+c')x+c'(b-p)$$

Now $\because p^2-b'p+c'=0\ \therefore p^2=b'p-c'$

and $c'=-p^2+b'p=p(b'-p)$.

$\therefore$ substituting, the L.C.M. $=x^3+(b+b'-p)x^2+(bb'-p^2)x+(b-p)(b'-p)p$.

EXERCISE LXI.

1. If $x+a$ be the H.C.F. of px^2+qx+r and rx^2+qx+p.

show that $a=\dfrac{p+r}{r}$.

2. Find the values of a and b in order that $x^3+(a+1)x^2+(b-3)x-2$ may be divisible by the square of $x+2$.

3. If the H.C.F. of x^2+bx+c and x^2+cx+b be of the form $x+a$, then either $a+1=0$ or $b=c$.

4. Determine m and n so that x^4-3x^3+mx+n may be divisible by x^2-2x+4,

5. If $x+p$ be the H.C.F. of x^2+qx+c and x^2+ax+c, show that their L. C. M. $=x^3+(2q-p)x^2+\dfrac{c(q+p)}{p}x+\dfrac{c^2}{p}$.

6. There are two quantities A, B of which the L. C. M. $=$ L, the H. C. F. $=H$; if $L+H=m\,A+B/m$, prove that

$$L^3+H^3=m^3A^3+B^3/m^3.$$

CHAPTER XVI.

FRACTIONS.

1. If, when one quantity is divided by another, the operation of division is indicated by placing the dividend over the divisor with a line between them, the quotient is called a **Fraction**.

Thus $\dfrac{a}{b}$ is a fraction and means the same as $a\div b$.

In the fraction $\dfrac{a}{b}$, a is called the *numerator* and b the *denominator*; while a and b together are called *terms* of the fraction.

Since by definition the fraction $\dfrac{a}{b}$ is equivalent to $a\div b$, it follows that $\dfrac{a}{b}\times b=a$, whether a and b are positive, negative, integral or fractional, and we may regard this property as defining the fraction for all value of a and b.

Note. In Arithmtic a fraction like $\frac{2}{3}$ (of the unit) is defined to be a quantity formed by dividing unity into 3 equal parts and taking 2 of these parts. It is evident that with this definition we shall have $\frac{2}{3}\times 3=2$.

For, $\frac{2}{3}\times 3$ or $\frac{2}{3}$ taken 3 times means that unity is divided into 3 parts, and two of these parts are taken 3 times, *i.e.* 6 of these parts are taken; and it is clear that 6 such parts make up 2 units.

The definition of an algebraical fraction as given above is thus not inconsistent with the Arithmetical definition and is in fact more general, as it covers cases (not defined in Arithmetic) in which fractions have their numerators or denominators or both negative or fractional.

Obs. A quantity which is integral in form may be regarded as a fraction of which the denominator is 1. Thus we may regard a as $\dfrac{a}{1}$.

2. A fraction is said to be **proper** or **improper** according as the numerator is or is not of lower dimensions than the denominator.

Thus $\frac{x+1}{3x^2-x+2}$ is a proper fraction ; while $\frac{3x+y}{4x-1}$, $\frac{5x^2-2x+3}{3x+7}$ are improper fractions.

An expression is said to be **integral** in a letter if the letter does not occur in the denominator in any term.

Thus $x^3-\frac{1}{2}x^2+3x-\frac{5}{4}$ is an integral expression in x.

3. Theorem I. *If the signs of both the numerator and the denominator of a fraction are changed, the value of the fraction remains unchanged.*

To prove that $\frac{-a}{-b}=\frac{a}{b}$.

We have $\frac{-a}{-b}=(-a)\div(-b)$ by definition.

$=a\div b$ by rule of signs.

$=\frac{a}{b}$ by definition.

Similarly, $\frac{a}{-b}=\frac{-a}{b}$.

4. Theorem II. *If the sign of either the numerator or the denominator of a fraction is changed, the sign of the whole fraction is changed.*

To prove that $\frac{-a}{b}=-\frac{a}{b}$, $\frac{a}{-b}=-\frac{a}{b}$.

We have $\frac{-a}{b}=(-a)\div b$ by definition.

$=-(a\div b)$ by rule of signs.

$=-\frac{a}{b}$ by definition.

Also $\frac{a}{-b}=a\div(-b)$ by definition.

$=-(a\div b)$ by rule of signs.

$=-\frac{a}{b}$ by definition.

Ex. $\frac{a-b}{c-d}=-\frac{b-a}{c-d}$, changing the sign of numerator,

$=-\frac{a-b}{d-c}$, changing the sign of denominator.

$=\frac{b-a}{d-c}$, changing the sign of both numerator and denominator.

5. Theorem III. *The value of a fraction is not altered by multiplying its numerator and denominator by the same quantity.*

To prove that $\frac{a}{b}=\frac{am}{bm}$, for all values of m.

We have $\frac{a}{b}=a\div b$ by definition.

$=(a\div b)\times m\div m.$

$=a\times m\div b\div m$, commutative law.

$=(a\times m)\div(b\times m)$, associative law.

$=\frac{a\times m}{b\times m}$, by definition.

$=\frac{am}{bm}.$

6. Theorem IV. *The value of a fraction is not altered by dividing both its numerator and denominator by the same quantity.*

To prove that $\frac{a}{b}=\frac{a\div m}{b\div m}$.

We have $\frac{a}{b}=a\div b$ by definition.

$=(a\div b)\times m\div m.$

$=a\div m\div b\times m.$ commutative law.

$=(a\div m)\div(b\div m)$, associative law.

$=\frac{a\div m}{b\div m}$ by definition.

7. Reduction to lowest terms. A fraction is said to be *reduced to its lowest terms* when there is no factor common to the numerator and the denominator.

It is evident that to reduce a fraction to its lowest terms we are to *divide the numerator and the dinominator by their H.C.F.*

(See Theor. IV.)

Ex. 1. Reduce $\frac{8x^6y^4z^3}{12x^2y^7z^5}$ to lowest terms.

Here the H. C. F. of $8x^6y^4z^3$ and $12x^2y^7z^5$ is $4x^2y^4z^3$.

$$\therefore \frac{8x^6y^4z^3}{12x^2y^7z^5}=\frac{8x^6y^4z^3\div 4x^2y^4z^3}{12x^2y^7z^5\div 4x^2y^4z^3}=\frac{2x^4}{3y^3z^2}.$$

Ex. 2. Reduce $\frac{x^2+7xy+12y^2}{x^2+2xy-3y^2}$ to lowest terms.

The fraction $=\dfrac{(x+3y)(x+4y)}{(x+3y)(x-y)}$.

$=\dfrac{x+4y}{x-y}$, cancelling the common factor from the numerator and the denominator.

Ex. 3. Reduce $\dfrac{6x^3-5x^2+10x+7}{4x^3+16x^2-15x-11}$ to lowest terms.

Here the H. C. F. of the numerator and the denominator is not evident on inspection, and proceeding according to the usual method we find it to be $2x+1$.

$\therefore$ the fraction $=\dfrac{(6x^3-5x^2+10x+7)\div(2x+1)}{(4x^3+16x^2-15x-11)\div(2x+1)}$

$=\dfrac{3x^2-4x+7}{2x^2+7x-11}$.

Obs. Knowing that $2x+1$ divides the numerator and the denominator we may proceed factorizing them thus :—

$$6x^3-5x^2+10x+7=3x^2(2x+1)-4x(2x+1)+7(2x+1)$$
$$=(2x+1)(3x^2-4x+7).$$
$$4x^3+16x^2-15x-11=2x^2(2x+1)+7x(2x+1)-11(2x+1)$$
$$=(2x+1)(2x^2+7x-11).$$

Ex. 4. Reduce $=\dfrac{6a^5-7a^4b+6a^3b^2-6a^2b^3+b^5}{4a^5-15a^3b^2+13a^2b^3-ab^4-b^5}$ to lowest terms

The H. C. F. of the numerator and the denominator is found to be $2a^2-3ab+b^2$.

$\therefore$ fraction $=\dfrac{(6a^5-7a^4b+6a^3b^2-6a^2b^3+b^5)\div(2a^2-3ab+b^2)}{(4a^5-15a^3b^2+13a^2b^3-ab^4-b^5)\div(2a^2-3ab+b^2)}$

$=\dfrac{3a^3+a^2b+3ab^2+b^3}{2a^3+3a^2b-4ab^2-b^3}$

EXERCISE LXII.

Reduce to lowest terms

1. $\dfrac{20a^2bc^3}{15a^4b^2c}$. **2.** $\dfrac{33x^4y^3z^2}{55x^2y^5z^4}$. **3.** $\dfrac{12a^4b^3x^2y^4}{-16ab^2x^5y}$.

4. $\dfrac{-45p^4q^3r^2s^5}{27p^2qr^5s^9}$. **5.** $\dfrac{x^3-y^3}{x^3-xy^2}$. **6.** $\dfrac{x^2(ab+b^2)}{b^2(ax+bx)}$.

7. $\dfrac{a^4+ab^3}{a^2b-ab^2+b^3}$. **8.** $\dfrac{x^2-y^2z^2}{xyz+y^2z^2}$. **9.** $\dfrac{9a^2-16x^2}{9a^2+12ax}$.

10. $\dfrac{a^4b^2-a^2b^4}{a^4b^3+a^3b^4}$. **11.** $\dfrac{(a^2-b^2)^3}{(a^3-b^3)^2}$.

Reduce to lowest terms :—

12. $\dfrac{16x^3-54y^3}{16x^3+24x^2y+36xy^2}$.

13. $\dfrac{(3a^2b-6ab^2)^2}{3a^3b-12ab^3}$.

14. $\dfrac{a^2-b^2+c^2+2ac}{a^2+b^2-c^2+2ab}$.

15. $\dfrac{ax+ay-bx-by}{ax-ay-bx+by}$.

16. $\dfrac{6x^2-19x+15}{12x^2-29x+15}$.

17. $\dfrac{2x^2+11x+14}{6x^2+19x-7}$.

18. $\dfrac{12a^2+35ab-33b^2}{3a^2+17ab+22b^2}$.

19. $\dfrac{6x^2-7xy-3y^2}{4x^2-12xy+9y^2}$.

20. $\dfrac{x^2-(a-b)x-ab}{x^2-(a+c)x+ac}$.

21. $\dfrac{x^2+(a+b)xy+aby^2}{x^2+(b-c)xy-bcy^2}$.

22. $\dfrac{(x^2+7x+13)^2-(2x+7)^2}{(x^2+9x+21)^2-(2x+9)^2}$.

23. $\dfrac{a^4+a^2b^2+b^4}{a^4+a^2b^2+a^3b}$.

24. $\dfrac{(26x^2+14x+1)^2-(14x^2-8x-5)^2}{(18x^2+19x-1)^2-(2x^2-x+5)^2}$.

25. $\dfrac{(b+c-2a)^3-(c+a-2b)^3}{(c+a-2b)^3-(a+b-2c)^3}$.

26. $\dfrac{7x^3-4x^2-21x+12}{5x^3+2x^2-15x-6}$.

27. $\dfrac{a^3-4a^2b+9ab^2-10b^3}{a^3+2a^2b-3ab^2+20b^3}$.

28. $\dfrac{2x^3+5x^2y+5xy^2-4y^3}{4x^3+13x^2y+19xy^2+4y^3}$.

29. $\dfrac{a^5-a^3+8}{a^5-a^2+4}$.

30. $\dfrac{3a^4-13a^3b-40a^2b^2-9ab^3+3b^4}{3a^4-7a^3b-27a^2b^2-6ab^3+2b^4}$.

31. $\dfrac{3x^3-27ax^2+78a^2x-72a^3}{2x^3+10ax^2-4a^2x-48a^3}$. (C. E. 1889.)

32. $\dfrac{2x^4-x^3-9x^2+13x-5}{7x^3-19x^2+17x-5}$. (C. E. 1870.)

33. $\dfrac{6x^5+35x^4+59x^3+19x^2-17x-6}{6x^5-5x^4-41x^3+71x^2-37x+6}$.

34. $\dfrac{6x^5-4x^4y-11x^3y^2-3x^2y^3-3xy^4-y^5}{4x^4+2x^3y-18x^2y^2+3xy^3-5y^4}$.

35. $\dfrac{3a^5+a^4b-63a^2b^3-75ab^4-18b^5}{2a^5-3a^3b^2-39a^2b^3-5ab^4-39b^5}$.

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8. Reduction to a common denominator. Any number of fractions can be reduced to equivalent fractions having a common denominator, which is the L. C. M. of the denominators of the given fractions.

Thus, let $\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}$......be any number of fractions; and let x be the L. C. M. of $b, d, f,$...

Then $\dfrac{a}{b}=\dfrac{a\times(x\div b)}{b\times(x\div b)}$ (Theor IV) $=\dfrac{a\times(x\div b)}{x}$;

$\dfrac{c}{d}=\dfrac{c\times(x\div d)}{d\times(x\div d)}$ (Theor IV) $=\dfrac{c\times(x\div d)}{x}$;

$\dfrac{e}{f}=\dfrac{e\times(x\div f)}{f\times(x\div f)}=\dfrac{e\times(x\div f)}{x}$; and so on.

Hence we have the rule : *Find the L. C. M. of the denominators of the fractions, and multiply the numerator and denominator of each fraction by the quotient of this L. C. M. by the denominator of that fraction.*

Ex. 1. Reduce $\dfrac{2x}{ab}$, $\dfrac{3y}{a^2b}$, $\dfrac{5z}{ab^3}$ to a common denominator.

Here L. C. M. of the denominators $=a^2b^3$.

Thus $\dfrac{2x}{ab}=\dfrac{2x\times(a^2b^3\div ab)}{ab\times(a^2b^3\div ab)}=\dfrac{2xab^2}{a^2b^3}$;

$\dfrac{3y}{a^2b}=\dfrac{3y\times(a^2b^3\div a^2b)}{a^2b\times(a^2b^3\div a^2b)}=\dfrac{3yb^2}{a^2b^3}$;

$\dfrac{5z}{ab^3}=\dfrac{5z\times(a^2b^3\div ab^3)}{ab^3\times(a^2b^3\div ab^3)}=\dfrac{5za}{a^2b^3}$.

Hence the fractions are reduced to a common denominator.

Ex. 2. Reduce $\dfrac{a}{x-y}$, $\dfrac{ab}{y(x+y)}$, $\dfrac{a^2b}{x^2-y^2}$ to a common denominator.

Here L. C. M. of the denominators $=y(x^2-y^2)$.

Thus $\dfrac{a}{x-y}=\dfrac{a\times y(x+y)}{y(x^2-y^2)}=\dfrac{ay(x+y)}{y(x^2-y^2)}$,

$\dfrac{ab}{y(x+y)}=\dfrac{ab\times(x-y)}{y(x^2-y^2)}=\dfrac{ab(x-y)}{y(x^2-y^2)}$,

$\dfrac{a^2b}{x^2-y^2}=\dfrac{a^2b\times y}{y(x^2-y^2)}=\dfrac{a^2by}{y(x^2-y^2)}$.

Ex. 3. Reduce $\dfrac{a}{(a-c)(a-b)}$, $\dfrac{b}{(b-a)(b-c)}$, $\dfrac{c}{(c-a)(c-b)}$ to a common denominator.

Preserving cyclic order in the factors in the denominators, we can write the fractions as

$$-\frac{a}{(c-a)(a-b)},\quad -\frac{b}{(a-b)(b-c)},\quad -\frac{c}{(c-a)(b-c)},$$

This shows that L.C.M. of the denominators $=(b-c)(c-a)(c-b)$.

Hence first fraction $= -\dfrac{a(b-c)}{(b-c)(c-a)(a-b)}$,

second fraction $= -\dfrac{b(c-a)}{(b-c)(c-a)(a-b)}$,

third fraction $= -\dfrac{c(a-b)}{(b-c)(c-a)(a-b)}$.

EXERCISE LXIII.

Reduce to a common denominator

1. $\dfrac{x}{a}, \dfrac{y}{b}, \dfrac{z}{c}$.

2. $\dfrac{a}{bc}, \dfrac{b}{ca}, \dfrac{c}{ab}$.

3. $a^2, \dfrac{b^3}{a}, \dfrac{c^3}{b}$.

4. $\dfrac{a+b}{a-b}, \dfrac{a-b}{a+b}$.

5. $\dfrac{x-y}{a}, \dfrac{y-z}{b}, \dfrac{z-x}{c}$.

6. $\dfrac{a}{x}, \dfrac{x}{a}, \dfrac{x-a}{x+a}$.

7. $\dfrac{a}{x+1}, \dfrac{3a}{2x+2}, \dfrac{5a}{4x+4}$.

8. $\dfrac{x^2}{x-2y}, \dfrac{2xy^2}{(x-2y)^2}$.

9. $\dfrac{3x+4}{x(x+2)}, \dfrac{3}{x}, \dfrac{5}{x+2}$.

10. $\dfrac{1}{a+b}, \dfrac{3a}{ab-b^2}, \dfrac{2ab}{a^2-b^2}$.

11. $\dfrac{x+1}{x^2-x}, \dfrac{x+2}{x^2-1}, \dfrac{3}{x^3+1}$.

12. $\dfrac{3x-1}{2x^2-7x+3}, \dfrac{x-3}{6x^2-5x+1}$

13. $\dfrac{x+y}{a^2(a+b)}, \dfrac{x-y}{b^2(b-a)}, \dfrac{x^2-y^2}{ab(a^2-b^2)}$.

14. $\dfrac{x+1}{x^2+3x-10}, \dfrac{x+2}{x^2-3x+2}, \dfrac{x+3}{x^2+4x-5}$.

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9. From art. 6 Chap. X we have
$\dfrac{a+b+c+\dots}{m} = \dfrac{a}{m} + \dfrac{b}{m} + \dfrac{c}{m} + \dots\dots$, where $m, a, b, c\dots\dots$ may be positive, negative, integral or fractional.

Hence a fraction may be decomposed into as many partial fractions as there are terms in the numerator and conversely.

Ex. 1. $\dfrac{a+b+c}{abc} = \dfrac{a}{abc} + \dfrac{b}{abc} + \dfrac{c}{abc} = \dfrac{1}{bc} + \dfrac{1}{ca} + \dfrac{1}{ab}$.

Ex. 2. $\dfrac{3xy^3z^4-5x^3yz^3+4x^3y^2}{2x^2y^5z}$

$$=\frac{3xy^3z^4}{2x^2y^5z}-\frac{5x^3yz^3}{2x^2y^5z}+\frac{4x^3y^2}{2x^2y^5z}.$$

$$=\frac{3z^3}{2xy^2}-\frac{5xz^2}{2y^4}+\frac{2x}{y^3z}.$$

10. Addition and subtraction of fractions. Since we have $\dfrac{a}{m}+\dfrac{b}{m}+\dfrac{c}{m}+\dots\dots=\dfrac{a+b+c+\dots\dots}{m}$, we find that *the sum of a number of fractions having a common denominator is a fraction of which the denominator is the same but the numerator is the sum of the numerators of the given fractions.*

If the fractions have not the same denominator, they must be first reduced as in art. 8 to equivalent fractions having the same denominator, and then the preceding rule is to be applied.

Ex. 1. Add together $\dfrac{a+b}{a-b}$, $\dfrac{a-b}{a+b}$.

We are to reduce the fractions to a common denominator which in this case being the L. C. M. of $a+b$ and $a-b$ is a^2-b^2.

Thus $\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}=\dfrac{(a+b)^2}{(a-b)(a+b)}+\dfrac{(a-b)^2}{(a-b)(a+b)}\dots\dots(1).$

$$=\frac{(a+b))^2+(a-b)^2}{a^2-b^2}=\frac{2(a^2+b^2)}{a^2-b^2}.$$

In a practice step (1) is omitted and the next step is immediately written down.

Ex. 2. Subtract $\dfrac{x-14}{6x^2+19x+10}$ from $\dfrac{x+10}{6x^2+x-2}$.

Reqd. result $=\dfrac{x+10}{6x^2+x-2}-\dfrac{x-14}{6x^2+19x+10}$

$$=\frac{x+10}{(2x-1)(3x+2)}-\frac{x-14}{(2x+5)(3x+2)}$$

$$=\frac{(x+10)(2x+5)-(x-14)(2x-1)}{(2x-1)(3x+2)(2x+5)}$$

$$=\frac{(2x^2+25x+50)-(2x^2-29x+14)}{(2x-1)(3x+2)(2x+5)}$$

$$=\frac{18(3x+2)}{(2x-1)(3x+2)(2x+5)}$$

$$=\frac{18}{(2x-1)(2x+5)}$$

Ex. 3. Simplify $\dfrac{3x-4}{x^2-3x+2}+\dfrac{2(x-1)}{x^2-5x+6}+\dfrac{x-11}{x^2-4x+3}$.

The expr. $=\dfrac{3x-4}{(x-1)(x-2)}+\dfrac{2(x-1)}{(x-2)(x-3)}+\dfrac{x-11}{(x-3)(x-1)}$

$=\dfrac{(3x-4)(x-3)+2(x-1)^2+(x-11)(x-2)}{(x-1)(x-2)(x-3)}$

$=\dfrac{(3x^2-13x+12)+(2x^2-4x+2)+(x^2-13x+22)}{(x-1)(x-2)(x-3)}$

$=\dfrac{6x^2-30x+36}{(x-1)(x-2)(x-3)}=\dfrac{6(x-2)(x-3)}{(x-1)(x-2)(x-3)}$

$=\dfrac{6}{x-1}$.

Ex. 4. Simplify $\dfrac{a}{x+a}+\dfrac{b}{x-a}+\dfrac{ab-a^2}{a^2-x^2}$.

The expr. $=\dfrac{a}{x+a}+\dfrac{b}{x-a}-\dfrac{ab-a^2}{x^2-a^2}$

$=\dfrac{a(x-a)+b(x+a)-(ab-a^2)}{x^2-a^2}$

$=\dfrac{x(a+b)}{x^2-a^2}$, simplifying the numerator.

In some cases it is useful not to reduce the fractions to a common-denominator all at once but to proceed by *adding in groups*. This is illustrated in the next three examples.

Ex. 5. Simplify $\dfrac{2}{x-5}-\dfrac{5}{x-2}-\dfrac{2}{x+5}+\dfrac{5}{x+2}$

The expr. $=\left(\dfrac{2}{x-5}-\dfrac{2}{x+5}\right)-\left(\dfrac{5}{x-2}-\dfrac{5}{x+2}\right)$

$=\dfrac{2(x+5)-2(x-5)}{x^2-25}-\dfrac{5(x+2)-5(x-2)}{x^2-4}$

$=\dfrac{20}{x^2-25}-\dfrac{20}{x^2-4}=\dfrac{20(x^2-4)-20(x^2-25)}{x^4-29x^2+100}$

$=\dfrac{420}{x^4-29x^2+100}$.

Ex. 6. Simplify $\dfrac{1}{a-b}+\dfrac{1}{a+b}+\dfrac{2a}{a^2+b^2}+\dfrac{4a^3}{a^4+b^4}$

$$\text{The expr.} = \left(\frac{1}{a-b}+\frac{1}{a+b}\right)+\frac{2a}{a^2+b^2}+\frac{4a^3}{a^4+b^4}$$

$$=\frac{2a}{a^2-b^2}+\frac{2a}{a^2+b^2}+\frac{4a^3}{a^4+b^4}$$

$$=2a\left(\frac{1}{a^2-b^2}+\frac{1}{a^2+b^2}\right)+\frac{4a^3}{a^4+b^4}$$

$$=2a\frac{2a^2}{a^4-b^4}+\frac{4a^3}{a^4+b^4}$$

$$=4a^3\left(\frac{1}{a^4-b^4}+\frac{1}{a^4+b^4}\right)$$

$$=4a^3\cdot\frac{2a^4}{a^8-b^8}=\frac{8a^7}{a^8-b^8}.$$

Ex. 7. Simplify

$$\frac{1}{a+b}+\frac{2b}{a^2+b^2}+\frac{4b^3}{a^4+b^4}+\frac{8b^7}{a^8-b^8}.$$

The expression

$$=\frac{1}{a+b}+\frac{2b}{a^2+b^2}+4b^3\left(\frac{1}{a^4+b^4}+\frac{2b^4}{a^8-b^8}\right)$$

$$=\frac{1}{a+b}+\frac{2b}{a^2+b^2}+4b^3.\ \frac{a^4+b^4}{a^8-b^8}$$

$$=\frac{1}{a+b}+\frac{2b}{a^2+b^2}+\frac{4b^3}{a^4-b^4}$$

$$=\frac{1}{a+b}+2b\left(\frac{1}{a^2+b^2}+\frac{2b^2}{a^4-b^4}\right)$$

$$=\frac{1}{a+b}+2b.\ \frac{a^2+b^2}{a^4-b^4}$$

$$=\frac{1}{a+b}+\frac{2b}{a^2-b^2}$$

$$=\frac{a+b}{a^2-b^2}=\frac{1}{a-b}.$$

When fractions are not in their lowest terms they must be first reduced to their lowest terms before proceeding to addition and subtraction. This is shown in the next three examples.

*Ex. 8. Add together $\dfrac{a^2bz^2x}{ax^2yz^2}$, $-\dfrac{2bc^2xy}{cy^2zx}$, $\dfrac{ca^2y^2z}{ay^2z^2x}$.

Simplifying the fractions, the reqd. result.

$$=\frac{ab}{xy}-\frac{2bc}{yz}+\frac{ca}{zx}$$

$$=\frac{abz-2bcx+cay}{xyz}$$

Ex. 9. Simplify

$$\frac{ax-bx-ay+by}{x^2-y^2}+\frac{bx-cx}{x^2+xy}+\frac{cx+cy-ax-ay}{(x+y)^2}.$$

We have $ax-bx-ay+by$

$$=x(a-b)-y(a-b)=(a-b)(x-y).$$

Again, $cx+cy-ax-ay=c(x+y)-a(x+y)=(x+y)\,(c-a)$.

$$\therefore \text{ the expr. } =\frac{(a-b)\,(x-y)}{x^2-y^2}+\frac{x(b-c)}{x(x+y)}+\frac{(x+y)\,(c-a)}{(x+y)^2}$$

$$=\frac{a-b}{x+y}+\frac{b-c}{x+y}+\frac{c-a}{x+y}$$

$$=\frac{a-b+b-c+c-a}{x+y}=\frac{0}{x+y}=0.$$

Ex. 10. Simplify

$$\frac{(2x-3y)^2-x^2}{4x^2-(3y+x)^2}+\frac{4x^2-(3y-x)^2}{9(x^2-y^2)}+\frac{9y^2-x^2}{(2x+3y)^2-x^2}.$$

$$\text{First term}=\frac{(2x-3y+x)\,(2x-3y-x)}{(2x+3y+x)\,(2x-3y-x)}$$

$$=\frac{3x-3y}{3x+3y}=\frac{x-y}{x+y}.$$

$$\text{Second term}=\frac{(2x+3y-x)\,(2x-3y+x)}{9(x+y)\,(x-y)}$$

$$=\frac{(x+3y)\,(3x-3y)}{9(x+y)\,(x-y)}=\frac{x+3y}{3(x+y)}.$$

$$\text{Third term}=\frac{(3y+x)\,(3y-x)}{(2x+3y+x)\,(2x+3y-x)}$$

$$=\frac{(3y+x)\,(3y-x)}{(3x+3y)\,(3y+x)}=\frac{3y-x}{3(x+y)}.$$

$\therefore$ the given expression

$$=\frac{x-y}{x+y}+\frac{x+3y}{3(x+y)}+\frac{3y-x}{3(x+y)}.$$

$$=\frac{3x-3y+x+3y+3y-x}{3(x+y)}$$

$$=\frac{3(x+y)}{3(x+y)}=1.$$

EXERCISE XLIV.

Add together

1. $\dfrac{2}{3a}$, $\dfrac{5}{9a}$.

2. $\dfrac{3x-1}{10x}$, $-\dfrac{7x-4}{15x}$.

Add together

3. $\frac{a}{a+b}$, $\frac{b}{a-b}$.

4. $\frac{3a-5b}{3a+5b}$, $\frac{3a+5b}{3a-5b}$.

5. $\frac{a+b}{a-b}$, $-\frac{a-b}{a+b}$.

6. $\frac{1}{a(a-b)}$, $\frac{1}{b(a+b)}$.

7. $\frac{2}{x+4}$, $x-1-\frac{3}{x-2}$.

8. $\frac{x+6}{x+2}$, $\frac{x+4}{x+5}$.

9. $\frac{b-c}{bc}$, $\frac{c-a}{ca}$, $\frac{a-b}{ab}$.

10. $\frac{a^2+ab+b^2}{a+b}$, $\frac{a^2-ab+b^2}{a-b}$.

Simplify the following

11. $\frac{1}{(x-a)(x-b)}-\frac{a-c}{(x-a)(x-b)(x-c)}$.

12. $\frac{1}{x+3}+\frac{x+1}{x^2-3x+9}-\frac{2x^2+x+12}{x^3+27}$. (C. E. 1860.)

13. $\frac{a^2+ac}{a^2c-c^3}-\frac{a-c}{(a+c)c}-\frac{2c}{a^2-c^2}$. (C. E. 1869.)

14. $\frac{x+y}{y}-\frac{x}{x+y}-\frac{x^3-x^2y}{x^2y-y^3}$. (C. E. 1863.)

15. $\frac{3}{a-3x}+\frac{3}{a+3x}-\frac{6a}{a^2-9x^2}$.

16. $\frac{3+2x}{4-6x}-\frac{4-2x}{6+9x}+\frac{x-2}{9x^2-4}$.

17. $\frac{x-a}{x-b}+\frac{x-b}{x-a}-\frac{(a-b)^2}{(x-a)(x-b)}$. (C. E. 1879.)

18. $\frac{3x-1}{(4x-5)(3x+1)}-\frac{2x+1}{(3x+1)(4x-2)}$.

19. $\frac{4}{8x^2-10x-3}-\frac{2}{6x^2-11x+3}$.

20. $\frac{x+1}{x(x-3)}+\frac{x+2}{x(x-4)}+\frac{2x-3}{x^2-7x+12}$.

21. $\frac{x-b}{x(x-a)}+\frac{x+a}{x(x+b)}-\frac{a^2+b^2}{ab+(a-b)x-x^2}$.

22. $\frac{x-y+z}{x+y-z}+\frac{x+y-z}{x-y+z}-\frac{4yz}{(y-z)^2-x^2}$.

23. $\frac{1}{a^2+2ab-3b^2}+\frac{1}{b^2+2ab-3a^2}-\frac{2}{3a^2+3b^2+10ab}$.

24. $\frac{2}{(3x+1)(5x+1)}-\frac{2}{(4x+1)(2x+1)}+\frac{1}{(2x+1)(3x+1)}$.

Simplify the following

25. $\dfrac{x+9}{x^2+3x-4}-\dfrac{x+10}{x^2+2x-8}+\dfrac{x+12}{x^2+3x-10}$.

26. $\dfrac{3x-8y}{x^2-7xy+12y^2}-\dfrac{7x-20y}{x^2-6xy+8y^2}+\dfrac{5x-13y}{x^2-5xy+6y^2}$.

27. $\dfrac{1}{10(x-3)}+\dfrac{9}{10(x+2)}-\dfrac{1}{6(x+4)}-\dfrac{5}{6(x+1)}$.

28. $\dfrac{b-c}{a^2-(b-c)^2}+\dfrac{c-a}{b^2-(c-a)^2}+\dfrac{a-b}{c^2-(a-b)^2}$.

29. $\dfrac{1}{a-2b}-\dfrac{2}{a-b}+\dfrac{2}{a+b}-\dfrac{1}{a+2b}$.

30. $\dfrac{1}{x+y}-\dfrac{3}{x+3y}-\dfrac{1}{x+5y}+\dfrac{3}{x+4y}$.

31. $\dfrac{a^2+3ax+2x^2}{a+2x}+\dfrac{6a^2-7ax-3x^2}{2a-3x}+\dfrac{x^2+ax-2a^2}{x+2a}$.

32. $\dfrac{a-b}{a^3-b^3}-\dfrac{a+b}{a^3+b^3}+\dfrac{2(a^2+b^2)}{a^4+a^2b^2+b^4}$.

33. $\dfrac{x^2-(y-z)^2}{(z+x)^2-y^2}+\dfrac{y^2-(z-x)^2}{(x+y)^2-z^2}+\dfrac{z^2-(x-y)^2}{(y+z)^2-x^2}$. (C. E. 1866.)

34. $\dfrac{x^4-(x-1)^2}{(x^2+1)^2-x^2}+\dfrac{x^2-(x^2-1)^2}{x^2(x+1)^2-1}+\dfrac{x^2(x-1)^2-1}{x^4-(x+1)^2}$. (M. M. 1871.)

35. $\dfrac{(a-2b)^2-9c^2}{4b^2-(3c+a)^2}+\dfrac{(2b-3c)^2-a^2}{9c^2-(a+2b)^2}+\dfrac{(3c-a)^2-4b^2}{a^2-(2b+3c)^2}$.

36. $\dfrac{a-2b}{a+2b}-\dfrac{a+2b}{a-2b}+\dfrac{8ab}{a^2+4b^2}$.

37. $\dfrac{1}{4a^3(a+x)}+\dfrac{1}{4a^3(a-x)}+\dfrac{1}{2a^2(a^2+x^2)}$ (C. E. 1861.)

38. $\dfrac{a}{a-x}+\dfrac{a}{a+x}+\dfrac{2a^2}{a^2+x^2}+\dfrac{4a^4}{a^4+x^4}$.

39. $\dfrac{1}{a+b}+\dfrac{2a}{a^2+b^2}+\dfrac{4a^3}{a^4+b^4}+\dfrac{8a^7}{b^8-a^8}$.

40. $\dfrac{1+x}{1-x}+\dfrac{x-1}{x+1}+\dfrac{4x}{1+x^2}+\dfrac{8x}{1-x^4}$

11. Multiplication of fractions. To find the product of any number of fractions we proceed thus:

$$\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}$$

$=(a\div b)\times(c\div d)\times(e\div f)$, by definition

$=a\div b\times c\div d\times e\div f$, by associative law

$=a\times c\times e\div b\div d\div f$, by commutative law

$=(a\times c\times e)\div(b\times d\times f)$, by associative law

$=ace\div bdf$

$=\dfrac{ace}{bdf}$, by definition.

Thus we have the following rule: *The product of two or more fractions is equal to a fraction of which the numerator is the product of their numerators and the denominator is the product of their denominators.*

Note. Since we can write c as $\dfrac{c}{1}$, we have $\dfrac{a}{b}\times c=\dfrac{a}{b}\times\dfrac{c}{1}=\dfrac{ac}{b}$.

Also $a\times\dfrac{1}{b}=\dfrac{a}{b}$.

Ex. 1. Multiply together $\dfrac{a^2b}{c^2}$, $\dfrac{b^2c}{a^2}$, $\dfrac{c^2a}{b^2}$.

$$\text{Product}=\frac{a^2b\times b^2c\times c^2a}{c^2\times a^2\times b^2}=\frac{a^3b^3c^3}{a^2b^2c^2}=abc.$$

Ex. 2. Multiply together $\dfrac{x^2+xy}{x^2-xy}$, $\dfrac{xy-y^2}{xy+y^2}$, $\dfrac{xy-yz}{xy+yz}$.

$$\text{Product}=\frac{(x^2+xy)(xy-y^2)(xy-yz)}{(x^2-xy)(xy+y^2)(xy+yz)}$$

$$=\frac{x(x+y)y(x-y)y(x-z)}{x(x-y)y(x+y)y(x+z)}=\frac{x-z}{x+z}.$$

Ex. 3. Find the value of $\dfrac{a^2+2a-3}{a^2-6a+9}\times\dfrac{a^2-5a+6}{a^2+5a-6}$.

$$\text{Product}=\frac{(a^2+2a-3)(a^2-5a+6)}{(a^2-6a+9)(a^2+5a-6)}$$

$$=\frac{(a+3)(a-1)(a-2)(a-3)}{(a-3)(a-3)(a+6)(a-1)}$$

$$=\frac{(a+3)(a-2)}{(a-3)(a+6)}=\frac{a^2+a-6}{a^2+3a-18}.$$

Ex. 4 Find the value of $\left(\frac{a}{b}\right)^n$, n being a positive integer.

$$\left(\frac{a}{b}\right)^n=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}\times\ldots\ldots\text{to } n \text{ factors.}$$

$$=\frac{a\times a\times\ldots\text{to } n \text{ factors}}{b\times b\times\ldots\text{to } n \text{ factors}}=\frac{a^{1+1+1+\ldots\text{to } n \text{ terms}}}{b^{1+1+1+\ldots\text{to } n \text{ terms}}}=\frac{a^n}{b^n}.$$

Ex. 5. Simplify $\left(\frac{a^2b}{c}\right)^2\times\left(\frac{b}{ac}\right)^3\times\left(\frac{bc}{a}\right)^2$.

$$\text{Product}=\frac{(a^2b)^2}{c^2}\times\frac{b^3}{(ac)^3}\times\frac{(bc)^2}{a^2}$$

$$=\frac{a^4b^2\times b^3\times b^2c^2}{c^2\times a^3c^3\times a^2}=\frac{b^7}{ac^3}.$$

Ex. 6. Find $\left(\frac{a}{b}+\frac{b}{a}\right)^3$.

$$\text{Reqd. value}=\left(\frac{a}{b}\right)^3+\left(\frac{b}{a}\right)^3+3\times\frac{a}{b}\times\frac{b}{a}\left(\frac{a}{b}+\frac{b}{a}\right)$$

$$=\frac{a^3}{b^3}+\frac{b^3}{a^3}+3\left(\frac{a}{b}+\frac{b}{a}\right)$$

Obs. If $\frac{a}{b}+\frac{b}{a}=6$, then from the above we have

$$6^3=\left(\frac{a^3}{b^3}+\frac{b^3}{a^3}\right)+3\times 6,$$

whence $\frac{a^3}{b^3}+\frac{b^3}{a^3}=6^3-3\times 6=198.$

EXERCISE LXV.

Multiply together

1. $\frac{ax}{by}$, $\frac{ay}{bx}$.
2. $\frac{a^2}{bc}$, $\frac{b^2}{ca}$, $\frac{c^2}{ab}$.
3. $\frac{3x^3}{y^3z}$, $\frac{yz^2}{9xy^2}$.
4. $\frac{2a^4b^3c^2}{5x^2y^3z^3}$, $\frac{25xy^2z^4}{4ab^2c^4}$.
5. $\frac{15x^2yz}{8xy^3}$, $\frac{24xy^2}{5xyz^2}$.
6. $\frac{8ab^2}{5x^2y}$, $\frac{x^2y^3}{a^4b^3}$, $\frac{2x^2y^4}{5a^2b^2}$.
7. $\frac{3z^2x}{4xy^2}$, $\frac{16x^2y^3z^2}{9xy^2z}$, $\frac{3x^2y}{2z^2x}$.
8. $\frac{a^2-b^2}{c^2-d^2}$, $\frac{c-d}{a-b}$.
9. $\frac{x^2-7x+12}{x^2-5x+4}$, $\frac{x^2-3x+2}{x^2-8x+15}$

Find the value of

10. $\dfrac{6x^2+7x-3}{6x^2-5x+1}\times\dfrac{2x^2+3x-2}{2x^2+9x+9}$.

11. $\dfrac{(x+y+z)^2}{x^2-(y+z)^2}\times\dfrac{(x-y)^2-z^2}{(z+x)^2-y^2}$.

12. $\left(\dfrac{a+b}{a-b}\right)^3\times\dfrac{a^3-b^3}{a^3+b^3}\times\dfrac{(a+b)^2-3ab}{(a+b)^2-ab}$.

13. $\dfrac{a^4-b^4}{a^2-2ab+b^2}\times\dfrac{a-b}{a^2+ab}\times\dfrac{a+b}{a^2+b^2}$.

14. $\dfrac{2x^2+x-1}{x^2-3x-10}\times\dfrac{x^2-x-6}{2x^2-7x+3}\times\dfrac{x^2-7x+10}{x^2+3x+2}$.

15. $\dfrac{x^2-x(a+b)+ab}{x^2-x(a-c)-ac}\times\dfrac{x^2+x(a-c)-ac}{x^2+x(a-b)-ab}\div\dfrac{x^2-x(a+d)+ad}{x^2-x(a+c)+ac}$.

16. $\left(1+\dfrac{8x-4}{x^2-5x+6}\right)\times\left(1-\dfrac{9x+3}{x^2+5x+6}\right)$.

17. $\left\{\dfrac{x}{a}+\dfrac{2x^2}{a(b-x)}\right\}\left\{\dfrac{a}{x}-\dfrac{2ax}{x(b+x)}\right\}$ C. E. 1880.

18. $\left(\dfrac{x-y}{x+y}-\dfrac{x^3-y^3}{x^3+y^3}\right)\left(\dfrac{x+y}{x-y}+\dfrac{x^3+y^3}{x^3-y^3}\right)$ A. E. 1893.

19. Multiply $\dfrac{a^2}{b^2}+1+\dfrac{b^2}{a^2}$ by $\dfrac{a}{b}-\dfrac{b}{a}$.

20. Find (i) $\left(x+\dfrac{1}{x}\right)^2$ (ii) $\left(x-\dfrac{1}{x}\right)^3$.

21. Evaluate (1) $\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}$ when $\dfrac{a}{b}-\dfrac{b}{a}=5$.

(2) $x^2+\dfrac{1}{x^2}$ when $x-\dfrac{1}{x}=9$.

(3) $x^3+\dfrac{1}{x^3}$ when $x+\dfrac{1}{x}=12$.

(4) $x^3-\dfrac{1}{x^3}$ when $x-\dfrac{1}{x}=10$.

22. Simplify (1) $\left(\dfrac{a}{b}\right)^5\times\left(\dfrac{b}{c}\right)^6\times\left(\dfrac{c}{a}\right)^7$.

(2) $\left(\dfrac{x^2y}{cz}\right)\times\left(\dfrac{y^2z}{bx}\right)^2\times\left(\dfrac{acy}{zx}\right)^3$.

12. Division of fractions. To divide $\frac{a}{b}$ by $\frac{c}{d}$, we proceed thus :

$$\frac{a}{b} \div \frac{c}{d} = (a \div b) \div (c \div d) \text{ by definition.}$$
$$= a \div b \div c \times d \text{ by associative law}$$
$$= a \div b \times d \div c \text{ by commutative law}$$
$$= (a \div b) \times (d \div c) \text{ by associative law}$$
$$= \frac{a}{b} \times \frac{d}{c} \text{ by definition.}$$

Thus to divide $\frac{a}{b}$ by $\frac{c}{d}$ is to multiply $\frac{a}{b}$ by $\frac{d}{c}$. Two fractions, of which the numerator and the denominator of the one are respectively equal to the denominator and the numerator of the other, are called *reciprocal* ; thus, $\frac{c}{d}$ and $\frac{d}{c}$ are reciprocal. Hence we have :

The quotient of any fraction by another is obtained by multiplying the first by the reciprocal of the second.

Note. $\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \times \frac{1}{c} = \frac{a}{bc}$; also $a \div \frac{1}{b} = ab$.

Ex. 1. Divide $\frac{a^3b^4}{c^3d^2}$ by $\frac{a^2b}{cd}$.

$$\text{Quotient} = \frac{a^3b^4}{c^3d^2} \div \frac{a^2b}{cd} = \frac{a^3b^4}{c^3d^2} \times \frac{cd}{a^2b} = \frac{ab^3}{c^2d}.$$

Ex. 2. Find the value of

$$\frac{2x^2+x-1}{x^2-4x+3} \div \frac{6x^2+x-2}{2x^2-5x-3} \times \frac{3x^2-7x-6}{2x^2+3x+1}.$$

$$\text{Reqd. value} = \frac{2x^2+x-1}{x^2-4x+3} \times \frac{2x^2-5x-3}{6x^2+x-2} \times \frac{3x^2-7x-6}{2x^2+3x+1}.$$
$$= \frac{(x+1)(2x-1)}{(x-3)(x-1)} \times \frac{(x-3)(2x+1)}{(2x-1)(3x+2)} \times \frac{(x-3)(3x+2)}{(2x+1)(x+1)}$$
$$= \frac{(x+1)(2x-1)(x-3)(2x+1)(x-3)(3x+2)}{(x-3)(x-1)(2x-1)(3x+2)(2x+1)(x+1)} = \frac{x-3}{x-1}.$$

Ex. 3. Divide $\frac{a^3}{b^3}+\frac{b^3}{a^3}$ by $\frac{a}{b}+\frac{b}{a}$.

$$\text{We have } \frac{a^3}{b^3}+\frac{b^3}{a^3} = \left(\frac{a}{b}\right)^3 + \left(\frac{b}{a}\right)^3$$
$$= \left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{a^2}{b^2}-1+\frac{b^2}{a^2}\right)$$

$$\therefore \text{ reqd. quotient} = \frac{a^2}{b^2} - 1 + \frac{b^2}{a^2}.$$

EXERCISE LXVI.

Divide

1. $\dfrac{ab}{cd}$ by $\dfrac{ac}{bd}$. 2. $\dfrac{x^2y}{a^2b}$ by $\dfrac{ab^2}{xy^2}$ 3. $\dfrac{x^4y^2z^2}{z^4xy}$ by $\dfrac{y^2x^2}{z^2}$.

4. $\dfrac{10x^4y^4z^4}{3a^2bc^3}$ by $\dfrac{20x^3y^4z^2}{9ab^2c}$. 5. $\dfrac{21a^4bx^3}{4c^3d^2y^2}$ by $\dfrac{7a^3b^2x}{2cd^3y^3}$.

6. $\dfrac{7ab^2}{3c^2d}\times\dfrac{9cd^3}{2a^2b}$ by $\dfrac{14b^2c}{3a^2d^4}$. 7. $\dfrac{x^2+6x+5}{x^2-5x+6}$ by $\dfrac{x^2+7x+10}{x^2-4x+3}$.

8. $\dfrac{6x^2+7x-3}{6x^2+x-2}$ by $\dfrac{6x^2+x-1}{6x^2-5x-6}$.

Simplify

9. $\left(x+\dfrac{16x-27}{x^2-16}\right)\div\left(x-1+\dfrac{13}{x+4}\right)$ (B. M. 1885.)

10. $\left(\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}\right)\div\left(\dfrac{a-b}{a+b}-\dfrac{a^3-b^3}{a^3+b^3}\right)$ (C. E. 1868.)

11. $\left(\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}\right)\div\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)$ (C. E. 1867.)

12. $\dfrac{(x+y)^2+(x-y)^2}{(x+y)^2-(x-y)^2}\div\dfrac{x^4-y^4}{2xy(x-y)}$. (M. M. 1887.)

13. $\dfrac{x^2+3x+2}{x^2+8x+15}+\dfrac{x^2+8x+12}{x^2+7x+12}\times\dfrac{x^2+11x+30}{x^2+5x+4}$.

14. $\dfrac{6x^2-x-12}{15x^2+14x-8}\div\dfrac{8x^2-6x-9}{20x^2-13x+2}\div\dfrac{4x^2-9x+2}{4x^2-x-3}$.

15. Factorize (1) $\dfrac{a^3}{b^3}-\dfrac{c^3}{d^3}$. (2) $\dfrac{x^3}{y^3}+1$.

(3) $\dfrac{a^4}{b^4}+\dfrac{b^4}{a^4}+1$. (4) $\dfrac{a^3}{b^3}+\dfrac{b^3}{a^3}+2$.

16. Divide (1) $\dfrac{a^4}{b^4}-\dfrac{c^4}{d^4}$ by $\dfrac{a}{b}-\dfrac{c}{d}$.

(2) $\dfrac{a^3}{b^3}+\dfrac{b^3}{c^3}+\dfrac{c^3}{a^3}-3$ by $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}$.

13. Complex Fractions. A *complex fraction* is one which has a fraction in the numerator or in the denominator or in both.

Thus $\dfrac{a}{\frac{b}{c}}$, $\dfrac{\frac{a}{b}}{c}$, $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ are complex fractions. For compactness of printing, the fractions are written as

$$a\bigg/\frac{b}{c},\ \frac{a}{b}\bigg/c,\ \frac{a}{b}\bigg/\frac{c}{d},$$

or more shortly, $a/(b/c)$, $(a/b)/c$, $(a/b)/(c/d)$.

14. Since $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ means $\dfrac{a}{b}\div\dfrac{c}{d}$ by definition, it follows that $\dfrac{\frac{a}{b}}{\frac{c}{d}}=\dfrac{ad}{bc}$ *i.e. to simplify a complex fraction multiply the numerator by the reciprocal of the denominator.*

Thus, $\dfrac{a}{\frac{b}{c}}=a\times\dfrac{c}{b}=\dfrac{ac}{b}$; $\dfrac{\frac{a}{b}}{c}=\dfrac{a}{b}\times\dfrac{1}{c}=\dfrac{a}{bc}$;

$\dfrac{1}{\frac{a}{b}}=1\times\dfrac{b}{a}=\dfrac{b}{a}$; $\dfrac{a}{\frac{1}{b}}=a\times b=ab$;

$\dfrac{\frac{1}{a}}{\frac{1}{b}}=\dfrac{1}{a}\times\dfrac{b}{1}=\dfrac{b}{a}$.

15. Simplification of complex fractions.

Ex. 1. Simplify $\dfrac{\frac{a^3-b^3}{a^2b^2}}{\frac{a^2+ab+b^2}{ab^2}}$

The fraction $=\dfrac{a^3-b^3}{a^2b^2}\times\dfrac{ab^2}{a^2+ab+b^2}=\dfrac{a-b}{a}$.

When there are compound fractional expressions in the numerator and the denominator of a complex fraction, we may either make

the numerator and the denominator integral by multiplying them by the same quantity or proceed by simplifying the numerator and denominator separately.

Ex. 2. Simplify $\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}}$.

Multiplying the numerator and the denominator by x^2y^2, we have

the fraction $=\dfrac{xy^2-x^2y}{y^2-x^2}=\dfrac{xy\,(y-x)}{(y-x)(y+x)}=\dfrac{xy}{x+y}$.

Ex. 3. Simplify $\dfrac{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}{\dfrac{a^2+x^2}{a^2-x^2}-\dfrac{a^2-x^2}{a^2+x^2}}$.

$$\text{Numerator}=\frac{(a+x)^2-(a-x)^2}{a^2-x^2}=\frac{4ax}{a^2-x^2}.$$

$$\text{Denominator}=\frac{(a^2+x^2)^2-(a^2-x^2)^2}{a^4-x^4}=\frac{4a^2x^2}{a^4-x^4}.$$

$$\therefore \text{ the fraction}=\frac{\dfrac{4ax}{a^2-x^2}}{\dfrac{4a^2x^2}{a^4-x^4}}=\frac{4ax}{a^2-x^2}\times\frac{a^4-x^4}{4a^2x^2}$$

$$=\frac{a^2+x^2}{ax}.$$

16. Continued fractions. A fraction of the kind considered in the next example is called a *continued fraction.* It is simplified by beginning from the lowest complex denominator.

Ex. 4. Simplify $1-\dfrac{x}{x-\dfrac{x+1}{x-\dfrac{2}{x-1}}}$.

We have $x-\dfrac{2}{x-1}=\dfrac{x^2-x-2}{x-1}=\dfrac{(x+1)(x-2)}{x-1}$;

$$\therefore \frac{x+1}{x-\dfrac{2}{x-1}}=\frac{x+1}{\dfrac{(x+1)(x-2)}{x-1}}=\frac{(x+1)(x-1)}{(x+1)(x-2)}=\frac{x-1}{x-2}.$$

$$\therefore \text{ the given fraction} = 1 - \cfrac{x}{x - \cfrac{x-1}{x-2}}$$

$$= 1 - \frac{x}{\frac{x(x-2) - (x-1)}{x-2}}$$

$$= 1 - \frac{x(x-2)}{x^2 - 3x + 1}$$

$$= \frac{(x^2-3x+1) - x(x-2)}{x^2-3x+1} = \frac{1-x}{x^2-3x+1}.$$

After practice the above steps may be considerably shortened.

17. Division-transformation. In Chap. VIII. we mostly considered cases of *exact* divisions, *viz.*, when the quotient of one integral expression by another integral expression (of lower dimensions) is an integral expression and touched upon the subject of inexact division. But divisions are more often inexact than not; and in such cases we can determine as in art. 8, Chap. VIII. as many terms as we like in the *partial quotient* and the *corresponding remainder*.

Thus if N is divided by D and at any stage in the process of division Q is the partial quotient and C is the remainder then we have the identity $N = QD + C$ or $\frac{N}{D} = Q + \frac{C}{D}$.

The *complete quotient* of $\frac{N}{D}$ is therefore $Q + \frac{C}{D}$.

Ex. 1. Prove that $\frac{10x^2+7x+6}{x+2} = 10x - 13 + \frac{32}{x+2}$

$$= 3 + 2x + 4x^2 - \frac{4x^3}{2+x}.$$

Here we proceed, dividing $10x^2+7x+6$ by $x+2$, by arranging the expression (i) in descending powers of x (ii) in ascending powers of x.

$$\text{(i)}\quad x+2)\ 10x^2+7x+6\ (\ 10x-13$$
$$\underline{10x^2+20x}$$
$$-13x+6$$
$$\underline{-13x-26}$$
$$32$$

$$\therefore \frac{10x^2+7x+6}{x+2} = 10x - 13 + \frac{32}{x+2}$$

(ii)
$$
\begin{array}{r|l}
2+x & 6+7x+10x^2 \;\big(\,3+2x+4x^2 \\
 & \underline{6+3x} \\
 & 4x+10x^2 \\
 & \underline{4x+2x^2} \\
 & 8x^2 \\
 & \underline{8x^2+4x^3} \\
 & -4x^3
\end{array}
$$

$$\therefore \quad \frac{6+7x+10x^2}{2+x} = 3+2x+4x^2 - \frac{4x^3}{2+x}.$$

Ex. 2. Prove that $\dfrac{1}{1+x} = 1-x+x^2-x^3+\dfrac{x^4}{1+x}$

$$= \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^3(x+1)}.$$

$$
\begin{array}{r|l}
1+x & 1 \;\big(\,1-x+x^2-x^3 \\
 & \underline{1+x} \\
 & -x \\
 & \underline{-x-x^2} \\
 & +x^2 \\
 & \underline{+x^2+x^3} \\
 & -x^3 \\
 & \underline{-x^3-x^4} \\
 & +x^4
\end{array}
$$

$$\therefore \quad \frac{1}{1+x} = 1-x+x^2-x^3+\frac{x^4}{1+x}.$$

Again,
$$
\begin{array}{r|l}
x+1 & 1 \;\Big(\,\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3} \\
 & \underline{1+\frac{1}{x}} \\
 & -\frac{1}{x} \\
 & \underline{-\frac{1}{x}-\frac{1}{x^2}} \\
 & \phantom{1-\frac{1}{x}}+\frac{1}{x^2} \\
 & \phantom{1-\frac{1}{x}}\underline{+\frac{1}{x^2}+\frac{1}{x^3}} \\
 & \phantom{1-\frac{1}{x}+\frac{1}{x^2}}-\frac{1}{x^3}
\end{array}
$$

$$\therefore \quad \frac{1}{x+1} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^3(x+1)}.$$

EXERCISE LXVII.

Simplify

1. $\dfrac{\dfrac{a^2-x^2}{a}}{\dfrac{a+x}{b}}$.

2. $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{1-\dfrac{a}{b}}$.

3. $\dfrac{3+\dfrac{2x}{5y}}{x+\dfrac{3y}{2}}$.

4. $\dfrac{\dfrac{7x}{9y}-5a}{2x-\dfrac{3a}{2y}}$.

5. $\dfrac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{a}{b}-\dfrac{c}{d}}$.

6. $\dfrac{\dfrac{a}{x}+\dfrac{b}{y}}{\dfrac{a}{y}+\dfrac{b}{x}}$.

7. $\dfrac{a}{1+\dfrac{a}{b}}+\dfrac{b}{1+\dfrac{b}{a}}$.

8. $\dfrac{\dfrac{x}{6}+\dfrac{2}{x}+\dfrac{4}{3}}{\dfrac{x}{3}+2+\dfrac{8}{3x}}$.

9. $\dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}\times\dfrac{\dfrac{1}{a}-\dfrac{1}{b}}{\dfrac{1}{a}+\dfrac{1}{b}}$.

10. $\dfrac{\dfrac{x+a}{x+b}-\dfrac{1}{x+c}}{\dfrac{1}{x+b}+\dfrac{x+a}{x+c}}$.

11. $\dfrac{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}}$.

12. $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}$.

13. $\dfrac{\dfrac{a^2-b^2}{a^2+b^2}-\dfrac{a^2+b^2}{a^2-b^2}}{\dfrac{a-b}{a+b}-\dfrac{a+b}{a-b}}$.

14. $\dfrac{2-\dfrac{3(x-1)}{x+1}}{4+\dfrac{5(x-1)}{x+1}}$.

15. $\dfrac{1+\dfrac{a-b}{a+b}}{1-\dfrac{a-b}{a+b}}\div\dfrac{1+\dfrac{a^2-b^2}{a^2+b^2}}{1-\dfrac{a^2-b^2}{a^2+b^2}}$. C. E. 1859.

16. $\dfrac{\dfrac{a+b}{1-ab}+\dfrac{a-b}{1+ab}}{1-\dfrac{a+b}{1-ab}\cdot\dfrac{a-b}{1+ab}}$.

17. $\dfrac{\dfrac{a+b}{1-ab}-\dfrac{a+c}{1-ac}}{1+\dfrac{a+b}{1-ab}\cdot\dfrac{a+c}{1-ac}}$.

18. $\dfrac{\dfrac{x^2+y^2}{y}-x}{\dfrac{1}{y}-\dfrac{1}{x}}\times\dfrac{x^2-y^2}{x^3+y^3}$. (M. M. 1887).

Simplify

19. $\dfrac{1}{1+\dfrac{1}{a\div b}}+\dfrac{1}{1-\dfrac{1}{a\div b}}+\dfrac{2}{1+\dfrac{1}{a^2\div b^2}}$.

20. $\dfrac{\dfrac{a}{2a+b}+\dfrac{a}{2a-b}}{\dfrac{4}{4a^2-b^2}}-\dfrac{9a-\dfrac{1}{a}}{3+\dfrac{1}{a}}$.

21. $\dfrac{1-\dfrac{b-c}{b+c}}{1+\dfrac{b-c}{b+c}}\times\dfrac{1-\dfrac{c-a}{c+a}}{1+\dfrac{c-a}{c+a}}\times\dfrac{1-\dfrac{a-b}{a+b}}{1+\dfrac{a-b}{a+b}}$.

22. $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}-1}{\dfrac{a^2}{b^2}+\dfrac{a}{b}+1}\div\dfrac{a-b}{1+\dfrac{b}{a}}\times\dfrac{\dfrac{a^2}{b}-\dfrac{b^2}{a}}{1+\dfrac{b^3}{a^3}}$.

23. $\dfrac{\dfrac{1}{a}-\dfrac{a+b}{a^2+b^2}}{\dfrac{1}{b}-\dfrac{a+b}{a^2+b^2}}+\dfrac{\dfrac{1}{a}-\dfrac{a-b}{a^2+b^2}}{\dfrac{1}{b}-\dfrac{a-b}{a^2+b^2}}$.

24. $\dfrac{x-1}{x-2-\dfrac{x-1}{x+\dfrac{2x-1}{x-2}}}$.

25. $\dfrac{x}{x-2+\dfrac{x}{3+\dfrac{2}{x}}}$.

26. $3-\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x-1}}}}$.

27. $\dfrac{x-2}{x-2-\dfrac{x}{x-1-\dfrac{1}{x-2}}}$.

28. $\dfrac{11x+3}{3+\dfrac{2}{3+\dfrac{1}{x}}}-\dfrac{7x-3}{3-\dfrac{2}{3-\dfrac{1}{x}}}$.

29. Prove that $\dfrac{1-x}{1+x}=1-2x+2x^2-2x^3+\dfrac{2x^4}{1+x}$

$$=-1+\frac{2}{x}-\frac{2}{x^2}+\frac{2}{x^3}-\frac{2}{x^3(1+x)}.$$

30. Prove that $\dfrac{x}{x^2+a^2}=\dfrac{1}{x}-\dfrac{a^2}{x^3}+\dfrac{a^4}{x^5}-\dfrac{a^6}{x^7}+\dfrac{a^8}{x^7(x^2+a^2)}$

$$=\frac{x}{a^2}-\frac{x^3}{a^4}+\frac{x^5}{a^6}-\frac{x^7}{a^8}+\frac{x^9}{a^8(x^2+a^2)}.$$

31. Find what proper fraction must be subtracted from $\dfrac{5x^4-4x^3+3x^2+27x+60}{x^2-3x+5}$ to make it integral.

32. Find what proper fraction must be added to $\dfrac{10x^5+x^4+31x^2+5x-40}{5x^2-7x+13}$ to make it integral.

---

18. Theorems on equal fractions.

Theorem I. If $\dfrac{a}{b}=\dfrac{c}{d}$, the $ad=bc$.

$\because \dfrac{a}{b}=\dfrac{c}{d}$, hence multiplying both sides by bd,

$\dfrac{a}{b}\times bd=\dfrac{c}{d}\times bd$ or $ad=bc$.

This result is obtained by multiplying the terms of the two fractions cross-wise and equating the products, and hence referred to as **multiplying cross-wise**.

Theorem II. If $\dfrac{a}{b}=\dfrac{c}{d}$, then $\dfrac{b}{a}=\dfrac{d}{c}$.

$\because \dfrac{a}{b}=\dfrac{c}{d}$, $\therefore 1\div\dfrac{a}{b}=1\div\dfrac{c}{d}$, $\therefore \dfrac{b}{a}=\dfrac{d}{c}$.

Theorem III. If $\dfrac{a}{b}=\dfrac{c}{d}$, then $\dfrac{a}{c}=\dfrac{b}{d}$.

$\because \dfrac{a}{b}=\dfrac{c}{d}$, $\therefore \dfrac{a}{b}\times\dfrac{b}{c}=\dfrac{c}{d}\times\dfrac{b}{c}$, $\therefore \dfrac{a}{c}=\dfrac{b}{d}$.

Theorem IV. If $\dfrac{a}{b}=\dfrac{c}{a}$ then $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$.

Let $\dfrac{a}{b}=\dfrac{c}{d}=k$, so that $a=bk$, $c=dk$.

$\therefore \dfrac{a+b}{a-b}=\dfrac{bk+b}{bk-b}=\dfrac{b(k+1)}{b(k-1)}=\dfrac{k+1}{k-1}$;

also $\dfrac{c+d}{c-d}=\dfrac{dk+d}{dk-d}=\dfrac{d(k+1)}{d(k-1)}=\dfrac{k+1}{k-1}$.

$$\therefore \frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

This is called *componendo and dividendo.*

Theorem V. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then

$$\text{each}=\frac{pa+qc+re}{pb+qd+rf}; \; p, q, r \text{ being any quantities.}$$

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$, so that

$a=bk$, $c=dk$, $c=fk$.

$$\text{Then } \frac{pa+qc+re}{pb+qd+rf}=\frac{pbk+qdk+rfk}{pb+qd+rf}.$$

$$=\frac{k(pb+qd+rf)}{pb+qd+rf}=k.$$

$=$each of the given fractions.

A similar result is true in the case of any number of fractions, and we may state the theorem thus :

If any number of fractions are equal, then each is equal to a fraction of which the numerator is the sum of any multiples of the numerators of the given fractions and the denominator is the sum of the corresponding multiples of the denominators.

As a particular case we have, if $\frac{a}{b}=\frac{c}{d}$, then each $=\frac{a+c}{b+d}$, also $=\frac{a-c}{b-d}$ *i. e.*, if two fractions are equal, then

$$\text{each}=\frac{\text{sum of numerators}}{\text{sum of denominators}}=\frac{\text{diff. of numerators}}{\text{diff. of denominators}}.$$

We shall content ourselves by simply stating the above theorems, referring the student to the chapter on Ratio for further development.

19. The following are some illustrative examples.

Ex. 1. Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$ when $x=\frac{4ab}{a+b}$.

(C. E. 1865, B. M. 1883, A. E. 1892, P. E. 1899).

$$\text{The expression}=1+\frac{4a}{x-2a}+1+\frac{4b}{x-2b}$$

$$=2+4$$

$$=2+4.\frac{a(x-2b)+b(x-2a)}{(x-2a)(x-2b)}$$

$$=2+4.\frac{(a+b)x-4ab}{(x-2a)(x-2b)}$$

$=2+4.\ 0$ [$\because$ $(a+b)x=4ab$ from hypothesis]

$=2.$

Otherwise thus :—

From the given relation,

$$\frac{x}{2a}=\frac{2b}{a+b}\ldots(1),\quad \frac{x}{2b}=\frac{2a}{a+b}\ldots(2).$$

From (1) by Theor. 4, art. 18, $\frac{x+2a}{x-2a}=\frac{2b+(a+b)}{2b-(a+b)}=\frac{3b+a}{b-a}$.

Similarly from (2) $\frac{x+2b}{x-2b}=\frac{2a+(a+b)}{2a-(a+b)}=\frac{3a+b}{a-b}$.

$$\therefore \frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}=\frac{3b+a}{b-a}+\frac{3a+b}{a-b}=\frac{3b+a}{b-a}-\frac{3a+b}{b-a}$$

$$=\frac{3b+a-3a-b}{b-a}=\frac{2(b-a)}{b-a}=2.$$

Ex. 2 Simplify $\dfrac{a^2\left(\frac{1}{b}-\frac{1}{c}\right)+b^2\left(\frac{1}{c}-\frac{1}{a}\right)+c^2\left(\frac{1}{a}-\frac{1}{b}\right)}{a\left(\frac{1}{b}-\frac{1}{c}\right)+b\left(\frac{1}{c}-\frac{1}{a}\right)+c\left(\frac{1}{a}-\frac{1}{b}\right)}$.

$$\text{Numerator}=a^2\,\frac{c-b}{bc}+b^2\,\frac{a-c}{ac}+c^2\,\frac{b-a}{ab}$$

$$=-\frac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{abc}$$

$$\text{Denominator}=a\,\frac{c-b}{bc}+b\,\frac{a-c}{ac}+c\,\frac{b-a}{ab}$$

$$=-\frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{abc}$$

$$\therefore \text{ the fraction}=\frac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$$

$$=\frac{-(b-c)(c-a)(a-b)(a+b+c)}{-(b-c)(c-a)(a-b)}$$ [see Ex. 10, p. 147, also art. 15, p. 144.]

$$=(a+b+c)$$

Ex. 3. Simplify $\dfrac{8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3}{(2a+b+c)(2b+c+a)(2c+a+b)}$

We have $2(a+b+c)=(b+c)+(c+a)+(a+b)$

$\therefore$ cubing, $8(a+b+c)^3$

$=\{(b+c)+(c+a)+(a+b)\}^3$

$=(b+c)^3+(c+a)^3+(a+b)^3+3(2a+b+c)(2b+c+a)(2c+a+b)$

[see formula XIII.]

$\therefore$ transposing, $8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3$

$=3(2a+b+c)(2b+c+a)(2c+a+b)$

$$\therefore \frac{8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3}{(2a+b+c)(2b+c+a)(2c+a+b)}=3.$$

Ex. 4. If $2s=a+b+c$, prove that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{(s-a)(s-b)(s-c)}.$$

$$\text{Left side}=\left(\frac{1}{s-a}+\frac{1}{s-b}\right)+\left(\frac{1}{s-c}-\frac{1}{s}\right)$$

$$=\frac{2s-a-b}{(s-a)(s-b)}+\frac{s-s+c}{(s-c)s},$$

$$=\frac{c}{(s-a)(s-b)}+\frac{c}{(s-c)s}, \quad \because\ 2s=a+b+c$$

$$=\frac{c\{(s-c)s+(s-a)(s-b)\}}{s(s-a)(s-b)(s-c)}$$

$$=\frac{c\{2s^2-s(a+b+c)+ab\}}{s(s-a)(s-b)(s-c)}$$

$$=\frac{c\times ab}{s(s-a)(s-b)(s-c)}, \quad \text{for } s(a+b+c)=s.\,2s=2s^2$$

$$=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

Ex. 5. Prove that

$$\left(\frac{b}{c}+\frac{c}{a}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right) \quad \text{C. E. 1867.}$$

$$\text{Left side}=\left(\frac{b^2+c^2}{bc}\right)^2+\left(\frac{c^2+a^2}{ca}\right)^2+\left(\frac{a^2+b^2}{ab}\right)^2$$

$$=\frac{a^2(b^2+c^2)^2+b^2(c^2+a^2)^2+c^2(a^2+b^2)^2}{a^2b^2c^2}.$$

$$\text{Numerator}=a^2(b^2+c^2)^2+b^2(c^4+a^4)+c^2(a^4+b^4)+4a^2b^2c^2$$

$$=a^2(b^2+c^2)^2+a^4(b^2+c^2)+b^2c^2(b^2+c^2)+4a^2b^2c^2$$

$$=(b^2+c^2)\{a^4+a^2(b^2+c^2)+b^2c^2\}+4a^2b^2c^2$$

$$=(b^2+c^2)(a^2+b^2)(a^2+c^2)+4a^2b^2c^2, \text{ formula XVII.}$$

$$\therefore\ \text{left side}=\frac{4a^2b^2c^2+(b^2+c^2)(c^2+a^2)(a^2+b^2)}{a^2b^2c^2}$$

$$=\frac{4a^2b^2c^2}{a^2b^2c^2}+\frac{b^2+c^2}{bc}\cdot\frac{c^2+a^2}{ca}\cdot\frac{a^2+b^2}{ab}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right).$$

Ex. 6. Prove that $\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x}{z}+\frac{z}{y}+\frac{y}{x}\right)$

$$=1+\left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)$$ B. M. 1887

Let $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=s$;

then $\frac{x}{y}+\frac{y}{z}=s-\frac{z}{x}$, $\frac{y}{z}+\frac{z}{x}=s-\frac{x}{y}$, $\frac{z}{x}+\frac{x}{y}=s-\frac{y}{z}$.

$$\therefore \left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)$$

$$=\left(s-\frac{x}{y}\right)\left(s-\frac{y}{z}\right)\left(s-\frac{z}{x}\right)$$

$$=s^3-s^2\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)$$

$$+s\left(\frac{x}{y}\cdot\frac{y}{z}+\frac{y}{z}\cdot\frac{z}{x}+\frac{z}{x}\cdot\frac{x}{y}\right)-\frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{x}.$$

$$=s^3-s^2.\,s+s\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)-1$$

$$=s\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)-1$$

$$=\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x}{z}+\frac{z}{y}+\frac{y}{x}\right)-1.$$

$\therefore$ transposing, the required relation follows.

Ex. 7. Simplify

$$\frac{ac+b^2}{2bc}(b+c-a)+\frac{ab+c^2}{2ca}(c+a-b)+\frac{bc+a^2}{2ab}(a+b-c).$$

Here first term $=\left(\frac{ac}{2bc}+\frac{b^2}{2bc}\right)(b+c-a)$

$$=\left(\frac{a}{2b}+\frac{b}{2c}\right)(b+c-a)$$

$$=\frac{a}{2b}(b+c-a)+\frac{b}{2c}(b+c-a).$$

Similarly, second term $=\frac{b}{2c}(c+a-b)+\frac{c}{2a}(c+a-b)$;

third term $=\frac{c}{2a}(a+b-c)+\frac{a}{2b}(a+b-c)$.

Hence adding and collecting the co-efficients of $\frac{a}{2b}, \frac{b}{2c}, \frac{c}{2a}$ the expression

$$=\frac{a}{2b}\times 2b+\frac{b}{2c}\times 2c+\frac{c}{2a}\times 2a=a+b+c.$$

Ex. 8. Prove that

$$\frac{b-c}{1+bc}+\frac{c-a}{1+ca}+\frac{a-b}{1+ab}=\frac{b-c}{1+bc}\cdot\frac{c-a}{1+ca}\cdot\frac{a-b}{1+ab}.$$

Left side

$$=\frac{(b-c)(1+ca)(1+ab)+(c-a)(1+ab)(1+bc)+(a-b)(1+bc)(1+ca)}{(1+bc)(1+ca)(1+ab)}$$

$$\begin{aligned}
\text{Numerator} &= (b-c)\{1+a(b+c)+a.abc\} \\
&\quad +(c-a)\{1+b(c+a)+b.abc\} \\
&\quad +(a-b)\{1+c(a+b)+c.abc\} \\
&= (b-c)+a(b^2-c^2)+abc.a(b-c) \\
&\quad +(c-a)+b(c^2-a^2)+abc.b(c-a) \\
&\quad +(a-b)+c(a^2-b^2)+abc.c(a-b) \\
&= a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)
\end{aligned}$$

[adding column by column]

$=(b-c)(c-a)(a-b)$, formula XVII.

$$\therefore \text{ Left side}=\frac{(b-c)(c-a)(a-b)}{(1+bc)(1+ca)(1+ab)}.$$

Obs. In connection with this and similar examples it will be useful for the student to remember formula XVII and identities like the following :—

$$(b-c)+(c-a)+(a-b)=0.$$
$$a(b-c)+b(c-a)+c(a-b)=0.$$
$$(b+c)(b-c)+(c+a)(c-a)+(a+b)(a-b)=0.$$

Ex. 9. Simplify $\frac{bc}{(a-b)(a-c)}+\frac{ca}{(b-c)(b-a)}+\frac{ab}{(c-a)(c-b)}$

Preserving cyclic order in the factors in the denominators, *i.e.* changing them into the forms $a-b$, $b-c$, $c-a$,

$$\text{the fraction}=-\frac{bc}{(a-b)(c-a)}-\frac{ca}{(b-c)(a-b)}-\frac{ab}{(c-a)(b-c)}$$

$$=-\frac{bc(b-c)+ca(c-a)+ab(a-b)}{(b-c)(c-a)(a-b)}$$

$$=-\frac{-(b-c)(c-a)(a-b)}{(b-c)(c-a)(a-b)}, \text{ formula XVII}$$

$$=1.$$

Obs. The following results of which the last is proved above and the others follow in the same manner, are important and should be remembered by the student.

(i) $$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}=0.$$

(ii) $$\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}=0.$$

(iii) $$\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}=1.$$

(iv) $$\frac{bc}{(a-b)(a-c)}+\frac{ca}{(b-c)(b-a)}+\frac{ab}{(c-a)(c-b)}=1.$$

Ex. 10. If $a+b+c=0$, prove that

$$\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ca}+\frac{c^2}{2c^2+ab}=1.$$

We have $2a^2+bc=a^2+a^2+bc$

$$=a^2-a(b+c)+bc, \quad \because a=-(b+c)$$

$$=(a-b)(a-c).$$

Similarly, $2b^2+ca=(b-c)(b-a)$, $2c^2+ab=(c-a)(c-b)$.

$\therefore$ the given expression

$$=\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}$$

$$=-\frac{a^2}{(a-b)(c-a)}-\frac{b^2}{(b-c)(a-b)}-\frac{c^2}{(c-a)(b-c)}$$

$$=-\frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{(a-b)(b-c)(c-a)}$$

$$=-\frac{-(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)}$$

$$=1.$$

Ex. 11. Simplify

$$\frac{a^2-2bc}{(a-b)(a-c)}+\frac{b^2-2ca}{(b-c)(b-a)}+\frac{c^2-2ab}{(c-a)(c-b)}.$$

We have $$\frac{a^2-2bc}{(a-b)(a-c)}=\frac{a^2}{(a-b)(a-c)}-\frac{2bc}{(a-b)(a-c)}$$

$$\frac{b^2-2ca}{(b-c)(b-a)}=\frac{b^2}{(b-c)(b-a)}-\frac{2ca}{(b-c)(b-a)}$$

$$\frac{c^2-2ab}{(c-a)(c-b)}=\frac{c^2}{(c-a)(c-b)}-\frac{2ab}{(c-a)(c-b)}$$

$\therefore$ adding, the given expression

$$=\left\{\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}\right\}$$
$$-2\left\{\frac{bc}{(a-b)(a-c)}+\frac{ca}{(b-c)(b-a)}+\frac{ab}{(c-a)(c-b)}\right\}$$
$$=1-2\times 1 \text{ [by (iii) and (v) above]}$$
$$=-1.$$

Ex. 12. Simplify $\dfrac{(x+a)^2}{(a-b)(a-c)}+\dfrac{(x+b)^2}{(b-c)(b-a)}+\dfrac{(x+c)^2}{(c-a)(c-b)}$

$$\frac{(x+a)^2}{(a-b)(a-c)}=\frac{x^2+2ax+a^2}{(a-b)(a-c)}$$
$$=\frac{x^2}{(a-b)(a-c)}+2x\frac{a}{(a-b)(a-c)}+\frac{a^2}{(a-b)(a-c)}.$$
$$\frac{(x+b)^2}{(b-c)(b-a)}=\frac{x^2}{(b-c)(b-a)}+2x\frac{b}{(b-c)(b-a)}+\frac{b^2}{(b-c)(b-a)}.$$
$$\frac{(x+c)^2}{(c-a)(c-b)}=\frac{x^2}{(c-a)(c-b)}+2x\frac{c}{(c-a)(c-b)}+\frac{c^2}{(c-a)(c-b)}.$$

$\therefore$ adding column by column, the giving expression

$$=x^2\left\{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}\right\}$$
$$+2x\left\{\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}\right\}$$
$$+\left\{\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}\right\}$$
$$=x^2\times 0+2x\times 0+1 \text{ [Simplifying each part within \{ \}].}$$
$$=1.$$

Otherwise thus :—

The fraction $=-\dfrac{(x+a)^2}{(a-b)(c-a)}-\dfrac{(x+b)^2}{(b-c)(a-b)}-\dfrac{(x+c)^2}{(c-a)(b-c)}.$

$$=-\frac{(x+a)^2(b-c)+(x+b)^2(c-a)+(x+c)^2(a-b)}{(b-c)(c-a)(a-b)}.$$

Numerator $=(x^2+2ax+a^2)(b-c)+(x^2+2bx+b^2)(c-a)$
$+(x^2+2cx+c^2)(a-b)$

$$=\;x^2(b-c)+2x.a(b-c)+a^2(b-c)$$
$$+x^2(c-a)+2x.b(c-a)+b^2(c-a)$$
$$+x^2(a-b)+2x\,c(a-b)+c^2(a-b)$$

$=a^2(b-c)+b^2(c-a)+c^2(a-b)$, adding column by column

$=-(b-c)(c-a)(a-b)$, formula XVII.

$\therefore$ the fraction $=-\dfrac{-(b-c)(c-a)(a-b)}{(b-c)(c-a)(a-b)}=1.$

EXERCISE LXVIII.

1. If $x=\dfrac{2ab}{a+b}$, prove that $\dfrac{a}{x-a}+\dfrac{b}{x-b}=0$.

2. If $x=\dfrac{(3b-a)c}{2b}$, prove that $\dfrac{x-2c}{x-c}=\dfrac{a+b}{a-b}$.

3. If $x=\dfrac{b(a+c)}{a-c}$, prove that $\dfrac{x-a+b}{x-b-c}=\dfrac{a}{c}$.

4. Find the value of $\dfrac{x+2a}{2b-x}+\dfrac{x-2a}{2b+x}-\dfrac{4ab}{4b^2-x^2}$ when $x=\dfrac{ab}{a+b}$.

5. Find the value of $\dfrac{b+c}{bc-x}+\dfrac{c+a}{ca-x}+\dfrac{a+b}{ab-x}$

when $x=\dfrac{abc}{a+b+c}$.

6. Find the value of $\dfrac{x^2-y^2+x}{y^2-x^2+y}$ when $x=\dfrac{a-b}{a+b}$, $y=\dfrac{a+b}{a-b}$

(C. E. 1883).

7. Find the value of $\left(\dfrac{x}{x+1}\right)^2+\left(\dfrac{x}{x-1}\right)^2$ when $x^2=\dfrac{n-1}{n+1}$.

Simplify the following :—

8. $\left(1-\dfrac{1}{1+x}\right)\left(x+\dfrac{1}{2+x}\right)\times\dfrac{\dfrac{1}{x^2}-x}{1+\dfrac{1}{x}}\div\left(1+x+\dfrac{1}{x}\right)$ (C. E. 1171)

9. $\left\{\dfrac{x}{1-\dfrac{1}{x}}-x-\dfrac{1}{1-x}\right\}\div\left\{\dfrac{x}{1+\dfrac{1}{x}}+x-\dfrac{1}{1+x}\right\}$ (C. E. 1886).

10. $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}+\dfrac{b}{c}-\dfrac{c}{b}+\dfrac{c}{a}-\dfrac{a}{c}}{\dfrac{a^2}{b}-\dfrac{b^2}{a}+\dfrac{b^2}{c}-\dfrac{c^2}{b}+\dfrac{c^2}{a}-\dfrac{a^2}{c}}$

11. $\dfrac{a+b}{a-b+\dfrac{b^2}{a+b}}-\dfrac{a-b}{a+b+\dfrac{b^2}{a-b}}$ (M. M. 1897).

12. $\dfrac{x}{x-a}-\dfrac{x}{x+a}-\dfrac{\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}}{\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}}$ (C. E. 1869).

Simplify the following :—

13. $\left\{\dfrac{\frac{x}{y}+2}{\frac{x}{y}+1}+\dfrac{x}{y}\right\}\div\left\{\dfrac{x}{y}+2-\dfrac{\frac{x}{y}}{\frac{x}{y}+1}\right\}$ (P. E. 1894).

14. $\dfrac{\frac{a^3}{b^3}-\frac{b^3}{a^3}}{\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{a}{b}+\frac{b}{a}-1\right)}\times\dfrac{\frac{1}{b}-\frac{1}{a}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{ab}}$ (C. E. 1874).

15. $\dfrac{\frac{a}{a-b}-\frac{a}{a+b}}{\frac{b}{a-b}-\frac{b}{a+b}}\div\dfrac{\frac{a+b}{a-b}+\frac{a-b}{a+b}}{\frac{a+b}{a-b}-\frac{a-b}{a+b}}\times\dfrac{a^2}{a^2+b^2}$ (C. E. 1876).

16. $\left\{\dfrac{b+\frac{a-b}{1+ab}}{1-\frac{(a-b)b}{1+ab}}-\dfrac{a-\frac{a-ab}{1-ab}}{1-\frac{(a-b)a}{1-ab}}\right\}\div\left(\dfrac{a}{b}-\dfrac{b}{a}\right)$ (P. E. 1898).

17. $\dfrac{1-\frac{b}{a}+\frac{b^2}{a^2}}{1+\frac{b}{a}+\frac{b^2}{a^2}}\times\dfrac{\frac{a^3}{b^3}-1}{\frac{a^3}{b^3}+1}\div\dfrac{\left(\frac{1}{a}-\frac{1}{b}\right)^2}{\left(\frac{1}{a}+\frac{1}{b}\right)^2}.$

18. $\dfrac{x^{3n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}-\dfrac{1}{x^n-1}+\dfrac{1}{x^n+1}$ (C. E. 1885).

19. $\dfrac{x+2}{1+x+x^2}-\dfrac{x-2}{1-x+x^2}-\dfrac{2x^2-4}{1-x^2+x^4}$ (C. E. 1877).

20. $\dfrac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{bc(b-c)+ca(c-a)+ab(a-b)}.$

21. $\dfrac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)}.$

22. $\dfrac{a^3+b^3+c^3-3abc}{(b-c)^2+(c-a)^2+(a-b)^2}.$

23. $\dfrac{a(b-c)^3+b(c-a)^3+c(a-b)^3}{(b-c)^3+(c-a)^3+(a-b)^3}.$

If $2s+a+b+c$, prove that

24. $\sqrt{\left\{1-\left(\dfrac{b^2+c^2-a^2}{2bc}\right)^2\right\}}=\dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

25. $\dfrac{a}{s-a}+\dfrac{b}{s-b}+\dfrac{c}{s-c}+2=\dfrac{abc}{(s-a)(s-b)(s-c)}.$

26. Prove that $\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(z+\frac{1}{z}\right)^2$
$=4+\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right)$, if $xyz=1$.

Simplify the following :—

27. $\frac{2(a+b)}{ab}(a+b-c)+\frac{2(b+c)}{bc}(b+c-a)+\frac{2(c+a)}{ca}(c+a-b)$.

28. $\frac{a+b}{ab}(a^2+b^2-c^2)+\frac{b+c}{bc}(b^2+c^2-a^2)+\frac{a+c}{ac}(a^2+c^2-b^2)$.

29. $\frac{a^2+b^2}{2a^2b^2}(a^3+b^3-c^3)+\frac{b^2+c^2}{2b^2c^2}(b^3+c^3-a^3)+\frac{c^2+a^2}{2c^2a^2}(c^3+a^3-b^3)$.

30. $ab\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)+bc\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)$
$+ca\left(\frac{1}{c}+\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)$.

Simplify the following :—

31. $\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}$.

32. $\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}$.

33. $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}$.

34. $\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}$.

35. $\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-c)(b-a)}+\frac{c^3}{(c-a)(c-b)}$.

36. $\frac{a(b+c)}{(a-b)(a-c)}+\frac{b(c+a)}{(b-c)(b-a)}+\frac{c(a+b)}{(c-a)(c-b)}$.

37. $\frac{a^2(b+c)}{(a-b)(a-c)}+\frac{b^2(c+a)}{(b-c)(b-a)}+\frac{c^2(a+b)}{(c-a)(c-b)}$.

38. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}$.

39. $\frac{1}{a^2(a-b)(a-c)}+\frac{1}{b^2(b-c)(b-a)}+\frac{1}{c^2(c-a)(c-b)}$.

40. $\frac{bc(x+a)}{(a-b)(a-c)}+\frac{ca(x+b)}{(b-c)(b-a)}+\frac{ab(x+c)}{(c-a)(c-b)}$.

41. $\frac{a^2+bc}{(a-b)(a-c)}+\frac{b^2+ca}{(b-c)(b-a)}+\frac{c^2+ab}{(c-a)(c-b)}$.

Simplify the following :—

42. $\dfrac{a^3+bc}{(a-b)(a-c)}+\dfrac{b^3+ca}{(b-c)(b-a)}+\dfrac{c^3+ab}{(c-a)(c-b)}$.

43. $\dfrac{b-c}{b+c}+\dfrac{c-a}{c+a}+\dfrac{a-b}{a+b}$.

44. $\dfrac{a-b}{m+ab}+\dfrac{b-c}{m+bc}+\dfrac{c-a}{m+ca}$.

45. $\dfrac{x^2-yz}{(x-y)(x-z)}+\dfrac{y^2+zx}{(y+z)(y-x)}+\dfrac{z^2+xy}{(z-x)(z+y)}$ (C. E. 1865).

46. $\dfrac{1}{bc(a-b)(a-c)}+\dfrac{1}{ca(b-c)(b-a)}+\dfrac{1}{ab(c-a)(c-b)}$.

47. $\dfrac{a}{bc(a-b)(a-c)}+\dfrac{b}{ca(b-c)(b-a)}+\dfrac{c}{ab(c-a)(c-b)}$.

48. $\dfrac{a^3(b+c)}{(a-b)(a-c)}+\dfrac{b^3(c+a)}{(b-c)(b-a)}+\dfrac{c^3(a+b)}{(c-a)(c-b)}$.

49. $\dfrac{a^3}{(a^2-b^2)(a^2-c^2)}+\dfrac{b^3}{(b^2-c^2)(b^2-a^2)}+\dfrac{c^3}{(c^2-a^2)(c^2-b^2)}$.

50. $\dfrac{2a^2-bc}{(a-b)(a-c)}+\dfrac{2b^2-ca}{(b-c)(b-a)}+\dfrac{2c^2-ab}{(c-a)(c-b)}$.

51. $\dfrac{(a-1)^2}{(a-b)(a-c)}+\dfrac{(b-1)^2}{(b-c)(b-a)}+\dfrac{(c-1)^2}{(c-a)(c-b)}$.

52. $\dfrac{a^2+a+1}{(a-b)(a-c)}+\dfrac{b^2+b+1}{(b-a)(b-c)}+\dfrac{c^2+c+1}{(c-a)(c-b)}$.

53. $\dfrac{pa^2+qbc+r}{(a-b)(a-c)}+\dfrac{pb^2+qca+r}{(b-c)(b-a)}+\dfrac{pc^2+qab+r}{(c-a)(c-b)}$.

54. $\dfrac{ka^3+la^2+ma+1}{(a-b)(a-c)}+\dfrac{kb^3+lb^2+mb+1}{(b-a)(b-c)}+\dfrac{kc^3+lc^2+mc+1}{(c-a)(c-b)}$

55. If $a+b+c=0$, prove that

(i) $\dfrac{1}{2a^2+bc}+\dfrac{1}{2b^2+ca}+\dfrac{1}{2c^2+ab}=0$.

(ii) $\dfrac{a}{2a^2+bc}+\dfrac{b}{2b^2+ca}+\dfrac{c}{2c^2+ab}=0$.

(iii) $\dfrac{bc}{2a^2+bc}+\dfrac{ca}{2b^2+ca}+\dfrac{ab}{2c^2+ab}=1$.

(iv) $\dfrac{a^3}{2a^2+bc}+\dfrac{b^3}{2b^2+ca}+\dfrac{c^3}{2c^2+ab}=0$.

MISCELLANEOUS EXERCISE PAPERS II.

Paper I.

1. Simplify :— $5y-3[2y+9x-2\{3y-4(x-y)\}]$.
2. Divide $4{\cdot}182x^3+18{\cdot}44x^2+7{\cdot}388x-2{\cdot}4$ by $1{\cdot}02x+4$.

3. Express $8xy(x^2+y^2)$ as the difference of two squares.

4. Resolve into factors :—

(1) x^4+4y^4 (2) x^5-32y^5 (3) $(p^2+x^2)q-(q^2+x^2)p$

5. Find the L.C.M. of $6x^2-13x+6$, $6x^2+5x-6$, and $9x^2-4$.

6. Find the H C.F. of $6x^4-2x^3+9x^2+9x-4$ and $9x^4+80x^2-9$; and find such a value of x as will make both these expressions vanish. (B.M. 1895)

7. Simplify :—

(i) $\frac{1}{abx}+\frac{1}{a(a-b)(x-a)}+\frac{1}{b(b-a)(x-b)}$ C. E. 1881.

(ii) $\frac{1}{a+x}$ $\frac{1}{a+2x}$ $\frac{1}{a+3x}$ $\frac{1}{a+4x}$.

8. What number is that whose fourth, sixth and eighth parts together fall short of the whole by 110?

PAPER II.

Divide

(i) $\frac{3x^5}{4}-4x^4y+\frac{77}{8}x^3y^2-\frac{43}{4}x^2y^3-\frac{33}{4}xy^4+27y^5$

by $\frac{x^2}{2}-xy+3y^2$.

(ii) $(x^2-1)^4-3(x^2-1)^2+1$ by x^4-3x^2+1 M. M. 1898.

2. If $x^2+y^2=z^2$, find the value of

$$(x+y+z)(x+y-z)(y+z-x)(z+x-y).$$

3. Resolve into factors

(1) $x^3-12x+16$.

(2) $3x^4-3x^3-7x^2+7x+8$. (3) $(9a^2+2)^2+12$.

4. If $a+b=5$, $ab=6$, find the value of $a^4+a^2b^2+b^4$.

5. Find the H.C.F. of $x^5-3x^4y+4x^3y^2-5x^2y^3+3xy^4-2y^5$ and $2x^4-3x^3y-x^2y^2-2xy^3$.

6. Find the L.C.M. of

x^3-a^3, x^3+a^3, $x^4+a^2x^2+a^4$, $x^3-ax^2-a^2x+a^3$ and $x^3+ax^2-a^2x-a^3$

7. Simplify : $\frac{(1+ab)(1+ac)}{(a-b)(a-c)}+\frac{(1+bc)(1+ba)}{(b-a)(b-c)}+\frac{(1+ca)(1+cb)}{(c-a)(c-b)}$

8. A man sold a horse for *Rs.* 525 and half as much as he gave for it, and gained thereby *Rs.* 157 8*a*. What did he pay for the horse?

PAPER III.

1. If $x^2 + \frac{1}{x} = 2$, prove that $x(x+1) = 1$ or $x = 1$.

2. Divide $(4x^3 - 3a^2x)^2 + (4y^3 - 3a^2y)^2 - a^6$ by $x^2 + y^2 - a^2$. (B. M. 1884).

[Simplifying and arranging in ascending powers of, a the dividend $= 16(x^6 + y^6) - 24(x^4 + y^4)a^2 + 9(x^2 + y^2)a^4 - a^6$, and the divisor $= (x^2 + y^2) - a^2$; hence etc.].

3. Find the L.C.M. of
$6x^3 - x^2 - 2x$, $21x^4 - 17x^3 + 2x^2$, $14x^2 + 5x - 1$.

4. If $a + b + c = 1$, prove that $(a+b)(a+c) = a + bc$,
$(b+c)(b+a) = b + ca$, $(c+a)(c+b) = c + ab$.

5. If $x^3 - bx^2 + 2apx + ap^2$ be divisible by $x + p$,
show that either $p = 0$ or $a + b + p = 0$.

6. Show that $(x+y+z)(a^2+b^2+c^2) = a^2x + b^2y + c^2z$,
if $x^2 - yz = a^2$, $y^2 - xz = b^2$ and $z^2 - xy = c^2$.

7. Simplify :—

(i) $$\frac{\frac{2}{15}(x-1) - \frac{1}{10}\left(2 - \frac{x}{3}\right)}{\frac{1}{30} - \frac{1}{10}(1-x)} - \frac{\frac{1}{6}\left(1 - \frac{x}{2}\right) + \frac{1}{4}\left(\frac{x}{3} - 1\right)}{\frac{1}{4}(1-3x) + \frac{1}{6}\left(\frac{3x}{2} + \frac{1}{2}\right)} - 1\cdot 5.$$

(ii) $$\frac{b^2+c^2-a^2}{(a-b)(a-c)} + \frac{c^2+a^2-b^2}{(b-a)(b-c)} + \frac{a^2+b^2-c^2}{(c-a)(c-b)}.$$

8. One tenth of the less of two consecutive numbers exceeds the one seven-teenth of the other by 2. Find the numbers.

PAPER IV.

1. Find the H.C.F. of $x^3 + 3x^2 - 9x + 5$ and $x^3 - 19x + 30$.

2. Divide $x(1+y^2)(1+z^2) + y((1+z^2)(1+x^2) + z(1+x^2)(1+y^2)$
$+ 4xyz$ by $1 + yz + zx + xy$. C. E. 1878.

[Multiplying out and arranging in descending powers of x, the dividend $= x^2(y+z)(1+yz) + x(1+y^2+z^2+y^2z^2+4yz) + (y+z)(1+yz)$, and the divisor $= x(y+z) + (1+yz)$; hence etc.].

3. Resolve into factors (1) $x^{12} - a^{12}$.
(2) $x^3(y-z) + y^3(z-x) + z^3(x-y)$.

4. If $ax + by = 1 = mx + ny$ and $xy = \frac{1}{an + mb}$,

show that $\frac{a}{m} + \frac{m}{a} + \frac{b}{n} + \frac{n}{b} = 4$.

5. Prove that $16s(s-a)(s-b)(s-c)$
$=2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$ when $2s=a+b+c$.

6. Express (i) $(a^2+b^2)(c^2+d^2)$, and
(ii) $(a^2+b^2)(c^2+d^2)(x^2+y^2)$ as the sum of two squares.

7. Simplify :—

(a) $\dfrac{48x^5-144x^4+108x^3}{72x^4-216x^3+162x^2}$. (b) $\dfrac{(3x^2-4x-5)^2-(x^2-5x-4)^2}{(3x^2+2x+2)^2-(x^2+5x+1)^2}$.

8. If I add 15 to the square of a number, I obtain the square of the next higher number. What is the number?

PAPER V.

1. Find the H.C.F. of $2x^5-11x^2-9$ and $4x^5+11x^4+81$.
A.E. 1898.

2. Express $(3a+5b+4c)^2-4(a+4b+c)(2a+b+3c)$ as a perfect square.

3. If $a+b+c$ divide $ab+ac+bc$ exactly, it will also divide $a^2+b^2+c^2$.

4. Find the condition that x^2+ax+b and $x^2+a'x+b'$ may have a common measure of the form $x+c$.

5. If $a=b+c$ show that $(a-b)^2+(a-c)^2=a^2-2bc$.

6. If $my+nx=m$ and $ny-mx=n$, show that $x^2+y^2=1$.

7. Simplify :—

$$\frac{7x^2+16xy-11y^2-\dfrac{14y^3}{x}-\dfrac{3y^4}{x^2}}{28x^3+71x^2y-35xy^2-69y^3-\dfrac{18y^4}{x}}$$

8. A person gives away 4 shillings more than $\frac{1}{5}$ of his money and has left 11*s*. more than $\frac{1}{2}$ of it : how much had he at first?

PAPER VI.

1. Resolve into factors :—
(i) $9a^2(x^3+12ay^2)-(4y^2x^3+243a^5)$
(ii) $x(x-a-b)+ab$.

2. If $x+m$ be the H.C.F. of ax^2+bx+c and cx^2+bx+a,
show that $m=\dfrac{a+c}{b}$.

3. Find the G.C.M. and L.C.M. of
$abx^2+x(b^2+ac)+bc$, $abx^2+x(b^2-ac)-bc$,
$acx^2+x(ab+bc)+b^2$, $acx^2+x(bc-ab)-b^2$.

4. If $a+b=1$, show that $(a^2-b^2)^2=a^3+b^3-ab$.

5. Resolve into factors

$$(1)\quad x^2-\left(\frac{c}{d}-\frac{d}{c}\right)x-1.$$

$$(2)\quad x^2z^2+y^2w^2-x^2y^2-z^2w^2.$$

6. From the relation $x^3-3x=c^3+\frac{1}{c^3}$, *deduce* that *one* value of x is $c+\frac{1}{c}$.

7. Find the value of x for which the expression $(4m+3)x^2-8m^2x-4m^2+4m+1$ will be divisible by $4m+1$.

8. Simplify

$$\frac{(x+b)(x+c)}{(a-b)(a-c)}+\frac{(x+c)(x+a)}{(b-c)(b-a)}+\frac{(x+a)(x+b)}{(c-a)(c-b)}.$$

Paper VII.

1. Divide $(a+b)x^4+(a^2+2ab+b^2-1)x^3-(a+b)(1-2ab)x^2+(a+b-2ab)x-1$ by $(a+b)x-1$ without removing the brackets.

2. If $2t^2=a^2+b^2+c^2$ and $2s=a+b+c$, prove that $(t^2-a^2)(t^2-b^2)+(t^2-b^2)(t^2-c^2)+(t^2-c^2)(t^2-a^2)=4s(s-a)(s-b)(s-c)$.

3. Find the factors of $xyz(x^3+y^3+z^3)-y^3z^3-z^3x^3-x^3y^3$, a^8-16b^8.

4. For what value of a is $x^2+y^2+2xy-a$ divisible by $x+y-2$?

5. Find the H.C.F. of $x^4-9a^2x^2+10a^3x$ and $ax^3-a^2x^2-4a^4$.

6. Find the L.C.M. of $1+4x+4x^2-16x^4$ and $1+2x-8x^3-16x^4$

7. Find the condition that x^3-px^2+qx-r may be divisible by $x-a$.

8. Simplify :—

$$\frac{1}{(x+3)(x+4)}+\frac{1}{(x+4)(x+5)}+\frac{1}{(x+5)(x+6)}+\frac{1}{(x+6)(x+7)}.$$

Paper VIII.

1. Show that $8(a+b+c)^3=(2a-b+c)^3+(2b-c+a)^3$
$$+(2c-a+b)^3+3(3a+b)(3b+c)(3c+a).$$

2. Resolve into factors :—

(i) $8x^3-5x+3$ (ii) $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$.

3. Find the H.C.F. of $4x^7-4x^6-22x^4+22x^3-18x^2+18x$ and $12x^5+9x^4-66x^3+162x+567$.

4. Find the L.C.M. of x^2+1, $(x-1)^3$, (x^3+1), $(x+1)^2$.

5. If $c^2=a^2+b^2$, prove that

$$(a+b+c)(a+b-c)(a-b+c)(b+c-a)=4a^2(c^2-a^2).$$

6. Find the difference between the squares of

$$x^5+10x^3y^2+5xy^4 \text{ and } 5x^4y+10x^2y^3+y^5.$$

7. If $x+d$ be the H.C.F. of x^2+px+c and x^2+qx+c, their L.C.M. will be $x^3+(2p-d)x^2+\dfrac{c(p+d)}{d}x+\dfrac{c^2}{d}$.

8. Simplify (i) $\dfrac{x+\dfrac{1}{9x}-\dfrac{2}{3}}{x^2-\dfrac{x}{4}-\dfrac{1}{8}}\times\dfrac{x-\dfrac{1}{12x}-\dfrac{1}{3}}{x+\dfrac{1}{15x}-\dfrac{8}{15}}\times\dfrac{x^2+\dfrac{x}{20}-\dfrac{1}{20}}{x-\dfrac{1}{18x}-\dfrac{1}{6}}$.

(ii) $\dfrac{bc(x-a)^2}{(a-b)(a-c)}+\dfrac{ca(x-b)^2}{(b-c)(b-a)}+\dfrac{ab(x-c)^2}{(c-a)(c-b)}$.

PAPER IX.

1. Find what quantity not involving higher powers of x than the second should be added to $x^8-3x^7-5x^5+2x^4+5x^3+4x^2+1$ to make it exactly divisible by x^3+2x-1. B.M. 1897.

2. If $2s=a+b+c$, show that $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3$.

3. Resolve into factors : (1) $4x^3-4x^2(a+b)+x(a^2+4ab)-a^2b$.

(2) $x^2y-x^2z-y^3+z^3+3yz(y-z)$.

4. Show that $(a^3-b^3)^3+(b^3-c^3)^3+(c^3-a^3)^3$ is divisible by

$$a^2+ab+b^2, \text{ or } a^2+ac+c^2, \text{ or } b^2+bc+c^2.$$

5. Find the G.C.M. of

$$112x^4+130x^3+98x^2+20x \text{ and } 231x^5+276x^4+270x^3+60x^2.$$

6. If x^3+px+q and x^3+rx+s have a common factor $x+a$ show that $a=\dfrac{q-s}{p-r}$.

7. Find p when $(x-m)(x-3m)(x+m)(x+3m)+p$ is a perfect square.

8. Simplify (i) $\dfrac{x-5-\dfrac{8}{x+2}}{x+2+\dfrac{6}{x-5}}\times\dfrac{x+4+\dfrac{16}{x-6}}{x-3-\dfrac{12}{x+1}}$.

(ii) $\dfrac{(b+c-a)^2}{(a-b)(a-c)}+\dfrac{(c+a-b)^2}{(b-c)(b-a)}+\dfrac{(a+b-c)^2}{(c-a)(c-b)}$.

PAPER X.

1. If $bc+ca+ab=1$, prove that $(a+b)(a+c)=1+a^2$, $(b+a)(b+c)=1+b^2$, $(c+a)(c+b)=1+c^2$.

2. If $\left(a+\frac{1}{a}\right)^2=3$, prove that $a^3+\frac{1}{a^3}=0$.

3. Resolve $4x^3+\{(x+y)^2-z^2\}(y+5x-z)+4x^2(y+z)$ and $4(a^4+b^4)^2+9a^4b^4$ into factors.

4. Find the H. C. F. of $x^5+y^5+3xy(x^3+y^3)+x^2y^2(x+y)$ and $11x^5-5y^5+xy(11x^3-5y^3)+6x^2y^2(x+y)$.

5. If $x+c$ be the H. C. F. of x^2+ax+q and x^3+rx^2+x+d, show that their L. C. M. will be

$$(x+c)(x+a-c)\left\{x(x-c)+rx+\frac{d}{c}\right\}.$$

6. Show that $(x+y)^3+(y+z)^3+(z+x)^3-3(y+z)(z+x)(x+y)$ $=2(x^3+y^3+z^3-3xyz)$.

7. Simplify (i) $\left(\frac{a+b}{a-b}+\frac{a^2+b^2}{a^2-b^2}\right)\times\left(\frac{a+b}{a-b}+\frac{a^3+b^3}{a^3-b^3}\right)$

(ii) $\frac{(b+c)^2}{(a-b)(a-c)}+\frac{(c+a)^2}{(b-c)(b-a)}+\frac{(a+b)^2}{(c-a)(c-b)}$

8. If $\frac{a}{b}+\frac{c}{d}=\frac{b}{a}+\frac{d}{c}$, then

(i) $\frac{a^2}{b^2}+\frac{b^2}{a^2}=\frac{c^2}{d^2}+\frac{d^2}{c^2}$

(ii) $\frac{a^4}{b^4}+\frac{b^4}{a^4}=\frac{c^4}{d^4}+\frac{d^4}{c^4}$.

PAPER XI.

1. Prove that $(7x^2-10x+3)^3+(5x^2-3x-7)^3$ is divisible by the product of $3x-4$ and $4x+1$.

2. Find the H. C. F. and L. C. M. of x^4-5x^3+4 and $x^3-11x+10$.

3. Find the continued product of $a-b+c+d$, $a+b-c+d$, $a+b+c-d$, $-a+b+c+d$.

4. Prove that $(x+a)^3+(x+b)^3+(x+c)^3-3(x+a)(x+b)(x+c)$ $=(3x+a+b+c)(a^2+b^2+c^2-ab-ac-bc)$.

5. Factorize :—

(i) $3x^3-x^2-6x+2$

(ii) $(2a-5)^4+(2a-5)^2(3a+4)^2+(3a+4)^4$

(iii) $3x^2+7x-(3a+1)(a-2)$

6. Express $(a^2+b^2+c^2)(x^2+y^2+z^2)$ as the sum of four squares.

7. Simplify

(i) $$\frac{bc(x-a)}{(a-b)(a-c)}+\frac{ca(x-b)}{(b-a)(b-c)}+\frac{ab(x-c)}{(c-a)(c-b)}.$$ (C. E. 1896.)

(ii) $$(ab+bc+ca)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

8. Simplify

$$\frac{6x^2y^2}{m+n}\div\left[\frac{3(m-n)x}{7(r+s)}\div\left\{\frac{4(r-s)}{21xy^2}\div\frac{r^2-s^2}{4(m^2-n^2)}\right\}\right].$$ (B.M. 1892).

PAPER XII.

1. Divide (i) $(ax+by+cz)^3+(cx-by+az)^3$ by $(a+c)(x+z)$.

(ii) the product of $ab(x^2+1)+(a^2+b^2)x$ and x^3+1 by that of $(x+1)$ and $(ax+b)$. (M. M. 1890).

2. Prove that

$$\{(ax+by)^2+(ay-bx)^2\}\{(ax+by)^2-(ay+bx)^2\}=(a^4-b^4)(x^4-y^4).$$

3. Find the H. C. F. of $6x^5+35x^4+59x^3+19x^2-17x-6$ and $6x^5-5x^4-41x^3+71x^2-37x+6$.

4. Find the L. C. M. of $x^3-2x^2-19x+20$, $x^3+2x^2-23x-60$, and $x^4+7x^3-4x^2-52x+48$. (B. M. 1891).

5. Reduce to a single term

$$\frac{1}{\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right)}+\frac{1}{\left(1-\frac{c}{b}\right)\left(1-\frac{a}{b}\right)}+\frac{1}{\left(1-\frac{a}{c}\right)\left(1-\frac{b}{c}\right)}$$

6. Simplify

$$\frac{2x^2+5x-3}{3x^2+2x-1}\times\frac{3x^2+5x-2}{6x^2+17x-3}\div\frac{2x^2+3x-2}{6x^2+5x-1}$$

7. Simplify

$$\frac{a+\dfrac{b^2-c^2}{a+2b}}{b+\dfrac{c^2-a^2}{b+2c}}\times\frac{1+\dfrac{b}{a+b}}{1+\dfrac{c}{b+c}}\times\frac{1-\dfrac{a}{b+c}}{1-\dfrac{c}{a+b}}.$$

8. Prove that

(i) $$\frac{2}{b-c}+\frac{2}{c-a}+\frac{2}{a-b}+\frac{(b-c)^2+(c-a)^2+(a-b)^2}{(b-c)(c-a)(a-b)}=0.$$

P. E. 1888.

(ii) $$\left(\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right)^2=\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}$$

CHAPTER XVII.

INVOLUTION AND EVOLUTION.

1. Involution is the process of involving or raising a quantity to a given power, and **evolution** is the inverse process of evolving or extracting any given root of a quantity.

INVOLUTION.

2. The student will see from the rule of signs that *any even power of a quantity, positive or negative, is positive and can never be negative, and that any odd power of a quantity has the same sign as the quantity itself.*

Thus $(+a)^4=+a^4$, $(-a)^4=+a^4$,
$(+a)^3=+a^3$, $(-a)^3=-a^3$.

Also $(-1)^n=+1$ or -1 according as n is even or odd.

3. The following theorems will be useful in involution of simple expressions.

(i) $(a^m)^n=a^{mn}$ where m and n are positive integers.

For, $(a^m)^n=a^m\times a^m\times\ldots$to n factors
$=a^{m+m+\ldots\text{to } n \text{ terms}}$ by index law.
$=a^{mn}$

(ii) $(ab)^n=a^nb^n$.

For, $(ab)^n=ab\times ab\times\ldots$to n factors.
$=(a\times a\times\ldots\text{to } n \text{ factors})\times(b\times b\times\ldots\text{to } n \text{ factors.})$
$=a^n\times b^n$
$=a^nb^n$.

Similarly, $(abc...)^n = a^n b^n c^n ...$

(iii) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

For, $\left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times ...$to n factors

$= \frac{a \times a \times ...\text{to } n \text{ factors}}{b \times b \times ...\text{to } n \text{ factors}}$

$= \frac{a^n}{b^n}$.

Similarly, $\left(\frac{abc...}{xyz...}\right)^n = \frac{a^n b^n c^n ...}{x^n y^n z^n ...}$

From the above we get the following rule :—

To obtain any power of a simple expression multiply the index of every letter in the expression by the index of the power and prefix the proper sign by art. 2.

Obs. When we have to raise a negative quantity to a given power, say, to find $(-a)^n$ we may put $-a = a \times (-1)$. Then $(-a)^n = a^n \times (-1)^n$ by (ii) above

$= (-1)^n a^n$, whether n is odd or even.

Ex. 1. $(2a^2b^3)^2 = 2^2(a^2)^2(b^3)^2$ by (ii)

$= 4a^4b^5$ by (i)

Ex. 2. $\left(-\frac{3a^3b}{c^2d^2}\right)^3 = -\left(\frac{3a^3b}{c^2d^2}\right)^3$ by the rule of signs.

$= -\frac{(3a^3b)^3}{(c^2d^2)^3}$ by (iii)

$= -\frac{3^3(a^3)^3b^3}{(c^2)^3(d^2)^3}$ by (ii)

$= -\frac{27a^9b^3}{c^6d^6}$ by (i)

EXERCISE LXIX.

Write down the squares, as well as the cubes of

1. $3a^3b^2$. **2.** $-2x^3y^2z^4$. **3.** $\frac{2}{3}a^3x^4b^2y$. **4.** $-a^2b^6c^3d^4$.

5. $\frac{2a^2}{bc^2}$. **6.** $-\frac{3a^3b^2}{x^2y^4}$. **7.** $\frac{2a^2b^3c^4}{x^2yz^3}$. **8.** $-\frac{3a^2xy^2}{2b^3c^2x^4}$.

Find the value of

9. $(a^3b^4)^5$ **10.** $(-2x^3y^2)^6$. **11.** $(-a^3b^4c^5)^{10}$

Simplify

12. $\left(\frac{bc}{a}\right)^7 \times \left(\frac{ca}{b}\right)^8 \times \left(\frac{ab^2}{c}\right)^9$ 13. $\left(\frac{ab}{cd}\right)^6 \times \left(\frac{aa}{bc}\right)^4 \times \left(\frac{ac}{bd}\right)^3$.

14. $\left(\frac{a^2b}{c^2d}\right)^5 \times \left(\frac{a^2c^3}{b^2d}\right)^3 \times \left(\frac{ad^2}{b^2c}\right)^4$.

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4. Involution of a binomial. We have already obtained the square or the cube of a bionomial. Thus

$$(a \pm b)^2 = a^2 \pm 2ab + b^2,$$
$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3,$$

upper and lower signs being taken together.

Also $(a+b)^4 = (a+b)^3(a+b)$

$$= (a^3 + 3a^2b + 3ab^2 + b^3)(a+b),$$

or $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Changing b into $-b$, we get

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

Again, $(a+b)^5 = (a+b)^4(a+b)$

$$= (a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)(a+b),$$

or $(a+b)^5 = = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

Changing b into $-b$, we get

$$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

Similarly by continued multiplication the student should get the following :—

$$(a \pm b)^6 = a^6 \pm 6a^5b + 15a^4b^2 \pm 20a^3b^3 + 15a^2b^4 \pm 6ab^5 + b^6.$$
$$(a \pm b)^7 = a^7 \pm 7a^6b + 21a^5b^2 \pm 35a^4b^3 + 35a^3b^4 \pm 21a^2b^5 + 7ab^6 \pm b^7.$$
$$(a \pm b)^8 = a^8 \pm 8a^7b + 28a^6b^2 \pm 56a^5b^3 + 70a^4b^4$$
$$\pm 56a^3b^5 + 28a^2b^6 \pm 8ab^7 + b^8.$$

Note 1. We do not propose to consider here higher powers of a binomial as they can be obtained without this tedious process of multiplication by the Binomial Theorem given in authors' *Intermediate Algebra*. The cases considered above will be sufficient for our present purposes. In the above expansions of the powers of $a \pm b$ the student will mark the following :—

(i) *The number of terms is one more than the index of the power* ;

(ii) the sum of the indices of a and b in any term is equal to the index of the power ; and that the power of a diminishes and that of b increases by one.

(iii) the co-efficients in the expansion of any power of $a-b$ are *numerically* the same as those in the expansion of the same power of $a+b$ but are alternately positive and negative.

(iv) As regards co-efficients of different terms let us observe one expansion, say $(a+b)^6$. The co-efficient of 1st term $a^6=1$, that of the 2nd term in which a^5 occurs$=\frac{1\times 6}{1}$, that of the 3rd term in which a^4 occurs in $\frac{6\times 5}{2}$ or 15 and that of the 4th term containing a^3 is $\frac{15\times 4}{3}$ or 20 and so on.

The co-efficient of the first term is 1; and that of any other term = co-eff. of preceding term × the index of the power of a there ÷ number of terms preceding. The co-efficients from the beginning and the end are equal. Same is the law as regards co-efficients for all binomial expansions for all positive integral powers.

Note 2. It may be observed that $(a+b)^4=\{(a+b)^2\}^2$ *i.e.* $(a+b)^4$ may be obtained by squaring the square of $a+b$ or by squaring $a^2+2ab+b^2$. Similarly $(a+b))^6$ may be obtained by squaring the cube of $a+b$ *i. e.* by squaring $a^3+3a^2b+3ab^2+b^3$.

Ex. $(2x-3y)^6=(2x)^6-6(2x)^5(3y)+15(2x)^4(3y)^2-20(2x)^3(3y)^3$ $+15\,(2x)^2(3y)^4-6(2x)(3y)^5+(3y)^6$

$=64x^6-576x^5y+2160x^4y^2-4320x^3y^3+4860x^2y^4-2916xy^5+729y^6.$

Observe that in place of a and b in the expansion of $(a-b)^6$ above we have here $(2x)$ and $(3y)$.

EXERCISE LXX.

Find the value of

1. $(x+2)^4$.	**2.** $(1+x)^5$.	**3.** $(x-2)^4$.	**4.** $(1-x)^5$.
5. $(a+2b)^4$.	**6.** $(2a-x)^4$.	**7.** $(2a+3b)^4$.	**8.** $(1-2x)^5$.
9. $(3x-4y)^4$	**10.** $(a+2b)^5$.	**11.** $(1+x)^7$.	**12.** $(1-x)^8$.
13. $(2a-5b)^4$.		**14.** $(4x^2-3y^3)^6$.	

Simplify

15. $(a+b)^4+(a-b)^4$. **16.** $(a+b)^5-(a-b)^5$.

17. $(2x-3y)^4+(2x+3y)^4$. **18.** $(1+x)^6+(1-x)^6$.

5. Square of any polynomial. We have already found the square of a trinomial. Thus

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$$

Again, $(a+b+c+d)^2=\{(a+b)+(c+d)\}^2$

$$=(a+b)^2+2(a+b)(c+d)+(c+d)^2$$
$$=a^2+2ab+b^2+2ac+2ad+2bc+2bd+c^2+2cd+d^2$$
$$=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

It appears from the above that the square of a polynomial is equal to the sum of the squares of all its terms together with twice the product of every two of them.

This can be proved from the distributive law of multiplication thus :—

We have $(a+b+c+d+......)^2=(a+b+c+d+......)$
$\times(a+b+c+d+......)$;

and the product is formed by multiplying *each* term of the first expression $a+b+c+d......$by *each* term of the second expression $a+b+c+d+......$Thus, $a, b, c, d......$of the first expression multiplied by $a, b, c, d......$respectively of the second gives $a^2, b^2, c^2, d^2......$as terms of the product. Also a of the first expression multiplied by b of the second gives ab which is again obtained by multiplying b of the first expression by a of the second *i.e* ab occurs *twice* in the product. Similarly the product of any two of the letters, $a, b, c, d...$ occurs twice. Hence the above rule.

6. It is evident that twice the sum of the products of every two of the terms of a polynomial is secured by taking together twice the product of each term of the polynomial and the sum of all the terms following it. Hence the preceding rule for the square of a polyno- may be stated thus :

The square of a polynomial is equal to the sum of the squares of all its terms together with twice the product of each term and the sum of all the terms which follow it.

Thus $(a+b+c+d+e+......)^2$

$$=a^2+b^2+c^2+d^2+e^2+......$$
$$+2a(b+c+d+e+......)+2b(c+d+e+.........)$$
$$+2c(d+e+......)+2d(e+.........)+.........$$

Ex. 1. $(1+3x-2x^2)^2$

$$=1+9x^2+4x^4+2.1.(3x-2x^2)+2.3x(-2x^2)$$
$$=1+9x^2+4x^4+6x-4x^2-12x^3.$$
$$=1+6x+5x^2-12x^3+4x^4.$$

Ex. 2. $(2x^3-\frac{3}{2}x^2+x-3)^2=4x^6+\frac{9}{4}x^4+x^2+9$
$+2.2x^3(-\frac{3}{2}x^2+x-3)-2.\frac{3}{2}x^2(x-3)+2.x(-3)$
$=4x^6+\frac{9}{4}x^4+x^2+9-6x^5+4x^4-12x^3-3x^3+9x^2-6x$
$=4x^6-6x^5+\frac{25}{4}x^4-15x^3+10x^2-6x+9.$

Ex. 3. Find the co efficient of x^4 in
$(2-3x+3x^2-5x^3+6x^4+7x^5)^2$.

We have $(2-3x+3x^2-5x^3+6x^4+7x^5)^2$
$=4+9x^2+9x^4+\ldots+4(-3x+3x^2-5x^3+6x^4+\ldots\ldots)$
$-6x(3x^2-5x^3+\ldots\ldots)+\ldots\ldots\ldots$

We omit terms which contain higher powers of x than x^4. We find the terms containing x^4 are $9x^4$, $4\times 6x^4, -6x\times(-5x^3)$; hence the co-efficient of $x^4=9+24+30=63$.

7. Cube of a polynomial. We have already seen that

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3,$$
$$(a+b+c)^3=a^3+b^3+c^3+3(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc)$$
$$\text{or } =a^3+b^3+c^3+3(b+c)(c+a)(a+b).$$

Again, $(a+b+c+d)^3=\{(a+b)+(c+d)\}^3$
$=(a+b)^3+3(a+b)^2(c+d)+3(a+b)(c+d)^2+(c+d)^3$
$=a^3+3a^2b+3ab^2+b^3+3(a^2+2ab+b^2)(c+d)$
$+3(a+b)(c^2+2cd+d^2)+c^3+3c^2d+3cd^2+d^3$
$=a^3+b^3+c^3+d^3+3a^2(b+c+d)+3b^2(c+d+a)$
$+3c^2(d+a+b)+3d^2(a+b+c)+6(abc+bcd+cda+dab).$

EXERCISE LXXI.

Find the value of

1. $(2a-3b+4c-5d)^2$ **2.** $(1-2x+3x^2-4x^3)^2$

3. $\left(\frac{a}{b}+\frac{b}{a}-1\right)^2$. **4.** $\left(\frac{2x}{y}+\frac{y}{3x}+2\right)^2$.

5. $(1-a+b-c+d)^2$. **6.** $(2x^4-3x^3+x^2-2x+3)^2$

7. $\left(\frac{3x}{4y}-2\right)^3$. **8.** $\left(a-\frac{b^2}{c}\right)^3$.

9. $\left(x-\frac{2}{x}\right)^3$. **10.** $\left(\frac{a}{b}-x+\frac{b}{a}\right)^3$.

11. $(2x-3y+4z)^3$ **12.** $(1-2x+3x^2)^3$

13. $(a^2+ab+b^2)^3$. **14.** $(3x^3-4x^2+2x-1)^3$.

15. Find the co-efficient of x^5 in $(1-3x+5x^2-7x^3+9x^4-11x^5)^2$.

16. Find the co-efficient of x^4 in $(1-2x+3x^2-4x^3)^3$.

17. Find the value of $(a+b)^8$ by squaring the fourth power of $a+b$.

18. Find the value of $(a+b)^6$ by squaring the cube of $(a+b)$.

19. Prove that $(a+b+c)^3 + a^3+b^3+c^3-(b+c)^3-(c+a)^3$
$-(a+b)^3=6abc$.

20. Prove that $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3$
$=24abc$.

EVOLUTION.

8. We know any even power of a quantity, positive or negative is positive and cannot be negative; hence *an* **even** *root of a* **positive** *quantity may be positive or negative, while an* **even** *root of a* **negative** *quantity is an impossibility* and is called an unreal or imaginary quantity.

Again, we know that any odd power of a quantity has the same sign as the quantity; hence *an odd root of a quantity has the same sign as the quantity.*

Thus since $(+a)\times(+a)=a^2$, as also $(-a)\times(-a)=a^2$ we have $\sqrt{a^2}=+a$ or $-a$.

Also $\sqrt{(-a^2)}$ is an impossible quantity, for no quantity multiplied by itself can be negative $(-a^2)$. $\sqrt{-a^2}$ is therefore an unreal or imaginary quantity.

Again, since $(-a)\times(-a)\times(-a)=-a^3$, we have $\sqrt[3]{(-a^3)}=-a$.

9. Evolution of simple expressions. We know that

$$(a^m)^n=a^{mn}, \qquad \therefore \sqrt[n]{a^{mn}}=a^m;$$

$$(ab)^n=a^nb^n, \qquad \therefore \sqrt[n]{a^nb^n}=ab;$$

$$\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}, \qquad \therefore \sqrt[n]{\frac{a^n}{b^n}}=\frac{a}{b}.$$

Hence to find any root of a simple expression divide the index of every letter in the expression by the index of the root and prefix the proper sign or signs according to art. 8.

Ex. 1. $\sqrt[2]{(9x^4y^2z^6)}=\sqrt[2]{(3^2x^4y^2z^6)}=\pm 3x^2yz^3$.

Ex. 2. $\sqrt[3]{(-8a^3b^6c^9)}=\sqrt[3]{(-2^3a^3b^6c^9)}=-2ab^2c^3$.

Ex. 3. $\sqrt[2]{\left(\frac{16a^4b^2}{25c^6}\right)}=\pm\frac{4a^2b}{5c^3}$.

Note. In the remainder of the chapter when we have to extract an even root of a quantity we shall suppose the signs $\pm$ prefixed to the result.

EXERCISE LXXII.

Find the square root of

1. $25a^4b^2c^6$. 2. $121x^4y^2z^8$. 3. $16a^2b^2c^6$. 4. $9x^6y^4z^8$.

5. $\dfrac{9a^2b^2}{16x^2y^4}$. 6. $\dfrac{36x^6y^8}{49a^4z^2}$. 7. $\dfrac{169a^2b^4c^8}{4x^4y^2z^4}$. 8. $\dfrac{16x^4y^4z^6}{25a^2b^6c^4}$.

Find the cube root of

9. $8a^3b^6$ 10. $-27a^6z^3$ 11. $125a^3x^9$ 12. $-64a^9x^6$.

13. $-\dfrac{27x^6y^6}{125a^9z^3}$. 14. $\dfrac{216a^6x^9}{b^3y^{12}}$. 15. $-\dfrac{a^3b^6}{27c^9}$. 16. $\dfrac{64x^9y^3z^6}{343a^3b^6c^{12}}$.

Find the value of

17. $\sqrt[4]{(81x^4y^8)}$. 18. $\sqrt[5]{(32a^{10}y^{15})}$ 19. $\sqrt[5]{(-243a^5b^{10})}$.

20. $\sqrt[4]{\left(\dfrac{a^4b^{12}}{16c^8}\right)}$ 21. $\sqrt[5]{\left(\dfrac{3125a^{10}}{b^5c^{15}}\right)}$ 22. $\sqrt[6]{\left(\dfrac{64x^6y^{12}}{729z^{18}}\right)}$.

23. $\sqrt[4]{\left(\dfrac{625a^4y^8}{81x^4}\right)}$ 24. $\sqrt[7]{\left(\dfrac{128a^7}{b^{14}c^{21}}\right)}$ 25. $\sqrt[8]{\left(\dfrac{x^8y^{16}}{z^{24}}\right)}$.

---

10. Square roots of polynomials. We shall here use a method of inspection. We know that

$$(a+b)^2=a^2+2ab+b^2,\ (a-b)^2=a^2-2ab+b^2.$$

Hence we can extract the square root of an expression which can be put in the form—*sum of the squares of two quantities plus or minus twice their product.*

Ex. 1. Extract the square root of $4x^2-12xy+9y^2$.

The expr. $=(2x)^2-2.2x.3y+(3y)^2=(2x-3y)^2$

$\therefore$ the square root required $=2x-3y$.

Ex. 2. Extract the square root of $9a^2+b^2+4c^2-6ab+12ac-4bc$

The expr. $=9a^2-6a(b-2c)+(b^2-4bc+4c^2)$

[arranging in powers of a]

$=(3a)^2-2\ 3a(b-2c)+(b-2c)^2$

$=\{3a-(b-2c)\}^2=(3a-b+2c)^2$

$\therefore$ the square root required $=3a-b+2c$.

Ex. 3. Extract the sqnare root of $(a^2+b^2)^2+4ab(a^2-b^2)$.

We have $(a^2+b^2)^2=(a^2-b^2)^2+4a^2b^2$,

$\therefore$ the given expr. $=(a^2-b^2)^2+4a^2b^2+4ab(a^2-b^2)$

$=(a^2-b^2+2ab)^2$.

$\therefore$ the square root $=a^2+2ab-b^2$.

Ex. 4. Find the square root of

$$\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12 \qquad \text{(C. E. 1866).}$$

$$\text{The expr.}=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}+2\right)+12$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}\right)+4$$

$$=\left(x^2+\frac{1}{x^2}-2\right)^2$$

$\therefore$ the square root required $=x^2+\frac{1}{x^2}-2.$

Ex. 5. Find the square root of $(a+c)^2+4(a-2b)(2a-2b+c)$.

The expr. $=(a+c)^2+4(a-2b)\,\{(a+c)+(a-2b)\}$

$=(a+c)^2+4(a-2b)\,(a+c)+4(a-2b)^2$

$=\{(a+c)+2(a-2b)\}^2=(3a-4b+c)^2$

$\therefore$ the square root required $=3a-4b+c$.

Ex. 6. Prove that the product of any four consecutive integers increased by unity is a perfect square.

Let x be the smallest of the consecutive integers; then we are to prove that $x(x+1)\,(x+2)\,(x+3)+1$ is a perfect square.

We have $x(x+1)\,(x+2)\,(x+3)+1$

$=(x^2+3x)\,(\overline{x^2+3x}+2)+1$

$=(x^2+3x)^2+2\,(x^2+3x)+1$

$=(x^2+3x+1)^2=$ a perfect square.

EXERCISE LXXIII.

Find by inspection the square root of

1. $25a^2b^2+30abcd+9c^2d^2$.
2. $49a^4x^2-140a^2bxy^2+100b^2y^4$.
3. $(5x-2y)^2+40xy$. 4. $(7a+9b)^2-252ab$.
5. $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)$ (P. E. 1887.)
6. $(x^2+3xy+2y^2)(x^2+5xy+6y^2)(x^2+4xy+3y^2)$.
7. $(x^2-3x+2)(x^2-x-2)(x^2-1)$.
8. $\left(x^2+\frac{1}{x^2}\right)^2-6\left(x+\frac{1}{x}\right)^2+21$.
9. $(ab+ac+bc)^2-4abc(a+c)$. (C. E. 1888.)

Find by inspection the square root of

10. $\dfrac{(a^2+b^2)^2}{(a^4+b^4-2a^2b^2)}+4\dfrac{a}{a+b}\times\dfrac{b}{a-b}$. (C. E. 1886).

11. $x^4+x^2yz+\dfrac{y^2z^2}{4}-2x^2z^2-yz^3+z^4$. (B. M. 1892).

12. $a^4+b^4+c^4+d^4-2(a^2+c^2)(b^2+d^2)+2a^2c^2+2b^2d^2$.

13. $3(3a^2-2ab+b^2)(a^2+3b^2)+b^2(a+4b)^2$. (M. M. 1898).

14. $(a^2+b^2+c^2)(x^2+y^2+z^2)-(ay-bx)^2-(cx-az)^2-(bz-cy)^2$.

15. $\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}+3+\dfrac{2a}{b}+\dfrac{2b}{a}$.

16. $(as+bc)(bs+ca)(cs+ab)$ where $s=a+b+c$.

17. $(x+1)(x+3)(x+5)(x+7)+16$.

18. $(x+2)(x+3)(x+8)(x+9)+9$.

19. $(x+1)(x+2)(x+6)(x+7)+(x+4)^2$.

20. $(ax+by+cz)^2-4ax(by+cz)$.

---

11. Square roots of polynomials. (*Continued*).

Let us consider the expression $a^2+2ab+b^2$, of which the square root is $a+b$ and try to discover the terms a and b of the square root.

The expression being arranged in descending powers of a, the first term a of the root is the square root of the first term a^2 of the expression. Subtract a^2, the square of this first term of the root, from the expresssion, and bring down the remainder $+2ab+b^2$, as shown above. Put down $2a$ (*i. e.* twice the first term of the root) as the first term of the divisor of the remainder, and divide by it the first term of the remainder *viz.* $2ab$, and set down the quotient b (the quotient of the first term of the remainder by the first term of the divisor) as the second term of the root, as also the second (and last) term of the divisor. Multiply the *complete* divisor $2a+b$ thus obtained by the second term b of the root, and subtract the product from the remainder, when nothing is left.

$$\begin{array}{r|l} & a^2+2ab+b^2\,(a+b \\ & a^2 \\ \hline 2a+b) & +2ab+b^2 \\ & 2ab+b^2 \\ \hline \end{array}$$

If there were more terms in the expression, we would proceed by regarding the first two terms of the root obtained by the previous method as a *single* term and treating it as we treated the first term in the above process ; and repeat the process until the square root is obtained.

Note. We shall not forget to arrange an expression of which the square root is wanted in ascending or descending powers of some letter, as the above method depends upon this arrangement.

Ex. 1. Extract the square root of $9a^2-30a+25$.

The expr. is already in descending powers of a.

$$\begin{array}{r|l} & 9a^2-30a+25\,(3a-5 \\ & 9a^2 \\ \hline 6a-5) & -30a+25 \\ & -30a+25 \\ \hline \end{array}$$

The first term of the root $=\sqrt{9a^2}=3a$. Subtracting its square or $9a^2$ from the expression, the remainder $=-30a+25$. The first term of the divisor $=2\times 3a=6a$, hence the second term of the root (and also of the divisor) $=-30a\div 6a=-5$. Thus tne complete divisor $=6a-5$, and multiplying it by -5 and setting down the prodnct below the remainder, we find, nothing is left on subtraction. Hence the root required $=3a-5$.

Ex. 2. Extract the squre root of
$4x^4+25x^2+16-12x^3-24x$.

We begin by arranging the expr. in descending powers of x.

$$\begin{array}{r|l} & 4x^4-12x^3+25x^2-24x+16\,(2x^2-3x+4 \\ & 4x^4 \\ \hline 4x^2-3x) & -12x^3+25x^2-24x+16 \\ & -12x^3+9x^2 \\ \hline 4x^2-6x+4) & +16x^2-24x+16 \\ & 16x^2-24x+16 \\ \hline \end{array}$$

Here we arrange the expression in descending powers of x. The first term of the root is $2x^2$, the square root of $4x^4$. Subtracting $4x^4$ from the given expression, the remainder $=-12x^3+25x^2-\ldots$The first term of the divisor is $2\times 2x^2$ or $4x^2$, and the second term of the root $=(-12x^3)\div 4x^2=-3x$, so that $4x^2-3x$, is the complete divisor. Multiply it by $-3x$, the second term of the root, and subtracting the product from the first remainder we get the second remainder $=16x^2-\ldots$Doubling $2x^2-3x$, the portion of the root already found, we get $4x^2-6x$, which is a part of the second divisor. The third term of the root $=$ quotient of the second remainder by the second partial divisor or more simply the quotient of their **first** terms and is in this case $16x^2\div 4x^2=4$. Hence the complete second divisor $=4x^2-6x+4$. Multiply this by 4, the third term of the root and subtract the product from the second remainder when nothing is left.

Thus the square root required $=2x^2-3x+4$.

Ex. 3. Extract the square root of

$$x^6-6x^5a+15x^4a^2-20x^3a^3+15x^2a^4-6xa^5+a^6.$$

$$\begin{array}{l|l}
 & x^6-6x^5a+15x^4a^2-20x^3a^3+15x^2a^4-6xa^5+a^6 \;\Big[\; x^3-3x^2a+3xa^2-a^3 \\
 & x^6 \\ \hline
2x^3-3x^2a & -6x^5a+15x^4a^2 \\
 & -6x^5a+\;\;9x^4a^2 \\ \hline
2x^3-6x^2a+3xa^2 & \quad 6x^4a^2-20x^3a^3+15x^2a^4 \\
 & \quad 6x^4a^2-18x^3a^3+\;\;9x^2a^4 \\ \hline
2x^3-6x^2a+6xa^2-a^3 & -2x^3a^3+6x^2a^4-6xa^5+a^6 \\
 & -2x^3a^3+6x^2a^4-6xa^5+a^6 \\ \hline
\end{array}$$

Here first term of the root $=\sqrt{x^6}$ or x^3; second term $=-6x^5a\div 2x^3=-3x^2a$; third term $=6x^4a^2\div 2x^3=3xa^2$; fourth term $=-2x^3a^3\div 2x^3=-a^3$. Thus we see that practically division of the first terms of the several remainders by the same quantity $(2x^3)$ determines the terms of the square root.

Ex. 4. Extract the square root of

$$\frac{4a^2}{b^2}+\frac{b^2}{9a^2}+\frac{16}{3}+\frac{8a}{b}+\frac{4b}{3a}.$$

We arrange the expression as below and the student will see afterwards (Chap. XXVI) that this arrangement is in descending powers of a; the terms containing $\frac{1}{a}$, $\frac{1}{a^2}$ come after $\frac{16}{3}$ which does not contain a.

$$\begin{array}{l|l}
 & \frac{4a^2}{b^2}+\frac{8a}{b}+\frac{16}{3}+\frac{4b}{3a}+\frac{b^2}{9a^2}\;\Big(\;\frac{2a}{b}+2+\frac{b}{3a}. \\
 & \frac{4a^2}{b^2} \\ \hline
\frac{4a}{b}+2 & +\frac{8a}{b}+\frac{16}{3} \\
 & \quad\frac{8a}{b}+4 \\ \hline
\frac{4a}{b}+4+\frac{b}{3a} & +\frac{4}{3}+\frac{4b}{3a}+\frac{b^2}{9a^2} \\
 & \quad\frac{4}{3}+\frac{4b}{3a}+\frac{b^2}{9a^2} \\ \hline
\end{array}$$

Here first term of the root $=\sqrt{\left(\frac{4a^2}{b^2}\right)}=\frac{2a}{b}$, second term

$$-\frac{8a}{b}\div\frac{4a}{b}=2, \text{ third term}=\frac{4}{3}\div\frac{4a}{b}=\frac{b}{3a}.$$

Ex. 5. For what value of x will $16x^4-24x^3+25x^2-15x-8$ be a perfect square?

We proceed to extract the square root.

$$
\begin{array}{r|l}
 & 16x^4-24x^3+25x^2-15x-8\ \big(\,4x^2-3x+2 \\
 & 16x^4 \\
\hline
8x^2-3x & -24x^3+25x^2 \\
 & -24x^3+\ \ 9x^2 \\
\hline
8x^2-6x+2 & 16x^2-15x-8 \\
 & 16x^2-12x+4 \\
\hline
 & -3x-12
\end{array}
$$

Hence $-3x-12=0$ or $x=-4$.

Ex. 6. Extract the square root of $4+x$ to 4 terms.

$$
\begin{array}{r|l}
 & 4+x\ \Big(\,2+\frac{x}{4}-\frac{x^2}{64}+\frac{x^3}{512} \\
 & 4 \\
\hline
4+\frac{x}{4} & +x \\
 & x+\frac{x^2}{16} \\
\hline
4+\frac{x}{2}-\frac{x^2}{64} & -\frac{x^2}{16} \\
 & -\frac{x^2}{16}-\frac{x^3}{128}+\frac{x^4}{4096} \\
\hline
4+\frac{x}{2}-\frac{x^2}{32}+\frac{x^3}{512} & \frac{x^3}{128}-\frac{x^4}{4096} \\
 & \frac{x^3}{128}+\frac{x^4}{1024}-\ldots \\
\hline
 & -\frac{5x^4}{4096}+\ldots
\end{array}
$$

Thus the required square root $=2+\frac{x}{4}-\frac{x^2}{64}+\frac{x^3}{512}$.

EXERCISE LXXIV.

Extract the square root of

1. $x^4+4x^3-2x^2-12x+9$.
2. $9x^4+24x^3+4x^2-16x+4$.
3. $\frac{4}{9}a^2+\frac{1}{25}b^2+\frac{1}{4}c^2-\frac{4}{15}ab-\frac{1}{5}bc+\frac{2}{3}ca$.
4. $1+4x-2x^2-12x^3+9x^4$.

Extract the square root of

5. $\frac{x^2}{9}+9y^2+\frac{9}{4}-2xy-x+9y.$

6. $49x^6+42x^5-47x^4-136x^3-32x^2+64x+64.$

7. $16x^6-40x^5y+73x^4y^2-116x^3y^3+106x^2y^4-84xy^5+49y^6.$

8. $\frac{9a^2}{b^2}+12+\frac{b^2}{4a^2}+\frac{18a}{b}+\frac{3b}{a}.$

9. $\frac{a^4}{b^4}+\frac{b^4}{a^4}-4\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+6.$

10. $x^6+2x^4+4x^3+x^2+4x-\frac{4}{x^2}-\frac{8}{x^3}+\frac{4}{x^6}.$ (M. M. 1895).

11. $\frac{a^6}{b^6}+\frac{b^6}{a^6}+6\left(\frac{a^4}{b^4}+\frac{b^4}{a^4}\right)+3\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)-20.$

12. $\frac{a^2}{4b^2}+\frac{4b^2}{9c^2}+\frac{9c^2}{a^2}+\frac{2a}{3c}+\frac{3c}{b}+\frac{4b}{a}.$

13. Find the condition that x^2+px+q may be a perfect square.

14. Determine the values of x for which the following expressions are perfect squares :—

(i) $9x^4-12x^3-2x^2+7x-9.$

(ii) $9x^6+12x^5-20x^4+14x^3+36x^2-45x+35.$

15. Extract the square root of

(i) $1+x$ to 5 terms (ii) $1-2x$ to 4 terms

(iii) $1-x+x^2$ to 4 terms (iv) $1+2x-3x^2$ to 4 terms

16. Find what term is wanting to make the following expression a complete square :—

$a^2x^4+64b^2-4(ax^2+8b)(a-b)x.$ (M. M. 1875).

---

12. Cube roots of polynomials. We here propose to find the cube root of a polynomial by inspection. We know

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$ or $a^3+b^3+3ab(a+b)$......(1)

$(a-b)^3=a^3-3a^2b+3ab^2-b^3$ or $a^3-b^3-3ab(a-b)$......(2)

Hence we can extract cube roots of expressions of the forms on the right of (1) and (2). Observe that if the expressions are arranged in ascending or descending powers of some letter, the terms of the cube root are the cube roots of the first and the last terms.

Ex. 1. Extract the cube root of $8x^3-36x^2y+54xy^2-27y^3$

The Expr. $=(2x)^3-3(2x)^2.3y+3.\ 2x(3y)^2-(3y)^3=(2x-3y)^3$

$\therefore$ the cube root required $=2x-3y$.

Ex. 2. Extract the cube root of

$a^3-8b^3+27c^3-6a^2b+12ab^2+36b^2c-54bc^2+9a^2c+27ac^2-36abc$.

Arranging in descending powers of a, the expr.

$$=a^3-3a^2(2b-3c)+3a(4b^2-12bc+9c^2)$$
$$-(8b^3-36b^2c+54bc^2-27c^3)$$
$$=a^3-3a^2(2b-3c)+3a(2b-3c)^2-(2b-3c)^3$$
$$=\{a-(2b-3c)\}^3=(a-2b+3c)^3.$$

$\therefore$ the required cube root $=a-2b+3c$.

13. Cube roots of polynomials. *(Continued).* Let us consider the expression $a^3+3a^2b+3ab^2+b^3$ and try to discover its cube root $a+b$.

$$\begin{array}{l|l}
 & a^3+3a^2b+3ab^2+b^3\ (a+b \\
 & a^3 \\
\hline
3\times a^2=3a^2 & \quad +3a^2b+3ab^2+b^3 \\
3a\times b=3ab & \\
\quad b^2=\qquad b^2 & \\
\hline
3a^2+3ab+b^2 & \quad +3a^2b+3ab^2+b^3 \\
\hline
\end{array}$$

The expr. being arranged in descending powers of a, the first term a of the cube root is the cube root of the first term a^3 of the expression. Substract a^3, the cube of the first term of the root, from the given expression and bring down the remainder $3a^2b+3ab^2+b^3$, as shown above. Put down $3a^2$ (*i.e.* 3 times the square of the first term of the root) as the first term of the divisor of the remainder, divide by it the first term of the remainder *viz.*, $3a^2b$ and put down the quotient b as the second term of the required root. Take $3ab$ (*i.e.* 3 times the product of the first and second terms of the required root) as the second term of the divisor and b^2 (*i.e.* the square of the second term ot the root) as the third (and last) term of the divisor, so that the *complete* divisor is $3a^2+3ab+b^2$. Multiply it by b, the second term of the root and subtract the product from the remainder when nothing is left.

If there were more terms in the expression, we would proceed by regarding the first two terms of the root obtained by the previous method as a *single* term and treating it as we treated the first term in the above process ; and repeat the process until the cube root is obtained.

Ex. 1. Find the cube root of $a^3-6a^2b+12ab^2-8b^3$

$$\begin{array}{l|l}
 & a^3-6a^2b+12ab^2-8b^3 \,\big(\, a-2b \\
 & a^3 \\ \hline
3\times a^2=3a^2 & -6a^2b+12ab^2-8b^3 \\
3a\times(-2b)=-6ab & \\
(-2b)^2=\ \ 4b^2 & \\ \hline
3a^2-6ab+4b^2 & -6a^2b+12ab^2-8b^3 \\ \hline
\end{array}$$

Here the first term of the root $=\sqrt[3]{a^3}=a$, the second term $=-6a^2b\div3a^2=-2b$.

Ex. 2. Find the cube root of $x^6-63x^5+33x^4-63x^3+66x^2-36x+8$

$$\begin{array}{l|l}
 & x^6-9x^5+33x^4-63x^3+66x^2-36x+8\,\big(\, x^2-3x+2 \\
 & x^6 \\ \hline
3\times(x^2)^2+3\times x^2 & -9x^5+33x^4-63x^3 \\
\times(-3x)+(-3x)^2 & -9x^5+27x^4-27x^3 \\
=3x^4-9x^3+9x^2 & \\ \hline
3(x^2-3x)^2+3(x^2-3x)\times2 & +6x^4-36x^3+66x^2-36x+8 \\
+2^2=3x^4-18x^3+33x^2 & +6x^4-36x^3+66x^2-36x+8 \\
-18x+4 & \\ \hline
\end{array}$$

Here the first term of the root $=\sqrt[3]{x^6}=x^2$; first remainder $=-9x^5+\ldots$; the second term of the root $=-9x^5\div\{3\times(x^2)^2\}=-3x$, the first complete divisor being thus $3\times(x^2)^2+3\times x^2\times(-3x)+(-3x)^2=3x^4-9x^3+9x^2$.

The second remainder $=6x^4-36x^3+\ldots$, the third term of the root $=$quotient of $6x^4-36x^3+\ldots$ by $3(x^2-3x)^2$ or more simply the quotient of their *first* terms and is therefore $6x^4\div3x^4$ or 2.

Hence the second complete divisor $=3(x^2-3x)^2+3(x^2-3x)\times2+2^2=3x^4-18x^3+33x^2-18x+4$. Hence etc.

14. From the known expansion of $(a+b)^5$ we can determine the fifth-root of an expression in the manner of articles 11 and 13 but the process will be tedious. It may be noted that the fourth root of an expression is the square root of its square root, the sixth root of an expression is the square root of its cube root or the cube root of its square root, and the eighth root of an expression is the square root of the square root of the square root of the expression.

Thus to find the sixth root of

$$x^6-6x^5a+15x^4a^2-20x^3a^3+15x^2a^4-6xa^5+a^6.$$

The square root of the expr. $=x^3-3x^2a+3xa^2-a^3$ (ex. 3, art. 11); hence its sixth root $=\sqrt[3]{(x^3-3x^2a+3xa^2-a^3)}$.

$$=x-a.$$

EXERCISE LXXV.

Find the cube root of

1. $8a^3-36a^2b+54ab^2-27b^3$.
2. $64x^3+144x^2y+108xy^2+27y^3$.
3. $8a^3-12a^2+6a-1$.
4. $x^3-6x+\dfrac{12}{x}-\dfrac{8}{x^3}$.
5. $\dfrac{8x^3}{27}-\dfrac{16x^2}{3}+32x-64$.
6. $\dfrac{x^3}{y^3}-\dfrac{y^3}{x^3}-3\left(\dfrac{x}{y}-\dfrac{y}{x}\right)$.
7. $\dfrac{a^6}{b^3}+\dfrac{6a^4c}{b^2}+\dfrac{12a^2c^2}{b}+8c^3$
8. $\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}-3\left(\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}\right)+5$. B. M. 1892.
9. $8x^6+x^3-64+6x(2x^2-7x-4)(x^2-2)$.
10. $x^6-3x^5+6x^4-7x^3+6x^2-3x+1$.
11. $64x^6-144x^5y+204x^4y^2-171x^3y^3+102x^2y^4-36xy^5+8y^6$.
12. $x^6+3x^5y+6x^4y^2+7x^3y^3+6x^2y^4+3xy^5+y^6$.
13. $\dfrac{x^3}{64}-\dfrac{3x^2}{16}+\dfrac{3x}{2}-7+\dfrac{24}{x}-\dfrac{48}{x^2}+\dfrac{64}{x^3}$.
14. $\dfrac{x^6}{y^6}-\dfrac{6x^5}{y^5}+21\dfrac{x^4}{y^4}-44\dfrac{x^3}{y^3}+63\dfrac{x^2}{y^2}-54\dfrac{x}{y}+27$.
15. Find to 3 terms the cube root of
    (i) $1+x$ (ii) $8-x$.
16. Find the value of x which makes $8x^3-12x^2+9x-10$ a perfect cube.
17. Find the fourth root of:—
    (i) $x^4+8x^3+24x^2+32x+16$
    (ii) $x^4+\dfrac{1}{x^4}+4\left(x^2+\dfrac{1}{x^2}\right)+6$.
18. Find the sixth root of :—
    (i) $x^6+6x^5+15x^4+20x^3+15x^2+6x+1$
    (ii) $x^6+\dfrac{64}{x^6}-12x^4-\dfrac{192}{x^4}+60x^2+\dfrac{240}{x^2}-160$.

15. The Arithmetical rules for extracting the square and cube roots of numbers are based upon the Algebraical methods of articles 11 and 13 respectively.

Suppose we want the square root of 401956. We proceed thus as in art. 11 :—

$$\begin{array}{r|l}
 & 401956\,(600+30+4 \\
 & \underline{360000} \\
1200+30 & 41956 \\
=1230 & \underline{36900} \\
1260+4 & 5056 \\
=1264 & \underline{5056}
\end{array}$$

Here the square root evidently consists of 3 figures. The greatest number of hundreds whose square is less than the given number is 6, so that 600 is the first term of the root. Subtracting its square from the given number, the remainder $=41956$. The second term of the root is the quotient of 41956 by 2×600 *i.e.* 30; hence the first divisor $=2\times600+30=1230$. The second remainder is 5056 and the third term of the root is the quotient of 5056 by $2(600+30)$ or by 1260; hence it is 4. Multiplying the second divisor 1264 by 4 and subtracting the product from the second remainder, nothing is left. Hence the square root $=600+30+4$ $=634$.

The above leads to the following compact process in Arithmetic :—

$$\begin{array}{r|l}
 & 4\dot{0}19\dot{5}6\,(634 \\
 & \underline{36} \\
123 & 419 \\
 & \underline{369} \\
1264 & 5056 \\
 & \underline{5056}
\end{array}$$

The dots placed over every second figure from the unit indicate the number of figures in the root.

CHAPTER XVIII.

SIMPLE EQUATIONS.

1. In Chap. X we have considered easy simple equations with one unknown quantity. We reproduce the general rule given there for the solution of a simple equation :—

(i) **Clear the equation of fractions (if necessary) by multiplying both sides by the L. C. M. of the denominators;**

(ii) transpose all the terms involving the unknown quantity to the left-hand side and the known quantity to the right-hand;

(iii) simplify both sides by collecting terms;

(iv) divide both sides by the co-efficient of the unknown quantity.

Thus, to solve $\frac{5(x-1)}{4}+\frac{x}{3}+1=\frac{7+x}{6}$.....................(1)

Multiplying both sides by 12 to clear of fractions,

$15(x-1)+4x+12=2(7+x)$, or $15x-15+4x+12=14+2x$,

$\therefore$ transposing, $15x+4x-2x=14+15-12$,

or $17x=17$, whence $x=1$.

2. Verification. The student should *verify* the solution of an equation which he works out, for that is the test of the accuracy of his work. To verify a solution is to substitute for the unknown quantity the value obtained and to prove that the two sides of the equation become equal.

Thus in the above equation, putting $x=1$,

left-side$=\frac{5\times 0}{4}+\frac{1}{3}+1=1\frac{1}{3}$; right-side$=\frac{7+1}{6}=\frac{8}{6}=\frac{4}{3}=1\frac{1}{3}$.

Hence $x=1$ satisfies the equation and the solution is correct.

Note. It is a common mistake with the beginner in solutions of equations, to put the sign = in the beginning of every line. Thus to solve the equation $3x-8=2-2x$ he is apt to write

$3x-8=2-2x$

$=3x+2x=2+8$

$=5x=10$, whence $x=2$.

Here the signs = in the beginning of the last two lines should be replaced by $\therefore$, or, *or*.

3. Literal equations. In equations known quantities may be denoted by letters and they are treated like ordinary numbers, as shown below.

Ex. Solve $bx-3a=6ax-2b$.

Transposing, $bx-6ax=-2b+3a$, or, $x(b-6a)=3a-2b$

$$\therefore\ x=\frac{3a-2b}{b-6a}.$$

EXERCISE LXXVI.

Solve

1. $x-a=2x-b$.
2. $a(x+b)=c(x+d)$.
3. $(m-n)(x-m)-(m-p)(x-n)=0$.

Solve

4. $(x+1)+(2x+a)=2(2x+2)-3$. 5. $(x-a)+(x-b)+(x-c)=0$.

6. $x(m+2)-6=m(x+6)-2$. 7. $ax-b^2=b(x-b)+a(a-b)$.

8. $a(x-a-b)=x+a(a+b-2)-2b$.

---

4. **Equations not involving fractions.** The following examples illustrate the method to be employed.

Ex. 1. Solve $(4x+3)^2-(2x-1)^2=12(x+2)^2-60$.

Expanding, $(16x^2+24x+9)-(4x^2-4x+1)=12(x^2+4x+4)-60$;

simplifying, $12x^2+28x+8=12x^2+48x-12$;

transposing, $12x^2+28x-12x^2-48x=-12-8$, or $-20x=-20$.

Dividing by -20, $x=1$, [verify].

The following example shows how an equation may be conveniently reduced to a simpler one

Ex. 2. Solve $(3x+1)^2+(2x-2)^2=(3x-2)^2+(2x+3)^2$

By transposition $(3x+1)^2-(3x-2)^2=(2x+3)^2-(2x-2)^2$

$$\therefore \{(3x+1)+(3x-2)\}\{(3x+1)-(3x-2)\}$$
$$=\{(2x+3)+(2x-2)\}\{(2x+3)-(2x-2)\}$$

$\therefore$ $(6x-1)3=(4x+1)5$, or, $18x-3=20x+5$

$\therefore$ $-2x=8$ (transposing) $\therefore$ $x=-4$.

Ex. 3. Solve $(x-a)(x-b)-c(b+c)=(x+c)(x-c)-bc$.

Multiplying out, $x^2-ax-bx+ab-bc-c^2=x^2-c^2-bc$,

$$\therefore \ -ax-bx=-ab \text{ or } -x(a+b)=-ab$$

$$\therefore \ x=\frac{-ab}{-(a+b)}=\frac{ab}{(a+b)}.$$

EXERCISE LXXVII.

Solve

1. $(x-3)^2+(x-4)^2+(x-5)^2=3x^2-22$.
2. $(x+3)^2+3(x+4)^2=4x^2+57$.
3. $(x+4)(x-3)+(x+5)(x-1)=2(x+5)(x-2)+2$.
4. $(2x+3)(3x-4)-(x+7)(2x-4)=(2x+1)(2x-2)+4$.
5. $(3x+5)^2-(3x-7)^2=(3x+9)^2-(3x-11)^2$.
6. $(2x+1)^3=2x(2x+1)(2x+2)$.
7. $(2x+1)^4=(4x^2-1)^2+16x^2(2x+2)+7$.

Slove

8. $(x+b)(x+a)=(x+c)(x+d)$. **9.** $(x+a)(a-bx)=(bx+a)(a-x)$

10. $a(2x+1)^2-(2ax+1)(2x+a)=0$.

11. $(x+a)(x-b)=(x-b+a)^2$, **12.** $(m+x)(n+x)=x(x-p)$.

---

5. Equations involving fractions. The following are some easy equations involving fractions. The convenient method here is, as we already know, to clear each equation of fractions by multiplying its terms by the L. C. M. of the denominators. When co-efficients are decimal fractions, they may be retained throughout the work if convenient or changed into vulgar fractions.

Ex. 1. Slove $\dfrac{2x-3}{6}+\dfrac{3x-8}{11}=\dfrac{4x+15}{33}+\dfrac{1}{2}$. (C. E. 1887.)

Multiplying both sides by 66, the L. C M. of denominators to clear of fractions,

$$22x-33+18x-48=8x+30+33.$$

Transposing, $22x+18x-8x=30+33+48+33$,

$$\text{or } 32x=144\ ;\ \therefore x=\frac{144}{32}=\frac{9}{2}=4\tfrac{1}{2}.$$

Ex. 2. Slove $\dfrac{5}{x^2+x-6}+\dfrac{7}{x^2-x-12}=\dfrac{4}{x^2-6x+8}$.

We have $\dfrac{5}{(x-2)(x+3)}+\dfrac{7}{(x+3)(x-4)}=\dfrac{4}{(x-4)(x-2)}$.

Multiplying by $(x-2)(x+3)(x-4)$ to clear of fractions, we have

$$5(x-4)+7(x-2)=4(x+3).$$
$$\therefore\ 5x-20+7x-14=4x+12,$$
$$\therefore\ 5x+7x-4x=12+20+14,$$
$$\therefore\ 8x=46\ ;\ \therefore\ x=\tfrac{23}{4}=5\tfrac{3}{4}.$$

Ex. 3. Slove $1{\cdot}\dot{3}x+{\cdot}5-\frac{2}{9}x=3{\cdot}125-1{\cdot}5x-\frac{2}{3}$.

Expressing the decimals as vulgar fractions,

$$\tfrac{4}{3}x+\tfrac{1}{2}-\tfrac{2}{9}x=\tfrac{25}{8}-\tfrac{3}{2}x-\tfrac{2}{3},$$

Multiplying both sides by 72,

$$96x+36-16x=225-108x-48,$$
$$\therefore\ 96x-16x+108x=225-48-36,$$
$$\therefore\ 188x=141 \text{ whence } x=\tfrac{141}{188}=\tfrac{3}{4}={\cdot}75.$$

Ex. 4. Solve $\cdot65x+\frac{\cdot585x-\cdot975}{\cdot6}=\frac{1\cdot56}{\cdot2}-\frac{\cdot39x-\cdot78}{\cdot9}$ (C.E. 1882).

Dividing each term of both sides by $\cdot13$,

$$5x+\frac{4\cdot5x-7\cdot5}{\cdot6}=\frac{12}{\cdot2}-\frac{3x-6}{\cdot9}$$

Multiplying both sides by $1\cdot8$, the L.C.M. of the denominators,

$$9x+13\cdot5x-22\cdot5=108-6x+12,$$
$$\therefore\ 9x+13\cdot5x+6x=108+12+22\cdot5$$
$$\text{or } 28\cdot5x=142\cdot5, \text{ whence}$$
$$x=5.$$

Ex. 5. Solve $ax-\frac{a^2-3bx}{a}+\frac{5a^2-6bx}{2a}=bx+ab^2-\frac{bx+4a}{4}$

Here $ax-a+\frac{3bx}{a}+\frac{5}{2}a-\frac{3bx}{a}=bx+ab^2-\frac{bx}{4}-a$

Cancelling and transposing $ax-\frac{3bx}{4}=ab^2-\frac{5}{2}a$.

Multiplying by 4, $4ax-3bx=4ab^2-10a$, or, $x(4a-3b)=4ab^2-10a$,

$$\therefore x=\frac{4ab^2-10a}{4a-3b}.$$

EXERCISE LXXVIII.

Solve

1. $\frac{2x+4}{3}-\frac{40-3x}{7}=9-\frac{11x-8}{4}$.

2. $\frac{5x-28}{7}+\frac{31}{4}+\frac{6x-21}{3}=\frac{7x-6}{4}-\frac{12-6x}{6}$.

3. $\frac{3x-2}{5}+\frac{4x-1}{7}-\frac{10x}{9}=5(x-9)+3-\frac{x}{3}$. (M. M. 1891.)

4. $x-\frac{x-2}{2}=5\frac{3}{4}-\frac{x+10}{5}+\frac{x-2}{4}$. (M. M. 1883.)

5. $\frac{21}{4}\left(\frac{2x}{3}-\frac{5}{18}\right)+\frac{7x-3\frac{3}{4}}{12}=2\frac{19}{144}-\frac{14-15x}{3}$. (B. M. 1900.)

6. $\frac{7x+9}{8}-\frac{3x+1}{7}=\frac{9x-13}{4}-\frac{249-9x}{14}$.

7. $\frac{7x+5}{23}+\frac{9x-1}{10}-\frac{x-9}{5}+\frac{2x-3}{15}=\frac{70}{3}$.

8. $\frac{15x-2}{13}+\frac{10x-1}{7}-1=\frac{\frac{45x}{2}}{35}-\frac{237}{455}$.

Solve

9. $\dfrac{\frac{9}{44}x-2}{21}-\dfrac{3+\frac{3}{11}x}{9}=\dfrac{\frac{x}{11}-8}{3}.$

10. $\dfrac{2}{x^2+5x+6}+\dfrac{3}{x^2+9x+18}+\dfrac{1}{x^2+8x+12}=0.$

11. $\dfrac{4}{(x-2)(x+4)}+\dfrac{3}{(x+4)(x-1)}+\dfrac{5}{(x-1)(x-2)}=0.$

12. $2{\cdot}4x-\dfrac{({\cdot}36x-{\cdot}1)}{{\cdot}5}={\cdot}8x+17{\cdot}8.$

13. $\dfrac{5{\cdot}2x}{13}-\dfrac{1{\cdot}2x}{5}\left(\dfrac{3}{5}-{\cdot}1\right)={\cdot}1x-\dfrac{5x-2}{4}+{\cdot}028.$

14. $1{\cdot}5(x-3{\cdot}\dot{6})-{\cdot}\dot{1}4285\dot{7}(5-4x)=x+12-\dfrac{3x-(3-{\cdot}\dot{6}x-2)}{6}.$

15. ${\cdot}5x+\dfrac{{\cdot}02x+{\cdot}07}{{\cdot}03}-\dfrac{x+2}{9}=9{\cdot}5.$ (C. E. 1866).

16. ${\cdot}011x+\dfrac{{\cdot}001x-{\cdot}125}{{\cdot}6}=\dfrac{5-x}{{\cdot}03}-{\cdot}145.$ (C. E. 1886).

17. $\dfrac{1{\cdot}05x+10}{50}+\dfrac{1{\cdot}35x-2}{20}-\dfrac{1{\cdot}5x-18}{10}+\dfrac{1{\cdot}5x-3}{15}=1{\cdot}854$ (B.M. 1902)

18. $\dfrac{a}{b}(x-a)+\dfrac{b}{a}(x-b)=x.$

19. $\dfrac{x-a_1}{a_1}+\dfrac{x-a_2}{a_2}+\dfrac{x-a_3}{a_3}=1.$

20. $a-\frac{1}{3}(x-a)=3b-\frac{1}{5}(x-b).$

21. $\dfrac{x^2-b^2}{ax}-\dfrac{b-x}{a}=\dfrac{2x}{a}-\dfrac{b}{x}.$

22. $\dfrac{x-(l+m+n)}{lmn}-\dfrac{x-l}{m}=\dfrac{x-m}{n}+\dfrac{x-n}{l}.$

23. $\dfrac{x+b}{a}+\dfrac{x+a}{b}+\dfrac{x+c}{a}+\dfrac{x+a}{c}+\dfrac{x+c}{b}+\dfrac{x+b}{c}=0.$

24. $\dfrac{a}{b}\left(1-\dfrac{a}{x}\right)+\dfrac{b}{a}\left(1-\dfrac{b}{x}\right)=1.$

25. $\frac{2}{3}\{x-(2a-3c)\}-\frac{5}{36}\{7a-5(x-2c)\}=\frac{1}{24}\{8(a+10c)-(2c-x)\}.$

6. Equations of the form $\frac{A}{B}=\frac{C}{D}$.

An equation of the type $\frac{A}{B}=\frac{C}{D}$ (both sides being multiplied by BD) can be transformed into AD=BC, a form in which fractions are avoided ; and since AD, BC are obtained by multiplying the terms of $\frac{A}{B}$, $\frac{C}{D}$ cross-wise, the process is called multiplying cross-wise (See Theor. 1, Art. 18, Chap. XVI).

Ex. Solve $\frac{5x-4}{3x-4}=\frac{10x-2}{6x-5}$.

Multiplying cross-wise

$$(5x-4)(6x-5)=(10x-2)(3x-4),$$

or $30x^2-49x+20=30x^2-46x+8$,

or $-49x+46x=-20+8$,

or $-3x=-12$.

$$\therefore\ x=\frac{-12}{-3}=4.$$

7. Judicious combination of terms.

In the following fractional equations terms having denominators which are the same or simple multiples of the same quantity are taken together and simplified.

Ex. 1. Solve $\frac{1}{x-1}-\frac{2}{x+7}=\frac{1}{7(x-1)}$.

By transposition, $\frac{1}{x-1}-\frac{1}{7(x-1)}=\frac{2}{x+7}$.

Simplifying, $\frac{6}{7(x-1)}=\frac{2}{x+7}$.

or $\frac{3}{7(x-1)}=\frac{1}{x+7}$.

Multiplying across, $7x-7=3x+21$

or $4x=28$, whence $x=7$.

Ex. 2. Solve $\frac{4(x+3)}{9}+\frac{7x-29}{5x-12}=\frac{8x+37}{18}$.

Transposing, $\frac{7x-29}{5x-12}=\frac{8x+37}{18}-\frac{4(x+3)}{9}$,

or $\frac{7x-29}{5x-12}=\frac{8x+37-8x-24}{18}$,

$$\therefore\ \frac{7x-29}{5x-12}=\frac{13}{18}.$$

$\therefore$ Multiplying cross-wise,

$$18(7x-29)=13(5x-12),$$
$$\therefore 126x-522=65x-156,$$
$$\therefore 126x-65x=522-156,$$
or $61x=366$ whence $x=6$.

Ex. 3. Solve

$$\frac{17x+27}{8}-\frac{5-34x}{21}=\frac{8x+5}{7}+\frac{6x+11}{4}.$$

Transposing, $\dfrac{17x+27}{8}-\dfrac{6x+11}{4}=\dfrac{5-34x}{21}+\dfrac{8x+5}{7}.$

$$\therefore \frac{17x+27-12x-22}{8}=\frac{5-34x+24x+15}{21}.$$

$$\therefore \frac{5x+5}{8}=\frac{-10x+20}{21}, \text{ or } \frac{x+1}{8}=\frac{-2x+4}{21}.$$

Multiplying cross-wise,

$$21x+21=-16x+32,$$
or $21x+16x=32-21$.

$\therefore 37x=11$ whence $x=\frac{11}{37}$.

EXERCISE LXXIX.

Solve :—

1. $\dfrac{24}{x-4}=\dfrac{45}{x+3}.$ 2. $\dfrac{6x-5}{x+1}=\dfrac{6(11x-5)}{11x+17}.$

3. $\dfrac{4x-5}{2x-3}=\dfrac{6x-7}{3x-4}.$ 4. $\dfrac{x-a}{x-b}=\dfrac{x-c}{x-d}.$

5. $\dfrac{x+a}{x+b}=\dfrac{x+3a}{x+a+b}.$

6. $\dfrac{7x+6}{28}-\dfrac{2x+4\frac{2}{7}}{23x-6}+\dfrac{x}{4}=\dfrac{11x}{21}-\dfrac{x-3}{42}.$

7. $\dfrac{4x+3}{9}+\dfrac{29-7x}{12-5x}=\dfrac{8x+19}{18}.$ (C. E. 1868.)

8. $\dfrac{10x+47}{18}-\dfrac{12x+38}{13x+23}=\dfrac{5x+11}{9}.$ (M. M. 1871.)

9. $\dfrac{29x-17}{9}-\dfrac{3x-4}{7}=\dfrac{27-19x}{35}+\dfrac{8x-5}{3}.$

10. $\dfrac{23x-51}{6}+\dfrac{8x+11}{5}=\dfrac{17x+29}{11}-\dfrac{61-25x}{7}.$

Solve :—

11. $\dfrac{2x+1}{29} - \dfrac{402-3x}{12} = \dfrac{6x-453}{2}$.

12. $\dfrac{6x+18}{13} - \dfrac{11-3x}{36} = 5x - 43\frac{1}{6} - \dfrac{13-x}{12} - \dfrac{21-2x}{18}$.

13. $\dfrac{9x+5}{14} + \dfrac{8x-7}{6x+2} = \dfrac{36x+15}{56} + \dfrac{10\frac{1}{4}}{14}$.

14. $\dfrac{6x-7\frac{1}{3}}{13-2x} + 2x + \dfrac{1+16x}{24} = 4\dfrac{5}{12} - \dfrac{12\frac{5}{8}-8x}{3}$

15. $\dfrac{x-\frac{1}{2}}{x-1} - \frac{3}{5}\left(\dfrac{1}{x-1} - \dfrac{1}{3}\right) = \dfrac{23}{10(x-1)}$. (A. E. 1891.)

16. $\dfrac{4-3x}{1\frac{1}{3}-x} + \dfrac{2\frac{1}{2}}{2x+1} = \dfrac{x+\frac{1}{3}}{2x+1} - \dfrac{21}{10}$.

17. $\dfrac{4\cdot05}{9x} - \dfrac{\cdot3}{\cdot8-2x} = \dfrac{1\cdot8}{x} - \dfrac{3\cdot6}{2\cdot4-6x}$. (C. E. 1881.)

18. $\dfrac{16x-27\frac{4}{5}}{3x-4} + \dfrac{77-x}{3(x-1)} = 5 + \dfrac{23}{x-1}$. (M. M. 1882.)

19. $\dfrac{5}{2x+7} - \dfrac{1}{4(5x-1)} = \dfrac{4}{5x-1} + \dfrac{3}{4(2x+7)}$.

20. $\dfrac{3}{x-\frac{7}{6}} - \dfrac{6+x}{3x+1\frac{3}{5}} = \dfrac{25-3x}{9x-7} - \dfrac{12}{5x+2\frac{2}{3}}$.

21. $\dfrac{bp-d}{p(cx+d)} + p - p\cdot\dfrac{x^2+x+1}{x(x+1)} = p\cdot\dfrac{x^2+x-1}{x(x+1)} - \dfrac{1}{p}$.

---

8. Judicious combination of terms (*continued*). The following examples illustrate further the process of judicious combination of terms by transposition, if necessary.

Ex. 1. Solve $\dfrac{3}{6x+1} + \dfrac{8}{4x+7} = \dfrac{6}{3x+5} + \dfrac{1}{2x+1}$.

Transposing, $\dfrac{3}{6x+1} - \dfrac{1}{2x+1} = \dfrac{6}{3x+5} - \dfrac{8}{4x+7}$.

$\therefore$ Simplifying, $\dfrac{2}{(6x+1)(2x+1)} = \dfrac{2}{(3x+5)(4x+7)}$.

$\therefore (6x+1)(2x+1) = (3x+5)(4x+7)$,

$12x^2 + 8x + 1 = 12x^2 + 41x + 35$.

$\therefore$ transposing and cancelling,

$-33x = 34$ or $x = -\dfrac{34}{33}$.

Obs. In the above example we so combine the terms in groups of two that each group when simplified contains no x in the numerator.

Ex. 2. Solve $\dfrac{m-n}{x+p}+\dfrac{p-q}{x+n}=\dfrac{m-n}{x+q}+\dfrac{p-q}{x+m}$.

Transposing, $\dfrac{m-n}{x+p}-\dfrac{m-n}{x+q}=\dfrac{p-q}{x+m}-\dfrac{p-q}{x+n}$,

or $\dfrac{(m-n)(q-p)}{(x+p)(x+q)}=\dfrac{(p-q)(n-m)}{(x+m)(x+n)}$.

Cancelling the factors $m-n$ and $q-p$,

$$\frac{1}{(x+p)(x+q)}=\frac{(-1)\times(-1)}{(x+m)(x+n)},$$

$$\text{or}\quad \frac{\quad}{(x+p)(x+q)}=\frac{\quad}{(x+m)(x+n)},$$

or $(x+p)(x+q)=(x+m)(x+n)$,

or $x^2+(p+q)x+pq=x^2+(m+n)x+mn$.

Transposing, $(p+q-m-n)x=mn-pq$.

$$\therefore\ x=\frac{mn-pq}{p+q-m-n}.$$

Ex. 3. Solve $\dfrac{1}{x^2+3x+2}+\dfrac{1}{x^2+4x+3}=\dfrac{1}{x^2+5x+4}+\dfrac{1}{x^2+5x+6}$.

Here $\dfrac{1}{(x+1)(x+2)}+\dfrac{1}{(x+1)(x+3)}$

$$=\frac{1}{(x+1)(x+4)}+\frac{1}{(x+2)(x+3)}.$$

$\therefore$ transposing,

$$\frac{\quad}{(x+1)(x+2)}-\frac{\quad}{(x+1)(x+4)}=\frac{\quad}{(x+2)(x+3)}-\frac{\quad}{(x+1)(x+3)}$$

$$\therefore\ \frac{(x+4)-(x+2)}{(x+1)(x+2)(x+4)}=\frac{(x+1)-(x+2)}{(x+1)(x+2)(x+3)},$$

$$\therefore\ \frac{2}{(x+1)(x+2)(x+4)}=\frac{-1}{(x+1)(x+2)(x+3)}.$$

$\therefore$ Multiplying by $(x+1)(x+2)$,

$\dfrac{2}{x+4}=\dfrac{-1}{x+3}$ or $2x+6=-x-4$;

$\therefore\ 3x=-10$, or $x=-3\frac{1}{3}$.

9. Judicious breaking up of terms. The following examples illustrate the method to be followed in some fractional equations.

Ex. 1. Solve $\dfrac{5}{x+3}+\dfrac{7}{x+6}=\dfrac{12}{x+11}$.

Here $\dfrac{5}{x+3}+\dfrac{7}{x+6}=\dfrac{5}{x+11}+\dfrac{7}{x+11}$,

or $\dfrac{5}{x+3}-\dfrac{5}{x+11}=\dfrac{7}{x+11}-\dfrac{7}{x+6}$,

or $5\left(\dfrac{1}{x+3}-\dfrac{1}{x+11}\right)=7\left(\dfrac{1}{x+11}-\dfrac{1}{x+6}\right)$,

or $\dfrac{5\times 8}{(x+3)(x+11)}=\dfrac{7\times(-5)}{(x+11)(x+6)}$.

$\therefore\ \dfrac{8}{x+3}=\dfrac{-7}{x+6}\left(\text{multiplying both sides by } \dfrac{x+11}{5}\right)$

$\therefore\ 8x+48=-7x-21$,
or $15x=-69$,

whence $x=-\dfrac{69}{15}=-\dfrac{23}{5}$.

Obs. In this and the following examples we so break up the term on the right that on transformation and grouping two together each group when simplified contains no x in the numerator.

Ex. 2. Solve $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{a+b}{x+c}$.

Here $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{a}{x+c}+\dfrac{b}{x+c}$,

Transposing, $\dfrac{a}{x+a}-\dfrac{a}{x+c}=\dfrac{b}{x+c}-\dfrac{b}{x+b}$.

or $a\left\{\dfrac{c-a}{(x+a)(x+c)}\right\}=b\left\{\dfrac{b-c}{(x+c)(x+b)}\right\}$.

Concelling $x+c$ from both sides and multiplying across,

$$a(c-a)(x+b)=b(b-c)(x+a).$$

$$\therefore\ (ac-a^2)x+ab(c-a)=(b^2-bc)x+ab(b-c).$$

Transposing, $(ac+bc-a^2-b^2)x=ab(b-2c+a)$,

or, $x=\dfrac{(b-2c+a)ab}{ac+bc-a^2-b^2}$.

N.B.—Examples like the above may be worked out also by ordinary method—by simplifying the left hand side and multiplying across.

Ex. 3. Solve $\frac{2}{2x-3}+\frac{6}{3x+2}=\frac{12}{4x+7}$.

Here $\frac{2}{2x-3}+\frac{6}{3x+2}=\frac{4}{4x+7}+\frac{8}{4x+7}$.

Transposing, $\frac{2}{2x-3}-\frac{4}{4x+7}=\frac{8}{4x+7}-\frac{6}{3x+2}$,

or $\frac{26}{(2x-3)(4x+7)}=\frac{-26}{(4x+7)(3x+2)}$.

Cancelling $\frac{26}{4x+7}$ from both sides, $\frac{1}{2x-3}=\frac{-1}{3x+2}$.

Multiplying across, $3x+2=-2x+3$.

Transposing, $5x=1$. $\therefore x=\frac{1}{5}$.

Ex. 4. Solve $\frac{pa}{ax+b}+\frac{qb}{bx+c}=\frac{pc+qc}{cx+d}$.

Here $\frac{pa}{ax+b}+\frac{qb}{bx+c}=\frac{pc}{cx+d}+\frac{qc}{cx+d}$,

or $\frac{pa}{ax+b}-\frac{pc}{cx+d}=\frac{qc}{cx+d}-\frac{qb}{bx+c}$.

or $\frac{pad-pbc}{(ax+b)(cx+d)}=\frac{qc^2-qbd}{(cx+d)(bx+c)}$.

$\therefore \frac{pad-pbc}{ax+b}=\frac{qc^2-qbd}{bx+c}$, multiplying both sides by $cx+d$,

$\therefore (bx+c)(pad-pbc)=(ax+b)(qc^2-qbd)$, multiplying across,

$\therefore x(pabd-pb^2c)+pacd-pbc^2=(qc^2a-qabd)x+b(qc^2-qbd)$,

Transposing, $x(pabd-pb^2c-qc^2a+qabd)$
$=(bqc^2-qb^2d-pacd+pbc^2)$.

$\therefore x=\frac{bqc^2-qb^2d-pacd+pbc^2}{pabd-pb^2c-qc^2a+qabd}$.

Ex. 5. Solve $\frac{10}{2x-1}+\frac{27}{3x+2}+\frac{14}{2x+1}=\frac{63}{3x+1}$.

Puttting $\frac{63}{3x+1}=\frac{15}{3x+1}+\frac{27}{3x+1}+\frac{21}{3x+1}$, and transposing,

$$\left(\frac{10}{2x-1}-\frac{15}{3x+1}\right)+\left(\frac{27}{3x+2}-\frac{27}{3x+1}\right)=\left(\frac{21}{3x+1}-\frac{14}{2x+1}\right).$$

$$\therefore \frac{25}{(2x-1)(3x+1)}-\frac{27}{(3x+2)(3x+1)}=\frac{7}{(3x+1)(2x+1)}.$$

$\therefore$ Multiplying by $3x+1$, the equation becomes

$$\frac{25}{2x-1}-\frac{27}{3x+2}=\frac{7}{2x+1}\ldots(1)$$

Again, putting $\frac{7}{2x+1}=\frac{25}{2x+1}-\frac{18}{2x+1}$ in (1) and transposing the given equation becomes

$$\frac{25}{2x-1}-\frac{25}{2x+1}=\frac{27}{3x+2}-\frac{18}{2x+1},$$

$$\text{or}\ \frac{50}{(2x-1)(2x+1)}=\frac{-9}{(3x+2)(2x+1)}.$$

Removing $2x+1$ from the bottom and multiplying cross-wise,

$$50(3x+2)=-9(2x-1),$$

$$\therefore\ 150x+100=-18x+9,$$

$$\therefore\ 168x=-91\ \text{ or }\ x=-\frac{13}{24}.$$

EXERCISE LXXX.

Solve

1. $\frac{1}{x+12}+\frac{1}{x+4}=\frac{1}{x+6}+\frac{1}{x+10}$.

2. $\frac{1}{x-6}+\frac{1}{x-3}=\frac{1}{x-2}+\frac{1}{x-7}$.

3. $\frac{8}{4x+3}+\frac{3}{6x+13}=\frac{1}{2x+5}+\frac{6}{3x+2}$.

4. $\frac{1}{2x-7}+\frac{3}{3x-5}=\frac{8}{8x-15}+\frac{6}{12x-37}$.

5. $\frac{18}{3x+11}+\frac{42}{6x+10}=\frac{7}{x+1}+\frac{12}{2x+9}$.

6. $\frac{51}{3x+7}+\frac{44}{4x-9}=\frac{33}{3x-11}+\frac{68}{4x+13}$.

7. $\frac{53}{4x-1}+\frac{160}{16x-9}=\frac{30}{3x-5}+\frac{159}{12x+7}$.

8. $\frac{1}{x+mb}+\frac{1}{x+md}=\frac{1}{x+mb+c}+\frac{1}{x+md-c}$.

9. $\frac{1}{x^2+5x+6}+\frac{1}{x^2+6x+8}=\frac{1}{x^2+7x+10}+\frac{1}{x^2+7x+12}$.

10. $\frac{1}{(x+3)(x+4)}+\frac{2}{(x+3)(x+5)}=\frac{1}{(x+3)(x+6)}+\frac{2}{(x+4)(x+5)}$,

Solve

11. $\dfrac{1}{(x+l)^2-m^2}+\dfrac{1}{(x+m)^2-l^2}=\dfrac{1}{x^2-(l+m)^2}+\dfrac{1}{x^2-(l-m)^2}$.

12. $\dfrac{2}{x+1}+\dfrac{6}{x+5}=\dfrac{8}{x+3}$.

13. $\dfrac{4}{x-4}+\dfrac{3}{x+2}=\dfrac{7}{x-2}$.

14. $\dfrac{3}{3x+1}-\dfrac{5}{7-3x}=\dfrac{8}{3x-19}$.

15. $\dfrac{18}{6-14x}+\dfrac{2}{14x+30}=\dfrac{16}{24-14x}$.

16. $\dfrac{21}{3x-7}-\dfrac{5}{x+3}=\dfrac{2}{x-5}$.

17. $\dfrac{12}{3x-8}=\dfrac{20}{4x-13}-\dfrac{1}{x+9}$.

18. $\dfrac{3}{10x+9}+\dfrac{4}{45x+2}=\dfrac{7}{18x+5}$. (M. M. 1884.)

19. $\dfrac{5}{5x+3}+\dfrac{8}{4x-1}=\dfrac{18}{6x+1}$.

20. $\dfrac{3}{3x-1}+\dfrac{5}{5x-1}+\dfrac{4}{1-2x}=0$.

21. $\dfrac{1}{x-1}+\dfrac{12}{2x+5}=\dfrac{2}{x+6}+\dfrac{25}{5x+2}$.

22. $\dfrac{18}{2x-1}+\dfrac{2}{2x+1}=\dfrac{1}{x-1}+\dfrac{27}{3x-1}$.

23. $\dfrac{a-b}{x-a}-\dfrac{b-a}{x-b}=\dfrac{2(a-b)}{x-(a+b)}$.

24. $\dfrac{2a-3b}{x-a+b}-\dfrac{2b-3a}{x+a-b}=\dfrac{5(a-b)}{x+a+b}$.

25. $\dfrac{a}{x+b}+\dfrac{b}{x+a}=\dfrac{a+b}{x+c}$.

26. $\dfrac{l}{lx+m}+\dfrac{m}{mx+n}=\dfrac{2}{x+p}$.

27. $\dfrac{b-c}{x+a}+\dfrac{a-b}{x+b}=\dfrac{a-c}{x+c}$.

28. $\dfrac{a-c}{2b+x}+\dfrac{b-c}{2a+x}=\dfrac{a+b-2c}{a+b+x}$.

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10. Reduction by Division.

In the following fractional equations each numerator (if not of lower dimension) is divided by the corresponding denominator.

Ex. 1. Solve $\dfrac{4x+6}{2x+1}-\dfrac{5x+3}{5x+2}=1$.

Here $\dfrac{4x+2+4}{2x+1}-\dfrac{5x+2+1}{5x+2}=1$,

or $2+\dfrac{4}{2x+1}-1-\dfrac{1}{5x+2}=1$.

Cancelling 1 from both sides, and transposing

$$\frac{4}{2x+1}=\frac{1}{5x+2}.$$

Multiplying across, $20x+8=2x+1$ or $18x=-7$; $\therefore\ x=-\frac{7}{18}$.

Ex. 2. Solve $\dfrac{3x+4}{3x+1}-\dfrac{12-3x}{7-3x}=\dfrac{8}{3x-19}$.

Here $1+\dfrac{3}{3x+1}-1-\dfrac{5}{7-3x}=\dfrac{8}{3x-19}$.

$\therefore\ \dfrac{3}{3x+1}-\dfrac{5}{7-3x}=\dfrac{8}{3x-19}$.

$\therefore$ by transposition $\dfrac{3}{3x+1}-\dfrac{3}{3x-19}=\dfrac{5}{7-3x}+\dfrac{5}{3x-19}$ [see art. 9].

$\therefore\ \dfrac{-60}{(3x+1)(3x-19)}=\dfrac{-60}{(7-3x)(3x-19)}$.

Cancelling $\dfrac{-60}{3x-19}$ from both sides, $\dfrac{1}{3x+1}=\dfrac{1}{7-3x}$.

$\therefore\ 3x+1=7-3x$, or $6x=6$.

$\therefore\ x=1$.

Ex. 5. Solve $\dfrac{x^2-1}{x-2}+\dfrac{2x^2-13x+11}{x-6}=\dfrac{3x^2+10x+11}{x+3}$.

Here $x+2+\dfrac{3}{x-2}+2x-1+\dfrac{5}{x-6}=3x+1+\dfrac{8}{x+3}$.

$\therefore\ \dfrac{3}{x-2}+\dfrac{5}{x-6}=\dfrac{8}{x+3}$ or $\dfrac{3}{x-2}-\dfrac{3}{x+3}=\dfrac{5}{x+3}-\dfrac{5}{x-6}$,

or $\dfrac{15}{(x-2)(x+3)}=\dfrac{-45}{(x+3)(x-6)}$.

Cancelling $\dfrac{15}{x+3}$ from both sides, $\dfrac{1}{x-2}=\dfrac{-3}{x-6}$.

$\therefore\ -3x+6=x-6$, or $-4x=-12$.

$\therefore\ x=3$.

Ex. 4. Solve $\dfrac{x-2}{x-3}-\dfrac{x-10}{x-11}=\dfrac{x}{x-1}-\dfrac{x-8}{x-9}$.

Here $1+\dfrac{1}{x-3}-1-\dfrac{1}{x-11}=1+\dfrac{1}{x-1}-1-\dfrac{1}{x-9}$.

$\therefore\ \dfrac{1}{x-3}-\dfrac{1}{x-11}=\dfrac{1}{x-1}-\dfrac{1}{x-9}$,

or $\dfrac{-8}{(x-3)(x-11)}=\dfrac{-8}{(x-1)(x-9)}$.

$\therefore\ (x-3)(x-11)=(x-1)(x-9)$,

or $x^2-14x+33=x^2-10x+9$,

or $-4x=-24$; $\therefore\ x=6$.

Ex. 5. Solve $\frac{x+2a}{x+a}+\frac{x+2b}{x+b}=\frac{2x+a+b+2c}{x+c}$.

Here $1+\frac{a}{x+a}+1+\frac{b}{x+b}=2+\frac{a+b}{x+c}$.

$\therefore \frac{a}{x+a}+\frac{b}{x+b}=\frac{a+b}{x+c}$.

Now see example 2, art. 9.

Ex. 6. Solve $\frac{x+a}{x+b}=\left(\frac{2x+a+c}{2x+b+c}\right)^2$.

Subtracting 1 from both sides,

$$\frac{x+a}{x+b}-1=\frac{(2x+a+c)^2-(2x+b+c)^2}{(2x+b+c)^2}.$$

$$\therefore \frac{a-b}{x+b}=\frac{(4x+a+b+2c)(a-b)}{(2x+b+c)^2}.$$

Multiplying both sides by $\frac{2x+b+c}{a-b}$,

$$\frac{2x+b+c}{x+b}=\frac{4x+a+b+2c}{2x+b+c}.$$

$$\therefore \text{ dividing, } 2+\frac{c-b}{x+b}=2+\frac{a-b}{2x+b+c}.$$

$$\therefore \frac{c-b}{x+b}=\frac{a-b}{2x+b+c}.$$

Multiplying cross-wise, $(c-b)(2x+b+c)=(a-b)(x+b)$.

$$\therefore (c-b)2x-b^2+c^2=(a-b)x+ab-b^2,$$

$$\therefore x(2c-b-a)=ab-c^2,$$

$$\therefore x=\frac{ab-c^2}{2c-b-a}.$$

EXERCISE LXXXI.

Solve

1. $\frac{x+4}{x+2}+\frac{x-2}{x-18}=\frac{2x-28}{x-18}$.

2. $\frac{x+1}{x-2}+\frac{x-1}{x-6}=\frac{2x+14}{x+3}$.

3. $\frac{x+1}{x-1}+\frac{1-x}{x+5}=\frac{8}{x+3}$.

4. $\frac{4-3x}{3x+20}+\frac{6x+26}{3x+8}=\frac{3x+74}{3x+40}$.

5. $\frac{x+2m}{x+m}-\frac{n+c-x}{x-c}=\frac{3m-3n+2x}{x+m-n}$.

6. $\frac{3x+16}{x+5}-\frac{12x-13}{3x-4}=\frac{5x+16}{x+3}-\frac{18x-57}{3x-10}$.

Solve

7. $\dfrac{m(x-a)}{x+a}+\dfrac{n(x-b)}{x+b}=m+n.$

8. $\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}=\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}.$

9. $\dfrac{x-a}{x-a-1}-\dfrac{x-a-1}{x-a-2}=\dfrac{x-b}{x-b-1}-\dfrac{x-b-1}{x-b-2}.$

10. $\dfrac{x-7}{x-9}-\dfrac{x-9}{x-11}=\dfrac{x-13}{x-15}-\dfrac{x-15}{x-17}.$ (P. E. 1890.)

11. $\dfrac{x}{x-1}-\dfrac{x+1}{x}-\dfrac{x+4}{x+3}+\dfrac{x+5}{x+4}=0.$

12. $\dfrac{3x-1}{3x-4}-\dfrac{6-x}{x-5}=\dfrac{12x+16}{3x+2\frac{1}{2}}-\dfrac{2x}{x-1}.$

13. $\dfrac{x+8a+b}{x+2a+b}+\dfrac{4x+2a+2b}{x+2a-b}=5.$

14. $\dfrac{x^2+x+1}{x+1}+\dfrac{x^2+2x+1}{x+2}=2x+\dfrac{2}{x+3}.$

15. $\dfrac{x^2+2x+2}{x+1}+\dfrac{x^2+8x+20}{x+4}=\dfrac{x^2+4x+6}{x+2}+\dfrac{x^2+6x+12}{x+3}.$

16. $\dfrac{x+2}{x+4}=\left(\dfrac{x+6}{x+7}\right)^2.$

17. $\dfrac{x-3}{x+5}=\left(\dfrac{2x-1}{2x+7}\right)^2.$

18. $\dfrac{x+p+m-n}{x+p}+\dfrac{n-p-m-x}{x+m}=\dfrac{m-p}{x-n}.$

19. $\dfrac{abx+a^2+a}{bx+a}+\dfrac{b^2x+bc+b}{bx+c}=\dfrac{(ab+b^2)x+ad+bd+a+b}{bx+d}.$

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11. Finding a common factor of the two sides of an equation.

Suppose an equation is reducible to the form $P.X=0$, where X contains the unknown quantity x.

Now $\because$ $P.X=0$, either $P=0$ or $X=0$.

If P does not contain x, then it being a constant quantity it cannot be 0, and the only solution is that obtained from $X=0$. If both contain x, we will get solutions both from $X=0$ and $P=0$

Ex. 1. Solve $\left(\frac{x+p}{x+q}\right)^3=\frac{x+2p-q}{x-p+2q}$.

Put $x+p=A$, $x+q=B$; then $x+2p-q=2A-B$ and $x-p+2q=2B-A$. Hence the equation becomes

$$\frac{A^3}{B^3}=\frac{2A-B}{2B-A}.$$

Adding 1 to both sides,

$$\frac{A^3+B^3}{B^3}=\frac{A+B}{2B-A},$$

$$\text{or } \frac{(A+B)(A^2-AB+B^2)}{B^3}=\frac{A+B}{2B-A}.$$

or transposing, $(A+B)\left\{\frac{A^2-AB+B^2}{B^3}-\frac{1}{2B-A}\right\}=0\ldots(1)$

$$\therefore\ A+B=0 \text{ which gives } 2x+p+q=0,$$

$$i.e., \ x=-\frac{p+q}{2}.$$

The other factor in (1) cannot be zero, for

$$\text{then } \frac{A^2-AB+B^2}{B^3}=\frac{1}{2B-A},$$

$$\text{or } (2B-A)(A^2-AB+B^2)=B^3,$$

$$\text{or } 2A^2B-2AB^2+2B^3-A^3+A^2B-AB^2=B^3,$$

$$\text{or } -A^3+3A^2B-3AB^2+B^3=0,$$

$$\text{or } -(A-B)^3=0, \text{ or } A-B=0,$$

or $p-q=0$, which is absurd, for p and q are unequal constants.

Ex. 2. Solve $\frac{3}{x-3}-\frac{4}{x+9}-\frac{5}{x-27}+\frac{6}{x-15}=0$. (M. M. 1873)

$$\text{Here } \left(\frac{3}{x-3}+\frac{6}{x-15}\right)-\left(\frac{4}{x+9}+\frac{5}{x-27}\right)=0.$$

$$\therefore\ \frac{3(x-15)+6(x-3)}{(x-3)(x-15)}-\frac{4(x-27)+5(x+9)}{(x+9)(x-27)}=0,$$

$$i.e.\ \left(\frac{9x-63}{x^2-18x+45}-\frac{9x-63}{x^2-18x-243}\right)=0,$$

$$i.e.\ (9x-63)\left(\frac{1}{x^2-18x+45}-\frac{1}{x^2-18x-243}\right)=0\ldots(1)$$

$\therefore\ (9x-63)=0$ whence $x=7$.

The other factor in (1) cannot be zero, for then

$$\frac{}{x^2-18x+45}=\frac{}{x^2-18x-243},$$

or $x^2-18x+45=x^2-18x-243$ which is impossible.

Ex. 3. Solve $\dfrac{x-4}{x-5}-\dfrac{x-5}{x-6}=\dfrac{x-7}{x-8}-\dfrac{x-8}{x-9}$.

Here $1+\dfrac{1}{x-5}-1-\dfrac{1}{x-6}=1+\dfrac{1}{x-8}-1-\dfrac{1}{x-9}$,

or $\dfrac{1}{x-5}-\dfrac{1}{x-6}=\dfrac{1}{x-8}-\dfrac{1}{x-9}$......(1)

or $\dfrac{1}{x-5}+\dfrac{1}{x-9}=\dfrac{1}{x-8}+\dfrac{1}{x-6}$, transposing

or $\dfrac{2x-14}{(x-5)(x-9)}=\dfrac{2x-14}{(x-8)(x-6)}$.

Hence either $2x-14=0$, giving $x=7$ or $(x-5)(x-9)=(x-8)(x-6)$ which on simplification will be found to give no root.

Note. From the stage (1) we might proceed without transposition by simplifying both sides as in art 8.

Ex. 4. Solve $\dfrac{x-a}{3b+5c}+\dfrac{x-3b}{5c+a}+\dfrac{x-5c}{a+3b}=3$. (C. E. 1896.)

We have $\left(\dfrac{x-a}{3b+5c}-1\right)+\left(\dfrac{x-3b}{5c+a}-1\right)+\left(\dfrac{x-5c}{a+3b}-1\right)=0$

$\therefore \dfrac{x-a-3b-5c}{3b+5c}+\dfrac{x-a-3b-5c}{5c+a}+\dfrac{x-a-3b-5c}{a+3b}=0$,

$\therefore (x-a-3b-5c)\left\{\dfrac{1}{3b+5c}+\dots\right\}=0$.

Hence $x-a-3b-5c=0 \quad \therefore x=a+3b+5c$.

Ex. 5. Solve $(x+a)^3+(x+b)^3+(x+c)^3=3(x+a)(x+b)(x+c)$.

Here $(x+a)^3+(x+b)^3+(x+c)^3-3(x+a)(x+b)(x+c)=0$.

$\therefore \frac{1}{2}(3x+a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}=0$

[See formula (A), p. 142.]

$\therefore 3x+a+b+c=0$, whence $x=-\frac{1}{3}(a+b+c)$.

EXERCISE LXXXII.

Solve

1. $\dfrac{x+b}{a-b}=\dfrac{x-b}{a+b}$.

2. $\dfrac{1}{x+7}+\dfrac{1}{x+5}=\dfrac{1}{x+8}+\dfrac{1}{x+4}$.

3. $\dfrac{1}{x+a}+\dfrac{1}{x-b}=\dfrac{1}{x+a+c}+\dfrac{1}{x-b-c}$.

4. $\dfrac{7}{x+16}+\dfrac{20}{x+7}=\dfrac{40}{x+5}-\dfrac{13}{x-4}$.

Solve

5. $\dfrac{61}{6(x-9)}+\dfrac{\quad}{x-4}=\dfrac{\quad}{x-8}+\dfrac{\quad}{6(x-3)}$.

6. $\left\{\dfrac{x+4}{x+3}\right\}^3=\dfrac{5+x}{2+x}$.

7. $\left\{\dfrac{x+a+2b}{x+a-2b}\right\}^3=\dfrac{x+a+6b}{x+a-6b}$.

8. $\dfrac{x-a}{b+c}+\dfrac{x-b}{c+a}+\dfrac{x-c}{a+b}=3$.

9. $\dfrac{x-a^3}{b^2-bc+c^2}+\dfrac{x-b^3}{c^2-ca+a^2}+\dfrac{x-c^3}{a^2-ab+b^2}=2(a+b+c)$.

10. $\dfrac{x-bc}{b+c}+\dfrac{x-ca}{c+a}+\dfrac{x-ab}{a+b}=a+b+c$. (C. E. 1905.)

11. $\dfrac{x-a^2}{b+c}+\dfrac{x-b^2}{c+a}+\dfrac{x-c^2}{a+b}=4(a+b+c)$. (C. E. 1908.)

12. $\dfrac{x-a}{b+c+2a}+\dfrac{x-b}{c+a+2b}+\dfrac{x-c}{a+b+2c}+3=0$. (C. E. 1898.)

13. $\dfrac{1}{x+a}+\dfrac{1}{x+b}=\dfrac{1}{x+a+b}+\dfrac{1}{x}$. (A. E. 1894.)

14. $\dfrac{6-5x}{5}-\dfrac{3(7-2x^2)}{14(x-1)}-1\dfrac{1}{10}=\dfrac{1+3x}{7}-x+\dfrac{1}{35}$. (M. M 1867.)

15. $\dfrac{3x-8}{x-3}-\dfrac{x-4}{x-5}=\dfrac{2x-3}{x-2}-\dfrac{1}{x-4}$.

16. $\dfrac{bc(ax+1)}{b+c}+\dfrac{ca(bx+1)}{c+a}+\dfrac{ab(cx+1)}{a+b}=-(a+b+c)$.

17. $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{a-c}{x+a-c}+\dfrac{b+c}{x+b+c}$. (B. M 1882.)

18. $\dfrac{2x+8\frac{1}{2}}{9}-\dfrac{13x-2}{17x-32}+\dfrac{x}{3}=\dfrac{7x}{12}-\dfrac{x+16}{36}$. (B. M. 1897.)

19. $\dfrac{x-4}{(x-1)(x-3)}+\dfrac{x-7}{(x-1)(x-6)}+\dfrac{x-9}{(x-3)(x-6)}=\dfrac{3}{x}$. (M. M. 1874.)

20. $\dfrac{x+4}{(x+1)(x+2)}+\dfrac{x+6}{(x+2)(x+3)}+\dfrac{x+5}{(x+3)(x+1)}=\dfrac{3}{x-1}$.

21. $\dfrac{a+c}{x-2b}-\dfrac{b+c}{x-2a}=\dfrac{a-c}{x+2b}-\dfrac{b-c}{x+2a}$. (M. M. 1888.)

22. $\dfrac{2x+11}{x+5}-\dfrac{9x-9}{3x-4}=\dfrac{4x+13}{x+3}-\dfrac{15x-47}{3x-10}$. (C. E. 1898.)

23. $(x-2a)^3+(x-2b)^3=2(x-a-b)^3$.

Solve

24. $\dfrac{(x-a)(x+b)}{x-a+b}=\dfrac{x(x-c)-b(x+c)}{x-b-c}$. (M. M. 1886.)

25. $\dfrac{6}{7-\dfrac{6}{7-\dfrac{6}{7-x}}}=1$. (B. M. 1891.)

26. $\dfrac{3\{ab-x(a+b)\}}{a+b}+\dfrac{(2a+b)b^2x}{a(a+b)^2}=\dfrac{bx}{a}-\dfrac{a^2b^2}{(a+b)^3}$.

27. $1-\dfrac{1-\dfrac{3}{3-x}}{3-x}=\dfrac{x}{x-3}$. (P. E. 1890).

28. $(x-3)^3+(x-4)^3+(x-5)^3=3(x-3)(x-4)(x-5)$.

29. $\dfrac{1}{2x+\dfrac{1}{1+\dfrac{2x+1}{3-2x}}}=\dfrac{10}{x+13}$.

30. $\dfrac{x-p+q}{(x-p)(x+q)}+\dfrac{x+q-r}{(x+q)(x-r)}=\dfrac{3}{x}-\dfrac{p+r-x}{(x-p)(x-r)}$.

31. $\dfrac{(m+n)x+m+n+a}{(m+n)x+a}-\dfrac{(m-n)x-m+n-a}{(m-n)x-a}=\dfrac{2a}{ax+m+n}$.

32. $\dfrac{1}{(a-b)(x-a)}-\dfrac{1}{(c-d)(x-c)}=\dfrac{1}{(a-b)(x-b)}-\dfrac{1}{(c-d)(x-a)}$.
(C. E. 1891.)

ANSWERS.

EXERCISE I (p. 2.)

2. 10. **3.** 126. **4.** $\frac{217}{136}$. **5.** 3 miles. **6.** 15 seers; $3\frac{3}{4}$.
7. 110. **8.** Rs. 2.3.7*p*.

EXERCISE II (pp. 6-7).

1. 46. **2.** $\frac{25}{6}$. **3.** 10. **4.** 13. **5.** 69. **6.** 67. **7.** 163.
8. 32. **9.** 100. **10.** 46. **11.** 345. **12.** 194. **13.** 472. **14.** 412.
15. 648. **16.** $4\frac{1}{2}$. **17.** 3888. **18.** 12. **19.** 27. **20.** 1728; 432.
21. 12. **22.** 3. **23.** $13\frac{1}{5}$. **24.** $\frac{17}{13}$. **25.** $\frac{42}{25}$. **26.** $\frac{105}{120}$. **27.** $\frac{7}{10}$.
28. 39. **29.** 102. **30.** 372. **31.** $6\frac{1}{4}$. **32.** 3. **33.** $4\frac{1}{2}$. **34.** 4.
35. $\frac{319}{900}$. **36.** 0. **37.** $\frac{113}{36}$. **38.** $4\frac{5}{6}$. **39.** $7\frac{1}{12}$. **40.** $5\frac{7}{12}$.

EXERCISE III (p. 9).

1. 1024. **2.** 768. **3.** 13824. **4.** $\frac{69984}{25}$. **5.** 2448. **6.** 608. **7.** 204.
8. 544. **9.** 3. **10.** $8\frac{1}{3}$. **11.** 20. **12.** 15. **13.** 2. **14.** 8.
15. 69. **16.** 4. **17.** $\frac{123}{50}$. **18.** $\frac{1769}{600}$. **19.** $\frac{123}{50}$. **20.** $\frac{85301}{27000}$.

EXERCISE IV (pp. 11-12).

1. $\frac{35}{24}$. **2.** 145. **3.** $\frac{275}{9}$. **4.** 18. **5.** 6. **6.** 6. **7.** 60.
8. 10. **9.** 18. **10.** 7. **11.** $\frac{5}{24}$. **12.** 287. **13.** 84. **14.** $\frac{1307}{240}$
15. $4\frac{2}{3}$. **16.** 7. **17.** 11. **18.** 14. **19.** 18. **20.** 185. **21.** 405.
22. 16. **23.** 298. **24.** $47\frac{1}{2}$. **25.** $8\frac{1}{5}$. **26.** $132\frac{5}{6}$. **27.** 180. **31.** (i) $\frac{77}{450}$. (ii) $\frac{1001}{450}$. **32.** (1) $(a+b)-c$. (2) $(a-b)c+d$. (3) $x-\{a\div b+c\times d\}$. (4) $(a\div b)\div c$. (5) $(a\div b)c$. (6) $\frac{a}{b\times c}$.

EXERCISE V (pp. 23-25).

5. 10·8; 11·1; 13·6. **6.** 60·8 miles nearly. **8.** 4·8; 4; 3·7; 3·2
9. 3·4 : 1·3. **12.** 3 **13.** 66. **14.** 3·7.

EXERCISE VI (p. 30).

1. 7. **2.** 2. **3.** 5. **4.** 5. **5.** 0. **6.** -9
7. 10. **8.** -16. **9.** -10. **10.** 2. **11.** -13. **12.** 19.
13. 1. **14.** -2. **15.** -5. **16.** 5. **17.** 0. **18.** 8.
19. -1. **20.** -3. **21.** -10. **22.** 3. **23.** 0. **24.** -12.
25. -10. **26.** 10. **27.** 10. **28.** 2. **29.** -2. **30.** -12.
31. 6. **32.** 7. **33.** -2. **34.** $-1\frac{1}{12}$. **35.** $-6\frac{7}{15}$. **36.** $1\frac{17}{36}$.
37. $-2\frac{4}{35}$. **38.** $7\frac{31}{45}$. **39.** $-3\frac{7}{12}$.

EXERCISE VII (p. 32).

1. 6. **2.** -5. **3.** -10. **4.** 3. **5.** 4. **6.** -2. **7.** -5. **8.** -5.

EXERCISE VIII (p. 34).

1. 15. **2.** -20. **3.** -35. **4.** -4. **5.** 24. **6.** -54. **7.** -48.
8. -21. **9.** 63. **10.** -42. **11.** -63. **12.** 121. **13.** 0. **14.** 42.
15. -90. **16.** -24. **17.** 210. **18.** -60. **19.** -60. **20.** -216.
21. -100. **22.** 2, 3. **23.** -2, -3. **24.** -5, 2. **25.** 5, -2.
26. -5, -2. **27.** 5, 2.

EXERCISE IX (p. 35).

1. 4. **2.** -4. **3.** 7. **4.** -9 **5.** -7. **6.** -9.
7. -8. **8.** 13. **9.** -1. **10.** 0. **11.** -3. **12.** 4.

EXERCISE X (pp. 36-37).

1. 1. **2.** 0. **3.** -18. **4.** 5. **5.** 5. **6.** -17. **7.** 0. **8.** 8.
9. -90. **10.** 114. **11.** 13. **12.** 324. **13.** 456. **14.** 797.
15. $\frac{8}{15}$. **16.** $\frac{91}{36}$. **17.** 0. **18.** $-8\frac{11}{14}$. **19.** 4. **20.** 4. **21.** 144.
22. 9. **23.** 44, 25, 12, 5, 4, 9, 20. **24.** $-6{\cdot}52$, $-7{\cdot}08$, $-7{\cdot}68$, $-8{\cdot}32$.

EXERCISE XI (pp. 38-39).

1. x. **2.** $-2xy$. **3.** $-\frac{47}{72}xyz$. **4.** $-\frac{9}{2}a^2bc$. **5.** 0. **6.** 0. **7.** $\frac{27}{2}abx$.
8. $2a^2b$. **9.** $8abcd$. **10.** $-\frac{25}{12}\sqrt{(ab)}$. **11.** $\frac{38}{35}\sqrt[3]{a}$. **12.** $53a^2c^2$.
13. $18(a^2-b^2)$. **14.** $2a+3b$. **15.** $4x-5y$. **16.** $-2x-5z$.
17. $-\frac{3}{2}a^2+\frac{2}{3}a+\frac{4}{5}$. **18.** $-2ab+3c^2-4xy$. **19.** $4b^3-\frac{3}{4}c^3+2b^2c$.
20. $5ab-2xy+\frac{1}{4}c^2-\frac{4}{5}d^2$.

EXERCISE XII (p. 42).

1. $ab+5ca$. **2.** $a+2c$. **3.** $9b^2$. **4.** $2abx+3bcy-8abz+2xyz$. **5.** $7a-4b$.
6. $7x-y$. **7.** $\frac{1}{6}ax+\frac{2}{15}by$. **8.** $-a^2-\frac{7}{40}b^2$. **9.** $5a+b-2c$. **10.** $x+y+4z$.

11. o. **12.** $a+b+c$. **13.** $2x+y-z$. **14.** $2a+2b+2c-2d$. **15.** o.
16. $-8x^2+2xy+4y^2$. **17.** $\frac{2}{3}a^2+\frac{1}{12}ax+\frac{13}{10}x^2+\frac{3}{2}xy$. **18.** $14x^3-10x^2$
$-3x+9$. **19.** $b^4+2b^3+b^2+20b-14$. **20.** $-\frac{1}{20}xy-\frac{1}{20}yz-\frac{1}{20}zx$.
21. $\frac{5}{12}x^4-\frac{17}{60}x^3+\frac{13}{60}x^2-\frac{8}{15}x$. **22.** $8a^2+6ab+ac+2b^2+5bc-4c^2$.
23. $12a^4+a^3b+3ab^3+b^4$. **24.** $11xy-7y^2$. **25.** $-\frac{1}{6}\sqrt{a}+\frac{17}{12}\sqrt{b}-\frac{1}{2}\sqrt{c}$.

EXERCISE XIII (pp. 44-45).

1. $-5x$. **2.** $\frac{1}{20}x$. **3.** $\frac{23}{21}a$. **4.** $-\frac{3}{4}a^2$. **5.** $-\frac{7}{8}b^2$. **6.** $\frac{7}{2}x$. **7.** $-ab$. **8.** c.
9. $-4xy$. **10.** $2b$. **11.** $a-b-c+d$. **12.** $\frac{1}{3}x-y$. **13.** $\frac{4}{5}a+\frac{1}{7}b$.
14. $2a-2b+c$. **15.** $\frac{1}{4}ab-\frac{7}{6}bc+\frac{3}{4}ca$. **16.** $\frac{28}{3}x^2+\frac{11}{3}y-10$. **17.** $14a^3-2a^2$
$-10a+1$. **18.** $6x^3-8x^2+12x+8$. **19.** $-9a^2-5ab+a+4b-2b^2$
20. $\frac{3}{5}x^2+\frac{7}{8}xy+\frac{4}{15}y^2$. **21.** $-5a^4+11a^3b-8a^2b^2+6ab^3+5b^4$.
22. $5x^6-10x^5y+x^4y^2-x^3y^3-6x^2y^4-4xy^5+2y^6$.

EXERCISE XIV (p 46).

1. $x-20$, $20-y$. **2.** $x-15$, $15-y$. **3.** $10-x$, $y-10$. **4.** $64a+4b$.
5. $-2x+3y-2z$. **6.** $2b-2c+2d$. **7.** $a+b-c+d$. **8.** $a-(x+y+z)$.
9. (i) $11x^3-13x^2+4x-15$. (ii) $14-16x+34x^2-14x^3$
10. (i) $46a^3-17a^2b-75ab^2+64b^3$. (ii) $-52a^3-50a^2b-47ab^2+74b^3$.
11. $4a^2-5ab+7b^2$. **12.** $-2b+2c$; $-a-b+c+d-e$. **13.** $2a^2-3b^2$
$+2c^2+bc-2ac+ab$. **14.** $2x^2+2ax-2a^2$. **15.** $-17a+9b+9c$.
16. $10x-14y-3z$. **17.** $19a^2-5ab-b^2$. **18.** $-151y-7y^2$.

EXERCISE XVI (p. 52).

1. $-12ab$. **2.** $-15xy$. **3.** $2ab^2c$. **4.** $-6x^2y^2$. **5.** $-3x^4y^3$ **6.** $15x^7$.
7. $12a^3b^5c^5$. **8.** $-2a^2b^2x^4yz$. **9.** $-36x^5y^7z^4$. **10.** $-6x^3y^2z^3$.
11. $24xy^2z^3$. **12.** $24a^2b^2c^2$. **13.** $-\frac{3}{4}x^4y^4$. **14.** $-6l^3m^4n^3$. **15.** $-\frac{3}{2}a^5b^6c^4$.
16. $-6a^2l^2b^{2m}c^{2n}$. **17.** $-2x^{a+b+c}y^{a+b+c}z^{a+b+c}$. **18.** x^6
19. x^6y^3. **20.** $81x^3y^8$. **21.** $4a^2b^6$. **22.** $-8a^{12}b^6$. **23.** $-\frac{8}{27}x^3y^3z^3$.
24. $60x^5y^5z^5$. **25.** $\frac{1}{2}a^5b^7c^9$. **26.** $-24x^5y^5z^5$. **27.** $-a^2b^5c^5$. **28.** $\frac{3}{4}x^{17}y^{11}z^{11}$.
29. $12a^{14}b^{21}c^{20}d^{17}$. **30.** x^7y^7. **31.** $-x^4y^{10}$. **32.** $-108x^{15}y^{11}z^2$.

EXERCISE XVII (p. 55).

1. $10x-15$. **2.** $6bc+8ac$. **3.** $-35xz+14yz$. **4.** $2a^2b^2c-3ab^2c^2$.
5. $9x^3y^4z^2-6xy^3z^3$. **6.** $-20lm+28mn-8nl$. **7.** $\frac{1}{2}xz-10yz-5z$.
8. $-4a^4bc^3+10a^2b^3c^3+2a^2bc^5$. **9.** $-a^2b^2x^2y+ab^2c^2xy^2-a^3bcxyz$.
10. $-\frac{7}{3}x^7+\frac{21}{8}x^6-\frac{7}{18}x^5+\frac{7}{5}x^4+2x^3-\frac{21}{4}x^2$.
11. $-4x^2y^4z^2w^4+8x^3y^3z^4w^2-6x^5y^5z^3w^4+2x^3y^4z^3w^3$.

EXERCISE XVIII. (p. 59).

1. $x^2+3x-10$. **2.** $x^2+5x-14$. **3.** $x^2-13x+36$.
4. $6x^2-11xy-35y^2$. **5.** $3a^2-ab-30b^2$. **6.** $10a^2-27ab+18b^2$.
7. $6ac-4ad+3bc-2bd$. **8.** $2ax-3ay-8bx+12by$.
9. $10al-14am+15bl-21bm$. **10.** $4x^2-9y^2$. **11.** $9a^2b^2-16c^2d^2$.
12. $\frac{7}{2}a^2x^2-\frac{97}{24}abxy+\frac{15}{16}b^2y^2$. **13.** $6a^4-25a^3b+46a^2b^2-44ab^3+15b^4$.
14. $8x^4+10x^3y-32x^2y^2+7xy^3+6y^4$.
15. $10x^4-41x^3y+33x^2y^2-48xy^3+21y^4$.
16. $6a^5-6a^4b-21a^3b^2+30a^2b^3-39ab^4+30b^5$.
17. $\frac{1}{4}a^3-\frac{11}{18}a^2+\frac{43}{72}a-\frac{1}{6}$. **18.** $\frac{1}{3}x^3-\frac{28}{45}x^2y+\frac{13}{30}xy^2-\frac{1}{5}y^3$.
19. $a^2-b^2+2bc-c^2$. **20** $-6x^5+17x^4-10x^3-9x^2+34x-8$.
21. $\frac{8}{9}x^4-\frac{2}{3}x^3-\frac{283}{72}x^2-x+\frac{5}{6}$. **22.** $6x^2+6y^2-20z^2-13xy+2yz+7zx$.
23. $3a^4-5a^3b-11a^2b^2+10ab^3-2b^4$. **24.** $16x^4-a^2x^2+9a^4$.
25. $9x^4+16x^3y+14x^2y^2-49xy^3+10y^4$. **26.** $27x^3+18xy-y^3+8$.
27. $15x^6-9x^5+22x^4-41x^3+39x^2-34x+8$.
28. $5a^6+2a^5-21a^4+38a^3-24a^2+14a-5$.
29. $1-6x+12x^2-17x^3+11x^4-13x^5+6x^6$.
30. $-4x^6+14x^5+11x^4-14x^3+7x^2+6x-8$.
31. $8a^3+18abc-27b^3+c^3$. **32.** $1+x^2+x^4+x^6+x^8$.
33. $-x^7+6x^6y-10x^5y^2+16x^3y^4-17x^2y^5+7xy^6-y^7$.
34. $x^8-116x^7+1789x^6-10460x^5+2502x^4-5382x^3+4633x^2-708x$
$+1189$. **35.** $9x^2+24xy+16y^2$. **36.** $25x^2-20xy+4y^2$.
37. $4a^2-12ab+20ac+9b^2-30bc+25c^2$. **38.** $81x^4-18x^2y^2+y^4$.
39. (i) -13; (ii) -11; (iii) 59; (iv) -89. **40.** -61; 34.

EXERCISE XIX (p. 60).

1. $x^3+6x^2+11x+6$. **2.** $x^3+5x^2-26x-120$.
3. $60x^3-26x^2-16x+6$. **4.** $48x^3-188x^2y+180xy^2-25y^3$.
5. $a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc$.
6 $ab^2-a^2b+bc^2-b^2c+ca^2-c^2a$. **7.** $12x^4-32x^3+31x^2-13x+2$.
8 $6a^4-a^3b-5a^2b^2+11ab^3-6b^4$. **9.** x^6-a^6.
10. x^8-1. **11.** x^8+x^4+1. **12.** $a^8+a^4b^4+b^8$.

EXERCISE XX (p. 63).

1. -4. **2.** $2x$. **3.** $-c$. **4.** $-\frac{8}{15}x^2$. **5.** $-\frac{28}{15}a^2$. **6.** $-\frac{3}{8}a^2b^2$.
7. $-9a^4b^4$. **8.** $-3x^2y^3$. **9.** $-3x^5y^4z^2$. **10.** $-4a^4b^3x^2$.

11 $4a^2b^2c^3x^3y^7z^9$, $6a^5b^4cx^2y^4z^5$, $-2a^3b^2cx^2y^3z^5$.

12. $\frac{l}{m}a^{2p}b^{q}c^{4r}$, $\frac{l}{n}a^{p}c^{2r}$, $\frac{l}{p}a^{3p}b^{q}c^{r}$. **13.** $4a^5b^5c^5$, $-2a^4b^4c^2$, $-\frac{1}{6}a^4b^3c$

EXERCISE XXI (pp. 64-65).

1. $2x-y$. **2.** $2x-3y$. **3.** $-3x^2+2x$. **4.** $\frac{3}{2}x-3y$.

5. $-3l-4al^2+5a^4l^3$. **6.** $9a^2x^2-6ax^3+12$.

7. $-2a^3b^2c^3+3a^2b^3c-5a^4b^5c^6$, $8abc^2-12b^2+20a^2b^4c^5$, $-4a^2b^4c^3+6ab^5c$ $-10a^3b^7c^6$. **8.** $-\frac{3}{2}a^2b^2c+\frac{5}{2}abc^2d^2-2b^3cd$.

9. $-\frac{p}{2}a+\frac{q}{2}a^2l-\frac{r}{2}a^3l^2$, $\frac{p}{3}a^2l^2-\frac{q}{3}a^3l^3+\frac{r}{3}a^4l^3$.

10. $-7x^4y^5z^4+3x^2y^2z^5-9x^3y^4z^3+12xy^3z^2$.

11. $a^2b(a-b)$, $(a+b)(c+d)$.

EXERCISE XXII (pp. 67-69).

1. $x-4$. **2.** $2x-1$. **3.** $5x+6y$. **4.** $a-5b$. **5.** $2x-3y$

6. $2+7x$. **7.** $18b-5$. **8.** $6l+35m$. **9.** $\frac{1}{3}a+\frac{3}{4}b$. **10.** $\frac{3}{4}x-\frac{5}{4}y$.

11 $3x^2-4x+5$. **12.** $5x^2+7x-3$. **13.** $7x^2+x-8$.

14. $2x^3-x^2+x-2$. **15.** $3a^2-4ab+2b^2$. **16.** $2a^3-3a^2+a-11$.

17. $5c^3+2c^2-3c-7$ **18.** $5a^3-3a^2-2a+3$.

19. $1+2x+3x^2+2x^3+x^4$. **20.** $2+4l+6l^2+3l^3$. **21.** $7x^2-7xy+5y^2$.

22. $2x^3+3x^2y-3xy^2+y^3$. **23.** $3a^2+4ab-2b^2$. **24.** a^2-ab+b^2.

25. a^2+ab+b^2. **26.** $a^2-2ab+2b^2$. **27.** $a^4-a^3b+a^2b^2-ab^3+b^4$.

28. a^2+ab+b^2. **29.** $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$.

30. $1-2x+3x^2-4x^3+5x^4$.

31. $a^7+a^6b+a^5b^2+a^4b^3+a^3b^4+a^2b^5+ab^6+b^7$.

32. $a^4+a^3b-ab^3-b^4$.

33. $a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6$.

34. $4a^2+6ab-10ac+9b^2+15bc+25c^2$.

35. $1-2x+6xy+4x^2+3y+9y^2$. **36.** $x^2-xy+x+y^2+y+1$.

37. $a^2-ab+ac+b^2+bc+c^2$. **38.** $a^2+2ab+b^2-c^2$. **39.** $\frac{1}{2}x^2-\frac{1}{4}x+6$.

40. $\frac{1}{3}x^2-\frac{1}{6}xy+\frac{1}{2}y^2$. **41.** $\frac{3}{2}x^2-xy-\frac{2}{3}y^2$. **42.** $\frac{1}{8}x^3+\frac{1}{6}x^2y+\frac{2}{9}xy^2+\frac{8}{27}y^3$.

43. $5x^2-7xy-3y^2$. **44.** $3a^2+ab-b^2$. **45.** $2x^4-5x^3-4x^2+x+2$.

46. $1-x^2+x^3$. **47.** $x^3+x^2a+xa^2+a^3$. **48.** $2x^4-5x^3-4x^2+x+2$.

49. x^2+b, remainder $=ab-c$.

50. (1) $1-7x+35x^2-175x^3$, remainder $=875x^4$;

(2) $1+x+0.\,x^2-2x^3$, remainder $=-4x^4+4x^5$.

EXERCISE XXIII (p. 71).

1. -8. **2.** $4x+3y-14z$. **3.** $-7x-5y-3z$. **4.** 2. **5.** $20x-2$.

6. $-15a^2+53ab-55b^2$. **7.** $-13x^2-32x-6$. **8.** $\frac{5x}{18}-\frac{4}{9}$. **9.** $\frac{2x}{3}-\frac{5}{3}$.

10. $a^2+20ab+4b^2-c^2$. **11.** $4a^3b-7a^2b^2-5ab^3+4a^2bc+5ab^2c+3ac^2b$.

12. 0. **13.** a^2d^2. **14.** $6x+24$.

EXERCISE XXIV (pp. 74—75).

1. $-24y$. **2.** $a-2b-2c$. **3.** $a+3b-3c-d$. **4.** $2x$.

5. $3a-6b-7c+9d$. **6.** $-5x+2y-z$. **7.** $-6a-8c$.

8. $-9x-3y+8z$. **9.** $2a-b+3c-3d$. **10.** $-13a+12b-26c$.

11. $-10a-3b+7c+5d$. **12.** $-17x+12y+54z$. **13.** $x-y$.

14. $153a+17b$. **15.** $6x-2y-12z$. **16.** $-17a-6b+2c$. **17.** 151

18. -68. **19.** $a^2+b^2+c^2$. **20.** $-23a+21b-8c$.

21. (i) $(c-9)+(l-7)x+(4-3b)x^2+(5-m)x^3+(a-7)x^4$,

(ii) $-(9-c)-(7-l)x-(3b-4)x^2-(m-5)x^3-(7-a)x^4$.

22. $-x^2(4b+c)-14x+(2c-2)$.

23. $hx^3+x^2(k+p-2q+l)+x(2m+l)-2n$.

24. $x^2(a+p+l)-x(b-q+m)+(c-r+n)$.

25. (i) $a+[-b-\{-c-(-d+\overline{-c+f-g})\}]$.

(ii) $a+[b-\{c-(-d+\overline{e-f+g})\}]$.

EXERCISE XXV. (pp. 76—78).

1. $x^3(p-a)+x^2(b-q)+x(r-c)+(d-s)$. **2.** $2ax^2-2bx+(2a+b)$.

3. $(5r-p-4q)x+(5r-q-5p)y+(p+4q-3r)z$.

4. $(6a-13b+9c)xy+(7a-12b+4c)yz+(9a-2b+6c)zx+(14c-5b-4a)$. **5.** $(6x-2z)a+(7x-2y-2z)b+(6x-3y-3z)c$.

6. $(3a-3b-c)xy+(4a-c+1)yz-(6a-2b+2c+4)zx$.

7. $(\frac{13}{2}x+\frac{5}{3}y-\frac{9}{4}z)a^2+(\frac{5}{2}x-\frac{10}{3}y+\frac{16}{3}z)b^2+(y-3x)c^2$.

8. $(3r-2q-2p)l+(7r-5q-2p)m+(5q-3p+7r)n+(8q-4r-4p)$.

9. $(3l-3m-n+8r)ax+(3r-n-3m)by+(2l+3m-n-2r)cz$.

10. $3ay-(x+2y+2z)b+(y-2z)c$. **11.** $x^2-(a+b)x+ab$.

12. $x^2+(a-b)x-ab$. **13.** $x^2-(a-b)x-ab$.

14. $alx^3+(2hly+am)x^2+(bly^2+2hmy)x+bmy^2$.

15. $plx^3-(pm+ql)x^2+(pn+qm)x-qn$.

16. $1+(a+1)x+(a+b+1)x^2+(a+b+c)x^3+(b+c)x^4+cx^5$.

17. $(a^2+2ab+b^2)x^4+(a^2-b^2+a+b)x^3+(a-b-a^2-b^2-2ab)x^2$
$+(a^2-b^2-a-b)x+(a-b)$.

18. $x^2-(p-r)x+r^2-pr+q$. 19. $x+b+c$. 20. $c+(a-b)$.
[In ex. 20 proceed by arranging in descending powers of c].

21. $a^2+ab-ac+b^2+bc+c^2$. 22. $c^2-c(a+b)+ab$.

23. $a^2-a(b+c)+bc$. 24. $c^2+c(a+b)+ab$.

25. $a^2(b-c)-a(b^2-c^2)+bc(b-c)$; $c^2-c(a+b)+ab$.

26. $c^2-c(a+b)+ab$. 27. $a(b+c)+bc$. 28. a^2-ab+b^2.

29. $(a+1)x^2+(a^2+1)x+a^3$. 30. $x^3-2(a+b+c)x^2+2a(b+c)x$.

31. $b+q-ap$.

EXERCISE XXVI. (p. 79).

1. 36. 2. 12. 3. 24. 4. $c\div ab$. 5. $16x$.

EXERCISE XXVII. (p. 82).

1. 4. 2. 7. 3. 8. 4. -1. 5. -3. 6. $-\frac{3}{2}$. 7. $-\frac{12}{5}$. 8. 3.
9. $\frac{6}{5}$. 10. -7. 11. -14. 12. $23\frac{1}{3}$. 13. -6. 14. $5\frac{5}{11}$. 15. $5\frac{4}{9}$.
16. $-6\frac{3}{7}$. 17. $\frac{3}{2}$. 18. $\frac{8}{5}$. 19. $\frac{3}{8}$. 20. $-\frac{4}{3}$. 21. -1. 22. $-\frac{22}{9}$.
23. $\frac{64}{75}$. 24. -1. 25. -5. 26. 4. 27. 7. 28. 3. 29. -4.
30. 3. 31. -9. 32. 5. 33. 6. 34. 13. 35. 8. 36. 5.

EXERCISE XXVIII. (p. 85).

1. 6. 2. 3. 3. 1. 4. 5. 5. 11. 6. $-\frac{9}{2}$. 7. 1. 8. $\frac{15}{2}$. 9. 3.
10. 1. 11. $\frac{5}{16}$. 12. $-\frac{27}{11}$. 13. 5. 14. 2. 15. $-\frac{18}{17}$. 16. 3.
17. 2. 18. $\frac{10}{11}$. 19. $\frac{114}{7}$. 20. $\frac{39}{5}$. 21. $\frac{13}{19}$. 22. -51. 23. 4.
24. -21. 25. $\frac{55}{14}$. 26. $\frac{37}{1}$. 27. $\frac{49}{3}$.

EXERCISE XXIX. (pp. 86—87).

1. $a-2x+y$. 2. $\frac{bx}{a}$. 3. $m(y+x)-x$.

4. $2x+1$, $2x+3$, $2x+5$; $2x$, $2x+2$, $2x+4$.

5. $y+(y-1)+(y-2)+(y-3)=x$. 6. $(x+1)^2-x^2=(x+1)+x$.

7. $x-y=z$. 8. $5(x-7)=2x+12$. 9. $2c(a+b)=x$. 10. $\frac{x}{y}=b$.

11. $x-b=a$; $x+c=y$. 12. $a-b-c-d=x$. 13. $ab=x$.

14. $ax=by$. 15. $5xy=z$. 16. $44a=11b$.

EXERCISE XXX. (pp. 90—92).

1. 27, 3. **2.** 15, 12. **3.** $22\frac{1}{2}$, $7\frac{1}{2}$. **4.** 23, 24, 25, 26, 27. **5.** 13, 12. **6.** 50, 32. **7.** 15. **8.** 12, 8. **9.** 15. **10.** 19, 20. **11.** 24 feet. **12.** £52, £2. 12*s*. **13.** 9 years. **14.** 9 **15.** 25. **16.** *A* Rs. 25, *B* Rs. 30, *C* Rs. 45. **17.** 48 years, 24 years **18.** 60 years, 20 years. **19.** 50 years. **20.** $6\frac{2}{3}$ days. **21.** 60 hours. **22.** 14, 16, 18. **23.** 20, 40, 60, 80. **24.** 20 years. **25.** 58 rupees, 42 two-anna pieces. **26.** 4, 24. **27.** 16, 10. **28.** 30, 20. **29.** 10, 23, 5, 40. **30.** 12, 35, 5, 75.

MISCELLANEOUS PAPERS I. (pp. 92—97).

Paper I.

1. (i) 3, (ii) 10. **2.** Distance between the 1st and 2nd points is 3·6, between 2nd and 3rd 12·4, between 1st and 3rd 11·7.
3. $-19a-6b+35c$. **4.** $\frac{1}{8}x^3-\frac{17}{20}x^2y+\frac{41}{60}xy^2-\frac{1}{15}y^3$.
5. $x^5+x^4+x^3+x^2+x-2$, **6.** $x^4-x^3y+x^2y^2-xy^3+y^4$.
7. $\frac{10}{7}$. **8.** 22 ; 8.

Paper II.

1. 36. **3.** -3. **4.** $5ax-9by+2cz$, $-5ax+9by-2cz$.
5. $4x+4y-2z$. **6.** Product $=a^3+4b^3-27c^3-a^2b-4ab^2+3a^2c-9ac^2-24b^2c+45bc^2+12abc$; quotient $=a^2-4b^2-9c^2-12bc$
7. 11. **8.** 5, 17, 2, 24.

Paper III.

2 $-12x+120y-34z$. **3.** 13 miles. **4.** $6a^2+10b^2+6c^2-24ab+12ac$.
5. $x^2+5(a-1)x-b$. **6.** $2bc+2ca+2ab-a^2-b^2-c^2$; 40.
7. $\frac{1}{2}x^3-\frac{7}{12}x^2y-\frac{4}{3}xy^2+y^3$. **8.** 18, 20.

Paper IV.

1. 94. **2.** $12a^2-8b^2-25c^2-29ab+45bc+5ca$.
3. $2x^4-5x^3-4x^2+x+2$. **4.** -12.
5. $x^2-\frac{31}{6}x-\frac{283}{12}$, $-\frac{1}{2}x^2+\frac{29}{6}x+\frac{143}{6}$. **6.** $2x+96b$. **7** 7. **8.** 96, 70

Paper V.

1. 34·59. **2.** -19. **3.** $-3a^2+30b^2-40c^2-13ab+23bc+23ca$, $\frac{77}{24}$.
4. $2x^2-2(a+b)x+ab$. **5.** $3a^2-12b^2-96c^2-3ab+4a-12b+5c$.
6. $7x-27y+12z$. **7.** 1. **8.** 20 maunds.

Paper VI.

1. 3   2. $\frac{5}{28}a - \frac{115}{14}b$.   3. $(c-r)x^3 + (2a-q)x^2 + (p-3b)x - d$.
4. $x^2 - 2ax - a^2 + 1$.   6. $4(ax+by+cz)$.   7. 13.   8. 25, 27

Paper VII.

2. $(4a+4b-5c)xy + (4b+4c-5a)yz + (4a+4c-5b)zx$.
3. $-6a + 15b - 9c$.   4. $-(m+n+r+y)$
5. $x^4 - 2qx^3 - 2(p^2-q^2)x^2 + (p^3+p^2q+pq^2-q^3)x - p^2q^2$;
$x^2 - (p+q)x + q^2$.   6. $a^2 - b^2 - c^2 - 2bc$.   7. 2.   8. £500, £250

Paper VIII.

1. $a + 3b + 2d$.   2. $-\frac{4}{3}a - b - 3c + 1$.   3. $\frac{1}{3}a^4 - \frac{11}{12}a^3b + \frac{43}{8}a^2b^2$
$- \frac{?}{4}ab^3 + 6b^4$.   4. $x^4 + 2x^3 + 3x^2 + 4x + 5$.   5. 0.
6. $7x^2 - 7xy + 5y^2$.   7. -5.   8. 35 years.

Paper IX.

2. $-75, -43, -21, -9, -7, -15, -33$ 3. $-187a + 316b + 32c$.
4. $x^2 + \frac{1}{3}xy - \frac{2}{5}y^2$.   6. $x^2 + 4xy + 7y^2$.   7. 1.   8. Rs. 400.

Paper X.

2. Square root $= 6319$.   3. $20ab - 3a^2 - 8a^3$.   4. $x^4 + 2x^3 + 3x^2 + 4x + 5$.
5. $3x^3 - 11x^2 - 36x + 16$.   6. 6.   7. 4.   8. 54.

EXERCISE XXXI. (pp. 98–99).

6. $a(a+b)$.   7. $a(a-2)$.   8. $a(3-5a)$.
9. $x(7x+1)$.   10. $5ax(1-2ax)$.   11. $3x(x-2)$.
12. $ab(a-b+c)$.   13. $a(a^2-ax+x^2)$.   14. $2a^2(1+2a-3a^3)$.
15. $(y-z)(x^2+y^2)$.   16. $(4a-x)(3a-b)$.   17. $(a+b+c)(a+b+c)$.

EXERCISE XXXII. (pp. 101–102).

1. $x^2 + 2x + 1$.   2. $x^2 + 6x + 9$.   3. $9x^2 + 24x + 16$.
4. $49x^2 + 84x + 36$.   5. $9x^2 + 30xy + 25y^2$.
6. $49x^2 - 126xy + 81y^2$.   7. $9a^2b^4 - 24ab^3c + 16b^2c^2$.
8. $a^2x^2 + 4abxy + 4b^2y^2$.   9. $l^2x^2 + 2lmxy + m^2y^2$.
10. $p^2x^2 - 2pqxy + q^2y^2$.   11. $a^4 - 2a^2bc + b^2c^2$.
12. $4l^2m^2 - 20lm^2n + 25m^2n^2$.   13. $\frac{4}{9}a^2 + ab + \frac{9}{16}b^2$.
14. $\frac{25}{49}x^2 - 2xy + \frac{49}{25}y^2$.   15. $\frac{1}{4}a^2b^2 - \frac{1}{3}abc + \frac{1}{9}c^2$.

16. $\frac{25}{4}a^4-4a^2b^2+\frac{16}{25}b^4$. **17.** 5625. **18.** 9604. **19.** 10404.
20. 247009. **21.** 63920025. **22.** 1·010025.
23. 80·946009. **24.** 9980·01.
25. (i) 45, (ii) 154, (iii) 274, (iv) -26. **26.** $a^2+b^2+c^2-2ab+2ac-2bc$.
27. $a^2+b^2+c^2+2ab-2ac-2bc$. **28.** $a^2+b^2+c^2-2ab-2ac+2bc$.
29. $4a^2+b^2+1-4ab-4a+2b$. **30** $4-12x-11x^2+30x^3+25x^4$.
31. $9x^4-30x^3+37x^2-20x+4$.
32. $\frac{25}{4}x^4-\frac{15}{4}x^3y+\frac{187}{48}x^2y^2-xy^3+\frac{4}{9}y^4$.
33. $a^2x^4+2abx^3+(b^2+2ac)x^2+2bcx+c^2$.
34. $\frac{4}{9}l^2+\frac{9}{16}m^2+\frac{1}{4}n^2-lm+\frac{2}{3}ln-\frac{3}{4}mn$. **35.** (i) 481 (ii) 1382.
36 (i) $162\frac{1}{2}$ (ii) 205. **38.** $4a^2+4b^2+4c^2$. **39.** $a^2+b^2+c^2$.
40. $(120)^2-(105)^2$; $(231)^2-(210)^2$; $(528)^2-(496)^2$.

EXERCISE XXXIII. (p. 103).

1. x^2-1. **2.** x^2-16 **3.** x^2-a^2. **4.** $4x^2-81$.
5. $9a^2-25b^2$ **6.** x^2-36y^2 **7.** x^2-4y^2 **8.** $\frac{4}{9}a^2-\frac{9}{16}b^2$.
9. $\frac{1}{4}a^2b^2-\frac{1}{9}b^2c^2$. **10.** $\frac{9}{25}x^2y^2-1$. **11.** a^2-2b. **12.** $49a^2-64$.
13. $4a^2-9b^2$. **14.** x^4-a^4. **15.** $a^2x^2-b^2y^2$. **16.** $p^2a^2-q^2b^2$.
17. $\frac{9x^2-25y^2}{3}$. **18.** $\frac{16x^2-y^2}{4}$. **19.** $16x^4-72x^2y^2+81y^4$.
20. $625a^4b^4-450a^2b^2c^2d^2+81c^4d^4$. **27.** $1-x^4$.
28. $16x^4-625y^4$. **29.** $m^4x^4-n^4$. **30.** x^8-256. **31.** a^8-b^8.
32. $a^8-2a^4b^4+b^8$. **33.** $256a^8-2592a^4b^4+6561b^8$. **34.** 0. **35.** 0.

EXERCISE XXXIV. (p. 105).

1. $a^2-b^2+2bc-c^2$. **2.** $b^2-c^2-a^2+2ac$.
3 $4a^2-9b^2+30bc-25c^2$. **4.** x^4+2x^2+9.
5. $49c^4-16b^4+24a^2b^2-9a^4$. **6** $16a^2b^4-36b^2c^2+108bc^2a-81c^2a^2$.
7. $9a^2x^2-6abxy+b^2y^2-4c^2z^2$. **8.** $l^2a^2+2lanc+n^2c^2-m^2b^2$.
9. $p^2x^4+2prx^2z^2+r^2z^4-q^2y^4$. **10.** $a^2-2ad+d^2-b^2+2bc-c^2$.
11. $a^2-b^2-c^2-d^2+2bc-2bd+2cd$.
12. $4x^2+16xz+16z^2-9y^2-30yw-25w^2$.
13. $9a^4-30a^2b^2+25b^4-49c^4+126c^2d^2-81d^4$.
14. $q^2y^2+s^2w^2+2qysw-p^2x^2-r^2z^2-2pxrz$.
15. $a^2b^2+b^2c^2+d^2a^2+2ab^2c+2a^2bd+2abcd-c^2d^2$.
16. $1+x^2+x^4+x^6+x^8$. **17.** $x^8+x^4y^4+y^8$. **18.** $x^{16}+x^8+1$.
19. $a^2+b^2-c^2+2ab$. **20.** $4ab-4cd$.
21. $2x^2+2y^2+2z^2-2xy-2yz-2zx$.

EXERCISE. XXXV. (pp. 106—107)

1. x^2+5x+6. 2. x^2+6x+5. 3. $x^2+3x-28$. 4. $a^2-8a-48$.
5. $y^2-2y-63$. 6. $-c^2+18c-65$. 7. a^2b^2-ab-6.
8. $25x^2-50x-24$. 9. $x^4+4x^2y^2+3y^4$. 10. $x^2-10xy+24y^2$.
11. $x^2-2xy-35y^2$. 12. $a^2+(b+c)ax+bcx^2$. 13. $a^2-4ax-5x^2$.
14. $9+12lm-5l^2m^2$. 15. $1-x(b+c)+bcx^2$ 16 $\frac{4}{9}x^2+\frac{4}{3}xy+\frac{15}{16}y^2$.
17. $9y^2-3yz-2z^2$ 18. $36x^2-24xy-5y^2$. 19. $9x^4+18x^2-7$.
20. $a^2x^2+axy(b+c)+bcy^2$. 21. x^4-13x^2+36.
22. $x^4-(a^2+576)x^2+576a^2$ 23. $a^4x^4-a^2x^2y^2(b^2+c^2)+b^2c^2y^4$.

EXERCISE XXXVI. (p. 108).

1. $6x^2+7x+2$. 2. $6x^2+31x+35$. 3. $12x^2-23x-24$.
4. $24x^2-2x-15$. 5. $8x^2+18x-35$. 6. $6x^2-x-12$.
7. $63x^2-53x+10$. 8. $6x^2-19x-11$. 9. $8x^2-60x+108$.
10. $24x^2+22x-35$. 11. $48x^2-106x+33$. 12. $14x^2-13x-12$.
13 $6a^2-23a+20$. 14. $33a^2-83a+14$. 15. $35a^2-78a+27$.
16. $10x^2-3xy-27y^2$. 17. $24a^2-34ab+5b^2$.

EXERCISE XXXVII. (p. 109)

1. $x^3+9x^2+26x+24$. 2. $x^3+14x^2+63x+90$.
3. $x^3-2x^2-55x+56$. 4. $x^3-15x^2+71x-105$.
5. $x^3+6x^2y+11xy^2+6y^3$. 6. $x^3+x^2y-14xy^2-24y^3$.
7. $a^3-15a^2b+71ab^2-105b^3$ 8. $a^3-3a^2b-10ab^2+24b^3$.
9. $8x^3-42x+20$. 10. $27a^3-63a^2b+42ab^2-8b^3$.

EXERCISE XXXVIII. (pp. 109-10).

1. x^3+1. 2. x^3-1. 3. $8x^3+125y^3$ 4. $27x^3-64y^3$.
5. $8x^3-y^3$. 4. $125x^3-8y^3$. 7. a^3+216b^3. 8. a^3b^3+8.
$125-x^3y^3$. 10. $\frac{a^3}{8}-\frac{8b^3}{27}$. 11. x^6-729. 12. $64x^6-y^6$.

EXERCISE XXXIX (p. 111).

1. $x^3+6x^2y+12xy^2+8y^3$. 2. $8x^3+12x^2y+6xy^2+y^3$.
3. $27x^3-108x^2+144x-64$. 4. $a^3+\frac{3}{2}a^2b+\frac{3}{4}ab^2+\frac{1}{8}b^3$.
5. $27x^6-108x^7+144x^8-64x^9$. 6. $a^6-6a^4b^2+12a^2b^4-8b^6$.
7. $a^3-b^3-c^3-3a^2b+3ab^2-3b^2c-3bc^2+3c^2a-3ca^2+6abc$.
8. $8a^3+b^3-27c^3+12a^2b+6ab^2-36a^2c+54ac^2-9b^2c+27bc^2-36abc$.

9. $a^3b^3+b^3c^3+c^3a^3+3a^2b^3c+3ab^3c^2+3a^3b^2c+3a^3bc^2+3b^2c^3a+3c^3a^2b+6a^2b^2c^2$. **10.** $a^6+b^6-c^6+3a^4b^2+3a^2b^4-3a^4c^2+3a^2c^4-3b^4c^2+3b^2c^4-6a^2b^2c^2$. **11.** $8l^3-125m^3+n^3-60l^2m+150lm^2+12l^2n+6ln^2+75m^2n-15mn^2-60lmn$. **12.** $27a^3-8b^3-125c^3-54a^2b+36ab^2-135a^2c+225ac^2-60b^2c-150bc^2+180abc$.
13. (i) 432, (ii) 259; (i) 72, (ii) 160 **20.** $2a^3+6ab^2$.
21. $6a^2b+2b^3$. **22.** $9(x+y)^2-27(x+y)$.

EXERCISE XL (pp. 112-13).

1. $a^3-b^3-c^3-3abc$. **2.** $x^3-y^3+1+3xy$. **3.** $x^3-y^3-1-3xy$.
4 $8x^3-27y^3+64z^3+72xyz$. **5.** $1+a^3-a^3b^3+3a^2b$.
6. $a^3b^3+b^3c^3+c^3a^3-3a^2b^2c^2$. **7.** $b^3c^3-c^3a^3+8+6bc^2a$.
8. x^6+45x^3-8. **9.** 1548. **10.** 3676. **12.** $\frac{481}{2}$.

EXERCISE XLIII (p. 122).

1. $ab^2(3a-10b)$. **2.** $2xy(z-3y)$. **3.** $2abc(4a-c)$.
4. $a^2b^2c^2(3ab^2-4bc+5a)$. **5.** $3x^3y^3(xy-3y^2+4x^2)$.
6. $5a^2b^2c^2d^2(ac-2bd+3a^2c^2-4a^3bd)$. **7.** $(x-a)(ab-cd)$.
8. $(y^2+z^2)(x^2-2yz)$.

EXERCISE XLIV (p. 123).

1. $(c-d)(a+b)$. **2.** $(c+d)(a-b)$. **3.** $(x-a)(x+b)$.
4. $(a+b)(a+c)$. **5.** $(x^2+1)(a^2+b^2)$. **6.** $(a-b)(x-z)$.
7. $(2x+y)(2x-m)$. **8.** $(x-2y)(a-b)$. **9.** $(x+1)(3x^3-5)$.
10. $(1+x)(3a+b)$. **11.** $(a-1)(a^2+1)$. **12.** $(x+y)(x^2+y^2)$.
13. $(x+ay)(ax+y)$. **14.** $(bp+aq)(ap+bq)$. **15** $b(5a-2c)(b-c)$.
15. $(x+3y)(2p-3q)$. **17.** $(x-y)(a-b+c)$ **18.** $(a+b)(ax+by+c)$.
19. $(b+c)(ab+bc+a^2)$. **20.** $(a-c)(l+mb)$. **21.** $(a+b+c)(x+y+z)$.

EXERCISE XLV (pp. 125-26).

1. $(x+1)^2$. **2.** $(x-2)^2$. **3.** $(3x+1)^2$. **4.** $(a-5b)^2$.
5. $(3-2x)^2$ **6.** $(3x-4y)^2$. **7.** $(1-xy)^2$. **8.** $x^2(x-3y)^2$.
9. $(a^2-bc)^2$. **10.** $\left(\frac{3x}{4}-\frac{4y}{3}\right)^2$. **11.** $\left(\frac{a}{3}+\frac{b}{2}\right)^2$ **12.** $9(a^3-b^3)^2$.
13. $(ax-by)^2$. **14.** $(4x-3a)^2$. **15.** $(2xyz-1)^2$. **16.** $(5x+y+4)^2$.
17. $(4a-6b-1)^2$. **18.** $4x^2$. **19.** $(a+b)(a+b+2c)$.

20. $(2x-3y)(2x-3y-1)$. 21. $(2-a)(2-a-b)$.
22. $(5x-1)(5x-1+a)$. 23. 1. 24. 100. 25. 0.
26. 144. 27. 49. 29. $81a^2-36ab+4b^2$. 30. $900x^2-660xy+121y^2$.
31. $4b^2-4bc+c^2$. 36. 4. 37 $\frac{49}{4}$. 38. $\frac{p^2}{4}$.
39. $\frac{49}{16}$. 40. $\frac{169}{100}$. 41. b^2.

EXERCISE XLVI (pp. 127-28).

1. $(x+1)(x-1)$. 2. $(x+2)(x-2)$. 3. $(2x+1)(2x-1)$.
4. $(5a-4)(5a+4)$. 5. $(xy+7)(xy-7)$. 6. $(ab^2+c^2)(ab^2-c^2)$.
7. $a^2(x^2+y)(x^2-y)$. 8. $(a^3+6)(a^3-6)$. 9. $(x^2+11)(x^2-11)$.
10. $(6a^2+13)(6a^2-13)$. 11. $(5x+4y)(5x-4y)$.
12. $(7x-8y)(7x+8y)$. 13. $x^2y^2(1+10z)(1-10z)$.
14. $4a^2b^2(2a+3b)(2a-3b)$. 15. $4c^2(2a^2+5)(2a^2-5)$.
16. $(lm+3a^2)(lm-3a^2)$. 17. $(11a-9x)(11a+9x)$.
18. $3(a+3b)(a-3b)$. 19. $x(2y+3z)(2y-3z)$. 20. $3a(2b+5c)(2b-5c)$.
21. $(ap+bq)(ap-bq)$. 22. to 24 Apply $a^2-b^2=(a+b)(a-b)$
25. $x^2(y^2+x^2)(y+x)(y-x)$. 26. $x(9+x^4)(3+x^2)(3-x^2)$.
27. $(a^4+b^4)(a^2+b^2)(a+b)(a-b)$. 28. $(a+b+c)(a+b-c)$.
29. $(a-b+c)(a-b-c)$. 30. $(a+b+c)(a \quad -c)$.
31. $(a+b+c+d)(a+b-c-d)$. 32. $(1+x+y+z)(1-x-y-z)$.
33. $(3x-4y+2a+b)(3x-4y-2a-b)$. 34. $(11x+14)(3x+4)$.
35 $(x+y+2a)(x-y)$. 36. $(7a+b)(a+7b)$.
37. $(x+y)^2(x-y)^2$. 38. $3(5x-19y)(x+9y)$.
39. $(2x+3y)(2x-3y-1)$. 40. $(a+2b)(a-2b+2c)$.
41. $(3a+b)(3a-b-4)$. 42. $(4x+5y)(4x-5y+z)$.
43. $4ab$. 44. $12x(4y-7z)$. 45. $-20a(9b-11c)$.
46. $(8a+11b+12c)(4a+7b)$. 47. $-8(a+2c)(3b+5d)$.
48. $8ac$.

EXERCISE XLVII. (pp 129-30).

1. $(x+y+1)(x-y-1)$. 2. $(x+y+3z)(x+y-3z)$.
3. $(x-2y+z)(x-2y-z)$. 4. $(x-3y+5)(x-3y-5)$.
5. $(2a-3b+2c)(2a-3b-2c)$. 6. $(3a+5b+10c)(3a+5b-10c)$.
7. $(x+y-z)(x-y+z)$ 8. $(a+b+c+d)(a+b-c-d)$.
9. $(2x-y+7z)(2x-y-7z)$. 10. $(2a+4b-1)(2a-4b+1)$.
11. $(3a+4b-5x-y)(3a-4b-5x+y)$.
12. $(2a-2b+3c-3d)(2a-2b-3c+3d)$.

13. $(5a-4b+c+7)(5a-4b-c-7)$. **14.** $(a^2+2a+2)(a^2-2a+2)$.
15. $(x^2+4x+8)(x^2-4x+8)$. **16.** $(9a^2+12ab+8b^2)(9a^2-12ab+8b^2)$.
17. $(2x^2+6xy+9y^2)(2x^2-6xy+9y^2)$.
18. $9(x^2+2xy+2y^2)(x^2-2xy+2y^2)$.
19. $(32+24a+9a^2)(32-24a+9a^2)$. **20.** $(a^2+a+1)(a^2-a+1)$.
21. $(x^2+3x+9)(x^2-3x+9)$. **22.** $(a^2+2ab+4b^2)(a^2-2ab+4b^2)$.
23. $(4x^2+6xy+9y^2)(4x^2-6xy+9y^2)$.
24. $(x^4-x^2+1)(x^2+x+1)(x^2-x+1)$.
25 $(a^4-a^2b^2+b^4)(a^2+ab+b^2)(a^2-ab+b^2)$. **26.** $5(2x^2+17y^2+6xy)$.
27. $x^2-3xy+9y^2$. **23.** $(x-y)^2-(x-y)z+z^2$.

EXERCISE XLVIII. (p. 131).

1. $(x+1)(x^2-x+1)$. **2.** $(x-1)(x^2+x+1)$.
3. $(3+2a)(9-6a+4a^2)$. **4.** $(3-2a)(9+6a+4a^2)$.
5. $(2x-y)(4x^2+2xy+y^2)$. **6.** $(3x+4)(9x^2-12x+16)$.
7. $(a+3b)(a^2-3ab+9b^2)$. **8.** $(2a-b^2)(4a^2+2ab^2+b^4)$.
9. $(xy-3z)(x^2y^2+3xyz+9z^2)$. **10** $(l-6m)(l^2+6lm+36m^2)$.
11. $x(2x-y)(4x^2+2xy+y^2)$. **12.** $(x+y)(x-y)(x^2+xy+y^2)$
$(x^2-xy+y^2)(x^2+y^2)(x^4-x^2y^2+y^4)$. **13.** $y(12x^2+6xy+y^2)$.
14. $(a^2+b^2-2x^2y^2)(a^4+b^4+2a^2b^2+2a^2x^2y^2+2b^2x^2y^2+4x^4y^4)$.
15. $27b(9a^2+3ab+7b^2)$. **16.** $(12b-a)(37a^2-6ab+36b^2)$.
17. $(a-b)(a^2+ab+b^2+m)$. **18.** $(3x+a)(9x^2-3ax+a^2+3)$.
19. $(x+y)(x^2-xy+y^2+2x+2y)$. **20.** $(x-2y)(x^2+2xy+4y^2+5)$.
21. $(x+y)^2+(x+y)2z+4z^2$. **22.** $4(19x^2+54xy+39y^2)$.
23. $(3x^2-5x+6)^2+(3x^2-5x+6)(2x^2+x-1)+(2x^2+x-1)^2$
24. $2(a+b)x$.

EXERCISE XLIX (p. 134-135).

1. $(x+1)(x+3)$. **2.** $(x+1)(x+5)$. **3.** $(x+5)(x+4)$. **4.** $(x-3)(x-5)$
5. $(x+5)(x+12)$. **6.** $(x-7)(x-4)$. **7.** $(x-3)(x-2)$.
8. $(x-8)(x-1)$. **9.** $(x+4)(x+6)$. **10.** $(x-4)(x-1)$.
11. $(x-4)(x+3)$. **12.** $(x+18)(x-1)$. **13.** $(x-9)(x+8.)$
14. $(x+12)(x+8)$. **15.** $(x+12)(x-7)$. **16.** $(x+10)(x-7)$.
17. $(x-12)(x-1)$. **18.** $(x-20)(x+19)$. **19.** $(x-15)(x+5)$.
20. $(x+14)(x-13)$. **21.** $(x+8)(x-5)$. **22.** $(a-10)(a+4)$.
23. $(x+10)(x+3)$. **24** $(x+20)(x-12)$. **25.** $(l+16)(l+3)$.
26. $(c-8)(c+7)$. **27.** $(a-26)(a+3)$. **28.** $(x+5y)(x-y)$.

29. $(x-8y)(x+6y)$. 30. $(x-15y)(x+10y)$.
31. $(x+20y)(x-12y)$. 32. $(x+6a)(x-5a)$.
33. $(a+15b)(a-4b)$. 34. $(ab+8)(ab+2)$. 35. $(p+12q)(p-2q)$.
36. $(l+9m)(l-7m)$. 37. $(c-15d)(c+3d)$.
38. $(h+11k)(h-10k)$. 39. $(a-27b)(a-2b)$.
40. $(m-14n)(m+6n)$. 41. $(a-36x)(a+2x)$.
42. $(h-16k)(h+5k)$. 43. $(x-16y)(x+15y)$.
44. $(x^2+11)(x^2+6)$. 45. $(m^2+9)(m^2+7)$.
46. $(c+4)(c-4)(c+5)(c-5)$.
47. $(a-2b)(a^2+2ab+4b^2)(a-b)(a^2+ab+b^2)$.
48. $(x^3-9)(x^3-6)$. 49. $(x^4-7y^4)(x^4+6y^4)$.
50. $(a^5-16b^5)(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$. 51. $2(x+2)(2x+5)$
52. $(5x-9y+20)(5x-9y-4)$. 53. $(x^2+5x-24)(x+1)(x+$ [illegible]
54. $(29b-13a)(23b-9a)$. 55. $3b(a+5b)$.
56. xb. 57. $(l^2+11m^2+10n^2)(l^2-8m^2-9n^2)$.
58. $-25(5q+2p)(q+p)$. 59. $(a+b)(a+b+1)$.
60. $(x+6b)(x-a+b)$ 61. $(x+a)(x-a+1)$.
62. $(x+3a)(x+5a-2)$. 63. $(x+5c-9d)(x-7c+10a)$.
64. $4(x-4)(x-3)(x-1)(x-2)$. 65. $25(x-2)(x+1)(x-$ [illegible]$)(x+2)$.

EXERCISE L (p. 136).

1. $(x+7)(x-2)$. 2. $(x-2)(x+1)$. 3. $(x+4)(x-3)$.
4. $(x+12)(x-3)$. 5. $(a-12)(a+4)$. 6. $(x-10)(x+3)$
7. $(x-9)(x+5)$. 8. $(a-9)(a-3)$. 9. $(a+9)(a+3)$.
10. $(a-5b)(a-4b)$. 11. $(c+2d)(c-d)$. 12. $(p-9)(p-8)$.
13. $(a+6b)(a+5b)$ 14. $(x+7y)(x-4y)$. 15. $(x-9y)(x+5y)$.

EXERCISE LI (pp. 138-39).

1. $(3x-4y)(x+y)$ 2. $(x-5)(3x+4)$. 3. $(4x+3)(x-10)$
4. $(4x-3)(3x+1)$. 5. $(2x-7y)(5x+3y)$. 6. $(a+2b)(6a-5b)$.
7. $(15a-8)(4a-3)$. 8. $(3x+4)(2x+3)$. 9. $(5p+21)(2p-1)$.
10. $(3y-2)(7y-1)$. 11. $(5a+6)(4a+5)$. 12. $(7ab+2)(14ab-3)$.
13. $(2a-7)(7a-8)$. 14. $(x+9)(16x+1)$. 15. $(3x+5)(4x+9)$.
16. $(63x+64)(x-1)$. 17. $(7x-12)(4x-3)$. 18. $3(x+1)(5x-24)$.
19. $(4x+5)(14x-11)$. 20. $(3x+4)(15x-7)$. 21. $(x^2+3)(2x^2+25)$.
22. $(x^3-2)(3x^3-1)$. 23. $(a^4-2)(2a^4-3)$.

24. $(a+b+1)(3a+3b-2)$.
25. $(3a-5b+2)(15a-25b+3)$.
26. $29y(8x+17y)$.
27. $-(10a-3b)(a+25b)$.
28. $-ab$.
29. $49(x-3)(x+2)(x+1)(x+4)$.
30. $25(x-4)(x+2)^2(x+1)$.

EXERCISE LII (p. 140).

1. $(5x-3)(3x-1)$.
2. $(x+3)(4x-1)$.
3. $(x+3)(3x+1)$.
4. $(4x-3)(2x+3)$.
5. $(2x+5)(2x-1)$.
5. $(3x+7)(2x-3)$.
7. $(3x-2y)(3x+y)$.
8. $(x+5a)(2x-5a)$.
9. $(x-3)(2x+3)$.
10. $(2c+3)(4c+5)$.
11. $(3a-4)(8a+1)$.
12. $(x-3)(9x-4)$.
13. $(3x+4)(15x-7)$.
14. $(4x+5)(14x-11)$.
15. $(18x-13)(6x-7)$.
16. $(5x+2)(3x+1)$.
17. $(2x-5)(7x-3)$.
18. $(x+1)(3x+2)$.
19. $(x+2)(2x-1)$.
20. $(x+2)(5x+1)$.
21. $(3x+17)(x-1)$.
22. $(2a+5b)(2a-b)$.
23. $(x-6y)(5x+y)$.
24. $(3x+2y)(5x-y)$.
25. $(3a-7)(2a+5)$.

EXERCISE LIII. (p. 141.)

1. $(2x+1)^3$.
2. $(x-4)^3$.
3. $(x+5)^3$.
4. $(4a-3b)^3$.
5. $18xy(2x+3y)$.
6. $-30ab(5a-2b)$.
7. $-3x(x+1)^2(x-1)^2$
8. $-3(2x-3y)(3y-z)(2x+$

EXERCISE LIV. (pp. 143-44).

1. $(x+y-1)(x^2+y^2+1-xy+x+y)$.
2. $(x-y-1)(x^2+xy+x+y^2-y+1)$.
3. $(x-y+z)(x^2+y^2+z^2+xy+yz-zx)$.
4. $(x+y-z)(x^2+y^2+z^2-xy+yz+zx)$
5. $(2a-b-c)(4a^2+b^2+c^2+2ab+2ac-bc)$.
6. $(3a-2b-1)(9a^2+4b^2+1+6ab+3a-2b)$.
7. $(a^2+b^2+c^2)(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)$.
8. $(xy+yz+zx)(x^2y^2+y^2z^2+z^2x^2-xy^2z-yz^2x-zx^2y)$.
9. $3(b-c)(c-a)(a-b)$.
10. $3(x-2y)(2y-z)(z-x)$.
19. $x^2+4y^2+9z^2-2xy+3xz+6yz$.
20. $a+2b+3c$.
21. $2a+2b+2c$.
22. 0.
23. 1.

EXERCISE LV. (pp. 149-51).

1. $(x+y-6)(x+y+3)$.
2. $(3a-4b-10)(3a-4b+6)$
3. $2(a-d)(a+b+c+d)$.
4. $2(2a-1)(2a^2+a+2)$.

5. $(ac+bd+ad-bc)(ac+bd-ad+bc)$. 6. $(a+b-3c)(a+b-3c-1)$.
7. $2(a-2b)(a^2-ab+4b^2)$. 8. $(a^2+2a+10)(a^2-2a+10)$.
9. $(2a-3b+3c-4d)(2a-3b-3c+4d)$.
10. $(x-1)(x+3)(x+4)(x+8)$. 11. $(x^2-x-1)(x^2+3x+1)$.
12. $(x+1)(x-3)(x^2+3)$. 13. $(x+1)(x-2)(x^2-x+4)$.
14. $(x^2+3x+4)(x^2+3x-1)$. 15. $(a+b)(a-b)(a^2+b^2+c^2)$.
16. $(a+b+c)(a+3b-c)$. 17. $(a+2b)(9a^2-2ab+4b^2)$.
18. $(x-2)(x^2-6x+4)$. 19. $(x-y+z)(x-7y-z)$.
20. (i) $x^2+25y^2+4+5xy-2x+10y$. (ii) $a^4-4a^2bc+7b^2c^2$.
(iii) $x^2+2xy+y^2+2zx+2zy+4z^2$.
(iv) $a^2x^2+b^2y^2+b^2x^2+a^2y^2+4abxy-abx^2-a^2xy-b^2xy-aby^2$.
21. $(ax+by+cz)^2-(ax+by+cz)(bx+cy+az)+(bx+cy+az)^2$.
22. $(x-a)(x-b)$. 24. $3x+a+b+c$.
28. (i) $(x^2+5x+3)(x^2+5x+7)$. (ii) $(x^2-x-16)(x^2-x-10)$.
(iii) $(x-4)(x+2)(x^2-2x-19)$. (iv) $(x^2+x-1)(9x^2+9x-7)$.
40. $(a+b+c)$. 41. $(a+b)(b+c)(c+a)$. 42. $(a-b)(b-c)(c-a)$.
43. $(a-b)(b-c)(c-a)$. 44. $(a-c)(b-a)(c-b)$.
45. $-(a-b)(b-c)(c-a)(3x+a+b+c)$. 46. $(a+b)(b+c)(c+a)$.
47. $(a+b+c)(ab+bc+ca)$. 48. $(x-2y+z)(x-3y+z)$.
49. $(2x+3y+4z+1)(2x-3y-4z)$. 50. $3(a+b-c)(2a+b+c)$.

EXERCISE LVI. (pp. 153—54).

1. (i) 4, 10. (ii) 8. (iii) 15 (iv) 1.
2. xy^2, x^2y^3. 3. $4x^3y$; $9x^5y^2z^6$ 4. $2ab^4c^3d^4e^5$; $7x$ 5. $4p^2qr$.
6. $4a^2x^3$. 7 $a-b$; $x-y$. 8 $a-2$. 9. $3x+2y$.
10. $xy-3$. 11. $a+b-c$. 12. $2x-y$. 13. $x-5$.
14. $x-1$. 15. $x+2$. 16. $x(x^2+x+1)$. 17 $x+y+z$.
18. $a-2b$. 19. $8(x-y)^2(x+1)^3$. 20. $x-y$. 21 x^2+x+1.
22. $2x+3y+z$. 23. $ax+by$. 24. $4x(2y+3x)$ 25. $ax+c$.
26. $x-2$. 27. $a+b+c$.

EXERCISE LVII. (pp. 159—60).

1 $x-1$. 2. $x-2$. 3. $x-3$. 4. $x-3$. 5. $x-4$ 6. $2x-1$.
7 x^4+x^2+1. 8. $2(3x-2)$. 9. x^2-x+1. 10. x^2+xy+y^2.
11. $xy(x+y)$. 12. $2x^2-xy+y^2$. 13. $(x-1)^2$. 14. $2x(2x-1)(x+2)$.

15. $4x^2-5xy-7y^2$. **16.** x^2-2x+3. **17.** x^2-4x+1.
18. $2x^2+3x-2$. **19.** $2x^2-x-2$. **20.** $(x-a)^2$. **21.** $3x^2+2x+1$.
22. $ax+b$. **23.** ax^2+by+c. **24.** $ax+by+cz$. **25.** $x-y$.
26. $3x^2-2x+4$. **27.** x^2+x+41. **28.** x^2+x-3. **29.** $2x-1$.
30. $x-2$. **31.** $x-$. ; 1.

EXERCISE LVIII. *(p. 162.)*

1. x^2-2x+3. *2.* No common factor. **3.** $x-3$. **4.** x^2-3x+4.
5. $x-1$. **6.** $2x+3$. **7.** $x-2$. **8.** x^2+x+3.

EXERCISE LIX. (pp. 163—64).

1. $12a^2b^2$. **2.** $72x^3y^2z^3$. **3.** $48a^4b^3c^5$. **4.** $720a^3b^3c^3$. **5.** $ab^2(a^2+x)\times(a-x)$. **6.** $2x^2(x-2)^2(x+2)$. **7.** $xy(2x+3y)(2x-3y)$.
8. $(x-6)(x-1)(x-4)$. **9.** $x^2(3x-7y)(x+y)(3x-11y)$.
10. $x(x+4)(x+3)(x^2-4x+16)$. **11.** $ab(a-2b)(a^2+2ab+4b^2)(a-3b)$.
12. $(x+y)(x^2+y^2)(x-y)$. **13.** $(x^2-a^2)(x^4+a^2x^2+a^4)$.
14. $(2x+3y+z)(2x-3y-z)(3y+2x-z)$.
15 $(x-3)(x-1)(x+2)(x-2)(x+4)$. **16.** $1+x^2+x^4$.
17. $16x^8-6561a^8$. **18.** $240(x-1)^2(x+1)^2(x^2+1)(x^2+x+1)(x^2-x+1)$.
19. $(x^2-3)(9x^2-1)(3x^2-1)(3x^2+1)$. **20** $(x-1)^3(x+1)$.

EXERCISE LX. (pp. 165—66).

1. $(2x^2+2x+1)(3x^3-2x-1)$.
2. $(x^2+x-1)(3x^3-8x^2-33x-10)$.
3. $(x^4-x^3-1)(x^3-1)(x+1)$.
4. $(7x-3y)(3x^2-2xy+y^2)(3x^2-5xy-2y^2)$.
5. $(x^2-2x+1)(x+5)(x^2-5x+6)$.
6. $(4y^5+11y^4x+81x^5)(3x^3-2x^2y+4xy^2-2y^3)$.
7. $(x^6-2x^4+x^2-1)(x^2+x+1)$.
8. $(x-1)(x-2)(x-3)(x+2)(x+4)$. **9.** $(9x^2-4)(4x^2-9)$.
10. $(5x-11)(7x^4-2x^2-9x-2)$.
11. $(3x^2-x-1)^2(3x^2-x+1)(3x^2+x-1)$.
12. $(x^3+x^2+x-6)(x^4-12x^3+42x^2-52x+21)(x^4-10x^3+33x^2-44x+20)$.
13. $(x-3)(4x^3-20x^2+17x-4)$
14. $(x^2-y^2)(x-2y)^2(x+4y)(x+5y)(x^2-5y^2)$.

EXERCISE LXI. (p. 167).

2. $a = \frac{5}{2}$, $b = 5$. 4. $m = 8$, $n = -24$.

EXERCISE LXII. (pp. 171–72).

1. $\frac{4c^2}{3a^2b}$. 2. $\frac{3x^2}{5yz^2}$. 3. $-\frac{3a^3by^2}{4z}$. 4. $-\frac{5p^2q^2}{3r^3s^4}$.

5. $\frac{x^2+xy+y^2}{x(x+y)}$. 6. $\frac{x}{b}$. 7. $\frac{a(a+b)}{b}$. 8. $\frac{x-yz}{yz}$.

9. $\frac{3a-4x}{3a}$. 10. $\frac{a-b}{ab}$. 11. $\frac{(a+b)^3(a-b)}{(a^2+ab+b^2)^2}$. 12. $\frac{2x-3y}{2x}$.

13. $\frac{3ab(a-2b)}{a+2b}$. 14. $\frac{a+c-b}{a+b-c}$. 15. $\frac{x+y}{x-y}$. 16. $\frac{2x-3}{4x-3}$. 17. $\frac{x+2}{3x-1}$.

18. $\frac{4a-3b}{a+2b}$. 19. $\frac{3x+y}{2x-3y}$. 20. $\frac{x+c}{x-c}$. 21. $\frac{x+ay}{x-cy}$. 22. $\frac{x+2}{x+6}$.

23. $\frac{a^2-ab+b^2}{a^2}$. 24. $\frac{3x+1}{2x+1}$. 25. $\frac{a-b}{b-c}$. 26. $\frac{7x-4}{5x+2}$. 27. $\frac{a-2b}{a+4b}$.

28. $\frac{2x-y}{4x+y}$. 39. $\frac{a^3-a^2-2a+4}{a^3-a^2-a+2}$. 30. $\frac{a^2-6ab-3b^2}{a^2-4ab-2b^2}$.

31. $\frac{3(x^2-7ax+12a^2)}{2(x^2+7ax+12a^2)}$. 32. $\frac{2x^2+3x-5}{7x-5}$. 33. $\frac{3x^3+10x^2+9x+2}{3x^3-10x^2+9x-2}$.

34. $\frac{3x^2+4xy+y^2}{2x+5y}$. 35. $\frac{3a^4+10a^3b+30a^2b^2+27ab^3+6b^4}{2a^4+6a^3b+15a^2b^2+6ab^3+13b^4}$.

EXERCISE LXIII (p. 174).

1. $\frac{bcx}{abc}$, $\frac{acy}{abc}$, $\frac{abz}{abc}$. 2. $\frac{a^2}{abc}$, $\frac{b^2}{abc}$, $\frac{c^2}{abc}$. 3. $\frac{a^3b}{ab}$, $\frac{b^4}{ab}$, $\frac{c^3a}{ab}$.

4. $\frac{(a+b)^2}{a^2-b^2}$, $\frac{(a-b)^2}{a^2-b^2}$. 5. $\frac{bc(x-y)}{abc}$; $\frac{ca(y-z)}{abc}$, $\frac{ab(z-x)}{abc}$.

6. $\frac{a^2(x+a)}{ax(x+a)}$, $\frac{x^2(x+a)}{ax(x+a)}$, $\frac{ax(x-a)}{ax(x+a)}$. 7. $\frac{4a}{4(x+1)}$, $\frac{6a}{4(x+1)}$, $\frac{5a}{4(x+1)}$.

8. $\frac{x^2(x-2y)}{(x-2y)^2}$, $\frac{2xy^2}{(x-2y)^2}$. 9. $\frac{3x+4}{x(x+2)}$, $\frac{3(x+2)}{x(x+2)}$, $\frac{5x}{x(x+2)}$.

10. $\frac{b(a-b)}{b(a^2-b^2)}$, $\frac{3a(a+b)}{b(a^2-b^2)}$, $\frac{2ab^2}{b(a^2-b^2)}$.

11. $\frac{(x+1)(x^3+1)}{x(x^3+1)(x-1)}$, $\frac{x(x+2)(x^2-x+1)}{x(x^3+1)(x-1)}$, $\frac{3x(x-1)}{x(x^3+1)(x-1)}$.

12. $\dfrac{(3x-1)^2}{(x-3)(2x-1)(3x-1)}$, $\dfrac{(x-3)^2}{(x-3)(2x-1)(3x-1)}$.

13. $\dfrac{b^2(x+y)(a-b)}{a^2b^2(a^2-b^2)}$, $-\dfrac{a^2(x-y)(a+b)}{a^2b^2(a^2-b^2)}$, $\dfrac{ab(x^2-y^2)}{a^2b^2(a^2-b^2)}$.

14. $\dfrac{(x+1)(x-1)}{(x+5)(x-2)(x-1)}$, $\dfrac{(x+2)(x+5)}{(x+5)(x-2)(x-1)}$, $\dfrac{(x+3)(x-2)}{(x+5)(x-2)(x-1)}$.

EXERCISE LXIV. (pp. 178–80).

1. $\dfrac{11}{9a}$. **2.** $\dfrac{1-x}{6x}$. **3.** $\dfrac{a^2+b^2}{a^2-b^2}$. **4.** $\dfrac{18a^2+50b^2}{9a^2-25b^2}$. **5.** $\dfrac{4ab}{a^2-b^2}$.

6. $\dfrac{a^2+b^2}{ab(a^2-b^2)}$. **7.** $\dfrac{x^3+x^2-11x-8}{(x+4)(x-2)}$. **8.** $\dfrac{2x^2+17x+38}{(x+2)(x+5)}$. **9.** 0.

10. $\dfrac{2a^3}{a^2-b^2}$. **11.** $\dfrac{1}{(x-b)(x-c)}$. **12.** 0. **13.** $\dfrac{3}{a+c}$. **14.** 1. **15.** 0.

16. $\dfrac{6x^2+65x+14}{6(4-9x^2)}$. **17.** 2. **18.** $\dfrac{4x^2-4x+7}{(3x+1)(4x-5)(4x-2)}$.

19. $\dfrac{2}{(4x+1)(3x-1)}$. **20.** $\dfrac{4x^2-7x-10}{x(x-3)(x-4)}$. **21.** $\dfrac{2x^2+x(a^2+b^2)-(a^2+b^2)}{x(x-a)(x+b)}$.

22. $-\dfrac{2(x^2+y^2+z^2)}{(x+y-z)(x-y+z)}$. **23.** 0. **24.** $\dfrac{1}{(4x+1)(5x+1)}$.

25. $\dfrac{x+11}{(x-1)(x+5)}$. **26.** $\dfrac{1}{x-3y}$. **27.** $\dfrac{x+11}{(x-3)(x+2)(x+4)(x+1)}$.

28. 0. **29.** $\dfrac{12b^3}{(a^2-b^2)(a^2-4b^2)}$. **30.** $\dfrac{y(x^2+10xy+33y^2)}{(x+y)(x+3y)(x+4y)(x+5y)}$.

31. $3(x+a)$. **32.** $\dfrac{2}{a^2+ab+b^2}$. **33.** 1. **34.** 1. **35.** 1.

36. $-\dfrac{64ab^3}{a^4-16b^4}$. **37.** $\dfrac{1}{a^4-x^4}$. **38.** $\dfrac{8a^7}{a^8-x^8}$. **39.** $\dfrac{1}{b-a}$.

40. $\dfrac{16x}{1-x^4}$.

EXERCISE LXV. (pp. 182—83).

1. $\dfrac{a^2}{b^2}$. **2.** 1. **3.** $\dfrac{x^2z}{3y^4}$. **4.** $\dfrac{5a^3bz}{2c^2xy}$. **5.** $\dfrac{9x}{yz}$. **6.** $\dfrac{16x^2y^6}{25a^5b^3}$.

7. $2x^2z$. **8.** $\dfrac{a+b}{c+d}$. **9.** $\dfrac{x-2}{x-5}$. **10.** $\dfrac{x+2}{x+3}$. **11.** 1. **12.** $\dfrac{(a+b)^2}{(a-b)^2}$.

13. $\frac{a+b}{a}$. 14. $\frac{x-2}{x+2}$. 15. $\frac{x-d}{x+c}$. 16. $\frac{x^2-1}{x^2+x-6}$. 17. 1.

18. $-\frac{4xy(x^2+y^2)}{x^4+x^2y^2+y^4}$. 19. $\frac{a^3}{b^3}-\frac{b^3}{a^3}$. 20. (i) $x^2+2+\frac{1}{x^2}$.

(ii) $x^3-3x+\frac{3}{x}-\frac{1}{x^3}$. 21. (1) 27, (2) 83, (3) 1692, (4) 970.

22. (1) $\frac{bc}{a^2}$. (2) $\frac{y^9a^3c}{xz^3b^2}$.

EXERCISE LXVI. (p. 185).

1. $\frac{b^2}{c^2}$. 2. $\frac{xy^3}{a^3b^3}$. 3. $\frac{x}{y}$. 4. $\frac{3xz^2b}{2ac^2}$. 5. $\frac{3ax^2dy}{2bc^2}$

6. $\frac{9a^3d^5}{4bc^3}$. 7. $\frac{x^2-1}{x^2-4}$. 8. $\frac{4x^2-9}{4x^2-1}$. 9. $\frac{x-3}{x-4}$.

10. $-\frac{a^4+a^2b^2+b^4}{ab(a-b)^2}$. 11. $\frac{xy}{x^2+y^2}$. 12. $\frac{1}{x+y}$. 13.

14. $\frac{x-1}{x-2}$. 15. (1) $\left(\frac{a}{b}-\frac{c}{d}\right)\left(\frac{a^2}{b^2}+\frac{ac}{bd}+\frac{c^2}{d^2}\right)$ (2) $\left(\frac{x^2}{y^2}-\frac{x}{y}+1\right)\left(\frac{x}{y}+1\right)$

(3) $\left(\frac{a^2}{b^2}+1+\frac{b^2}{a^2}\right)\left(\frac{a^2}{b^2}-1+\frac{b^2}{a^2}\right)$ (4) $\left(\frac{a}{b}+\frac{b}{a}-1\right)\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}+\frac{a}{b}+\frac{b}{a}\right)$.

16. (1) $\frac{a^3}{b^3}+\frac{a^2c}{b^2d}+\frac{ac^2}{bd^2}+\frac{c^3}{d^3}$ (2) $\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-\frac{a}{c}-\frac{b}{a}-\frac{c}{b}$.

EXERCISE LXVII. (pp. 190-92).

1. $\frac{b(a-x)}{a}$. 2. $-\frac{a+b}{a}$. 3. $\frac{2(15y+2x)}{5y(2x+3y)}$. 4. $\frac{2(7x-45ay)}{9(4xy-3a)}$.

5. $\frac{ad+bc}{ad-bc}$. 6. $\frac{ay+bx}{ax+by}$. 7. $\frac{2ab}{a+b}$. 8. $\frac{x+6}{2(x+4)}$. 9. $\frac{a^2+b^2}{(a+b)^2}$.

10. $\frac{x^2+ax+cx-x+ac-b}{x^2+ax+bx+x+ab+c}$. 11. $\frac{ac-bd}{ac+bd}$. 12. $\frac{2ab}{a^2+b^2}$.

13. $\frac{ab}{a^2+b^2}$. 14. $\frac{5-x}{9x-1}$. 15. $\frac{b}{a}$. 16. $\frac{2a}{1-a^2}$. 17. $\frac{b-c}{1+bc}$.

18. x. 19. $\frac{4a^4}{a^4-b^4}$. 20. a^2-3a+1. 21. 1. 22. 1.

23. $\frac{2b^3(a-b)}{a^2(a^2-ab+2b^2)}$. 24. $\frac{x^2-1}{x^2-2x}$. 25. $\frac{x(3x+2)}{4(x^2-x-1)}$.

26. $\frac{7x-2}{3x-1}$. 27. $\frac{x^2-3x+1}{x^2-4x+1}$. 28. 2. 31. $\frac{5x+5}{x^2-3x+5}$.

32. $\frac{3x+1}{5x^2-7x+13}$.

EXERCISE LXVIII (pp. 200—03).

4. 0. **5.** $\dfrac{(a+b+c)^2}{abc}$. **6.** $\dfrac{a^4-10a^3b-6ab^3-b^4}{a^4+10a^3b+6ab^3-b^4}$. **7.** $n(n-1)$.

8. $\dfrac{x(1-x)}{x+2}$. **9.** $\dfrac{x+1}{(x-1)(2x-1)}$. **10.** $\dfrac{1}{a+b+c}$. **11.** $\dfrac{4b}{a}$.

12. $\dfrac{4a^2x}{x^4-a^4}$. **13.** 1. **14.** $a-b$. **15.** $\dfrac{2a^4}{(a^2+b^2)^2}$. **16.** $\dfrac{ab}{a+b}$.

17. $\dfrac{a+b}{a-b}$. **18.** $x^{2n}+2$. **19.** $\dfrac{4x^4+8}{1+x^4+x^8}$. **20.** 1. **21.** $-(a+b+c)$.

22. $\frac{1}{2}(a+b+c)$. **23.** $a+b+c$. **27.** 12. **28.** $2(a+b+c)$. **29.** $a+b+c$.

30. 6. **31.** 0. **32.** 0. **33.** 1. **34.** 0. **35.** $a+b+c$. **36.** -1.

37. 0. **38.** $\dfrac{1}{abc}$. **39.** $\dfrac{ab+bc+ca}{a^2b^2c^2}$. **40.** x. **41.** 2.

42. $a+b+c+1$. **43.** $\dfrac{(b-c)(a-c)(a-b)}{(b+c)(c+a)(a+b)}$.

44. $\dfrac{m(a-b)(b-c)(c-a)}{(m+ab)(m+bc)(m+ca)}$. **45.** 0. **46.** 0. **47.** $\dfrac{1}{abc}$.

48. $bc+ca+ab$. **49.** $\dfrac{bc+ca+ab}{(a+b)(b+c)(c+a)}$. **50.** 1. **51.** 1.

52. 1. **53.** $p+q$. **54.** $k(a+b+c)+l$.

MISCELLANEOUS PAPERS II. (pp. 203—09).

PAPER I.

1. $41y-51x$. **2.** $4\cdot1x^2+2x-\cdot6$. **3.** $(x^2+2xy+y^2)^2-(x^2-2xy+y^2)^2$.

4. (i) $(x^2+2xy+2y^2)(x^2-2xy+2y^2)$; (ii) $(x-2y)(x^4+2x^3y+4x^2y^2+8xy^3+16y^4)$; (iii) $(p-q)(pq-x^2)$.

5. $(3x+2)(3x-2)(2x-3)(2x+3)$. **6.** $3x-1$; $\frac{1}{3}$.

7. (i) $\dfrac{1}{x(x-a)(x-b)}$. (ii) $\dfrac{2x^2(2a+5x)}{(a+x)(a+2x)(a+3x)(a+4x)}$. **8.** 240.

PAPER II.

1. (i) $\frac{3}{4}x^3-5x^2y+\frac{1}{4}xy^2+9y^3$. (ii) x^4-x^2-1. **2.** $4x^2y^2$.

3. (1) $(x-2)(x^2+2x-8)$; (2) $(x+1)^2(3x^2-9x+8)$.
(3) $(9a^2+6a+4)(9a^2-6a+4)$. **4.** 133. **5.** $x-2y$.

6. $(x^6-a^6)(x^2-a^2)$. **7.** -1. **8.** Rs. 735.

PAPER III.

2. $16(x^4-x^2y^2+y^4)-8(x^2+y^2)a^2+a^4$. **3.** $x^2(2x+1)(3x-2)(7x-1)$.

7. (i) $\dfrac{x-15}{6x-4}$; (ii) -2. **8.** 50, 51.

PAPER IV.

1. $(x+5)$. **2.** $x+y+z+xyz$. **3.** (1) $(x+a)(x-a)(x^2+ax+a^2)$
$\times(x^2-ax+a^2)(x^4-a^2x^2+a^4)(x^2+a^2)$ (2) $(x-y)(y-z)(x-z)$
$\times(x+y+z)$. **6.** See ex. 21, (i) and (ii). p. 119.

7. (a) $\dfrac{2x}{3}$; (b) $\dfrac{x-3}{x-1}$. **8.** 7.

PAPER V.

1. x^2+2x+3. **2.** $(a-3b+2c)^2$. **4.** $(c-a')(ab'-a'b)+(c-b')^2=0$.

7. $\dfrac{x^2+3xy+y^2}{x(4x^2+13xy+6y^2)}$. **8.** 50.

PAPER VI.

1. (i) $(3a+2y)(3a-2y)(x-3a)(x^2+3ax+9a^2)$; (ii) $(x-a)(x-b)$.
3 G.C.M.: $ax+b$; L.C.M.: $(ax+b)(bx+c)(cx+b)(x+a)(cx-b)$.
5. (1) $\left(x-\dfrac{c}{d}\right)\left(x+\dfrac{d}{c}\right)$; (2) $(x+w)(x-w)(z-y)(z+y)$.
7. $x=2m+1$. **8.** 1.

PAPER VII.

1. $x^3+(a+b)x^2+2abx+1$.
3. $(x^2-yz)(y^2-zx)(z^2-xy)$, $(a^2+2ab+2b^2)(a^2-2ab+2b^2)(a^2+2b^2)$
$\times(a^2-2b^2)$. **4.** $a=4$. **5.** $x-2a$. **6.** $1+4x-16x^3-32x^4+64x^6$.

7. $a^3-pa^2+qa-r=0$. **8.** $\dfrac{4}{(x+3)(x+7)}$.

PAPER VIII.

2. (i) $(x+1)(8x^2-8x+3)$; (ii) $(ab+bc+ca)(a+b+c)$.
3. x^2+2x+3. **4.** $(x^2+1)(x-1)^3(x+1)^2(x^2-x+1)$.
6. $x^{10}-5x^8y^2+10x^6y^4-10x^4y^6+5x^2y^8-y^{10}$. **8.** (i) 1, (ii) x^2.

PAPER IX.

1. $-5x$. 3. $(x-b)(2x-a)^2$; (2) $(y-z)(x+y-z)(x-y+z)$.
5. $(7x+2)x$. 7. $p=16m^4$. 8 (i) 1, (ii) 4.

PAPER X.

3. $(x+y+z)(3x+y-z)^2$; $(2a^2+2ab+b^2)(2a^2-2ab+b^2)(a^2+2ab+2b^2)(a^2-2ab+2b^2)$. 4. $x+y$. 7. (i) $\frac{4(a^2+b^2)}{(a-b)^2}$, (ii) 1.

PAPER XI.

2. $x-1$; $(x^3-4x^2-4x-4)(x^3-11x+10)$.
3. $8abcd-a^4-d^4-b^4-c^4+2a^2b^2+2a^2c^2+2a^2d^2+2b^2c^2+2b^2d^2+2c^2d^2$.
5. (i) $(3x-1)(x^2-2)$; (ii) $(7a^2+11a+61)(19a^2-3a+21)$;
(iii) $(x-a+2)(3x+3a+1)$. 6. See ex. 23, p. 119.
7. (i) x. (ii) $2(a+b+c)$ 8. $\frac{32}{3}$.

PAPER XII.

1. (i) $(a^2+c^2-ac)x^2+3b^2y^2+(a^2+c^2-ac)z^2+3b(a-c)xy+3b(c-a)yz+(4ac-a^2-c^2)zx$; (ii) $(x^2-x+1)(bx+a)$. 3. $2x^2+5x-3$.
4. $(x-1)(x-2)(x+3)(x+4)(x-5)(x+6)$. 5. 1. 6. 1. 7. 1.

EXERCISE LXIX. (pp. 212-13).

1. $9a^4b^4$; $27a^6b^6$.
2. $4x^4y^4z^8$; $-8x^6y^6z^{12}$.
3. $\frac{4}{9}a^6x^8b^4y^2$; $\frac{8}{27}a^9x^{12}b^6y^3$.
4. $a^4b^4c^6d^8$; $-a^6b^6c^9d^{12}$.
5. $\frac{4a^4}{b^2c^4}$; $\frac{8a^6}{b^3c^6}$. 6 $\frac{9a^4b^4}{x^4y^8}$; $-\frac{27a^6b^6}{x^6y^{12}}$. 7. $\frac{4a^4b^6c^8}{x^4y^2z^6}$; $\frac{8a^6b^9c^{12}}{x^6y^3z^9}$.
8. $\frac{9a^4x^2y^4}{4b^6c^4z^8}$; $-\frac{27a^6x^3y^6}{8b^9c^6z^{12}}$ 9. $a^{15}b^{21}$. 10. $64x^{18}y^{12}$. 11. $a^{30}b^{40}c^{50}$.
12. [illegible]. 13. $\frac{a^{13}}{bc^{7}d^{5}}$. 14. $\frac{a^{8}}{b^{9}c^{5}}$.

EXERCISE LXX. (p. 214).

1. $x^4+8x^3+24x^2+32x+16$.
2. $x^5+5x^4+10x^3+10x^2+5x+1$.
3. $x^4-8x^3+24x^2-32x+16$.
4. $1-5x+10x^2-10x^3+5x^4-x^5$.

5. $a^4+8a^3b+24a^2b^2+32ab^3+16b^4$.

6. $16a^4-32a^3x+24a^2x^2-8ax^3+x^4$.

7. $16a^4+96a^3b+216a^2b^2+216ab^3+81b^4$.

8. $1-10x+40x^2-80x^3+80x^4-32x^5$.

9. $81x^4-432x^3y+864x^2y^2-768xy^3+256y^4$.

10. $a^5+10a^4b+40a^3b^2+80a^2b^3+80ab^4+32b^5$.

11. $1+7x+21x^2+35x^3+35x^4+21x^5+7x^6+x^7$.

12. $x^8-8x^7+28x^6-56x^5+70x^4-56x^3+28x^2-8x+1$.

13. $16a^4-160a^3b+600a^2b^2-1000ab^3+625b^4$.

14. $4096x^{12}-18432x^{10}y^3+34560x^8y^6-34560x^6y^9$
$+19440x^4y^{12}-5832x^2y^{15}+729y^{18}$.

15. $2a^4+12a^2b^2+2b^4$. 16. $10a^4b+20a^2b^3+2b^5$.

17. $32x^4+432x^2y^2+162y^4$. 18. $2+30x^2+30x^4+2x^6$.

EXERCISE LXXI. (pp. 216–17).

1. $4a^2+9b^2+16c^2+25d^2-12ab+16ac-20ad-24bc+30bd-40cd$

2. $16x^6-24x^5+25x^4-20x^3+10x^2-4x+1$.

3. $\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}-\dfrac{2a}{b}-\dfrac{2b}{a}+3$. 4. $\dfrac{4x^2}{y^2}+\dfrac{y^2}{9x^2}+\dfrac{8x}{y}+\dfrac{4y}{3x}+\dfrac{16}{3}$.

5. $1+a^2+b^2+c^2+d^2-2a+2b-2c+2d-2ab+2ac-2ad-2bc$
$+2bd-2cd$.

6. $4x^8-12x^7+13x^6-14x^5+25x^4-22x^3+10x^2-12x+9$.

7. $\dfrac{27x^3}{64y^3}-8-\dfrac{27x^2}{8y^2}+9\dfrac{x}{y}$.

8 $a^3-\dfrac{b^6}{c^3}-\dfrac{3a^2b^2}{c}+\dfrac{3ab^4}{c^2}$. 9. $x^3-\dfrac{8}{x^3}-6x+\dfrac{12}{x}$.

10. $\dfrac{a^3}{b^3}-x^3+\dfrac{b^3}{a^3}+\dfrac{3a}{b}+\dfrac{3b}{a}-\dfrac{3a^2x}{b^2}+\dfrac{3ax^2}{b}-\dfrac{3b^2x}{a^2}+\dfrac{3bx^2}{a}-6x$.

11. $8x^3-27y^3+64z^3-36x^2y+54xy^2+48x^2z+96xz^2+108y^2z$
$-144z^2y-144xyz$.

12. $1-6x+21x^2-44x^3+63x^4-54x^5+27x^6$.

13. $a^6+3a^5b+6a^4b^2+7a^3b^3+6a^2b^4+3ab^5+b^6$.

14. $27x^9-108x^8+198x^7-235x^6+204x^5-132x^4+65x^3-24x^2+6x-1$.

15. -146. 16. 63.

17. $a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$.

18. $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

EXERCISE LXXII. (p. 218).

1. $5a^2bc^2$. 2. $11x^2yz^4$. 3. $4abc^3$. 4. $3x^3y^2z^4$.

5. $\frac{3ab}{4xy^2}$. 6. $\frac{6x^3y^4}{7a^2z}$. 7. $\frac{13ab^2c^4}{2x^2yz^2}$. 8. $\frac{4x^2y^2z^3}{5ab^3c^2}$.

[In the Answers from 1 to 8 the signs may be both + and −]

9. $2ab^2$ 10. $-3a^2z$. 11. $5ax^3$. 12. $-4a^3x^2$. 13. $\frac{-3x^2y^2}{5a^3z}$.

14. $\frac{6a^2x^3}{by^4}$. 15. $\frac{-ab^2}{3c^3}$. 16. $\frac{4x^3yz^2}{7ab^2c^4}$. 17. $3xy^2$. 18. $2a^2y^3$.

19. $-3ab^2$. 20. $\frac{ab^3}{2c^2}$. 21. $\frac{5a^2}{bc^3}$. 22. $\frac{2zy^2}{3x^3}$. 23. $\frac{5ay^2}{3x}$.

24. $\frac{2a}{b^2c^3}$. 25. $\frac{xy^2}{z^3}$. In answers to sums 17,20,22,23 and 28 the signs may be both positive and negative.

EXERCISE LXXIII (pp. 219-20).

1. $5ab+3cd$. 2. $7a^2x-10by^2$ 3. $5x+2y$. 4. $7a-9b$.

5. $x-\frac{1}{x}-2$. 6. $(x+y)(x+2y)(x+3y)$. 7. $(x-1)(x-2)(x+1)$.

8. $x^2+\frac{1}{x^2}-3$. 9. $ab-ac+bc$. 10. $1+\frac{2ab}{a^2-b^2}$. 11. $x^2+\frac{1}{2}yz-z^2$.

12. $a^2-b^2+c^2-d^2$. 13. $3a^2-ab+5b^2$. 14. $ax+by+cz$.

15. $\frac{a}{b}+\frac{b}{a}+1$. 16. $(a+b)(b+c)(c+a)$. 17. $x^2+8x+11$.

18. $x^2+11x+21$. 19. $x^2+8x+10$. 20. $by+cz-ax$.

EXERCISE LXXIV. (pp. 223-24).

1. x^2+2x-3. 2. $3x^2+4x-2$. 3. $\frac{2}{3}a-\frac{1}{5}b+\frac{1}{2}c$.

4. $1+2x-3x^2$. 5. $\frac{x}{3}-3y-\frac{3}{2}$. 6. $7x^3+3x^2-4x-8$.

7. $4x^3-5x^2y+6xy^2-7y^3$. 8. $\frac{3a}{b}+\frac{b}{2a}+3$. 9. $\frac{a^2}{b^2}+\frac{b^2}{a^2}-2$.

10. $x^3+x+2-\frac{2}{x^3}$. 11. $\frac{a^3}{b^3}+\frac{3a}{b}-\frac{3b}{a}-\frac{b^3}{a^3}$. 12. $\frac{a}{2b}+\frac{2b}{3c}+\frac{3c}{a}$.

13. $p^2=4q$. 14. (i) $x=\frac{10}{3}$ (ii) $x=2$.

15. (i) $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4$. (ii) $1-x-\frac{1}{2}x^2-\frac{1}{2}x^3$. (iii) $1-\frac{1}{2}x+\frac{3}{8}x^2+\frac{3}{16}x^3$. (i) $1+x-2x^2+2x^3$. 16. $4x^2(a+b)^2$.

EXERCISE LXXV. (pp. 227).

1. $2a-3b$. **2.** $4x+3y$. **3.** $2a-1$. **4.** $x-\frac{2}{x}$. **5.** $\frac{2x}{3}-4$.

6. $\frac{x}{y}-\frac{y}{x}$. **7.** $\frac{a^2}{b}+2c$. **8.** $\frac{a}{b}-\frac{b}{a}-1$. **9.** $2x^2+x-4$. **10.** x^2-x+1.

11. $4x^2-3xy+2y^2$. **12.** x^2+xy+y^2. **13.** $\frac{x}{4}-1+\frac{4}{x}$.

14. $\frac{x^2}{y^2}-\frac{2x}{y}+3$. **15.** (i) $1+\frac{1}{3}x-\frac{1}{9}x^2$. (ii) $2-\frac{x}{12}-\frac{x^2}{88}$. **16.** $x=3$.

17. (i) $x+2$, (ii) $x+\frac{1}{x}$. **18.** (i) $x+1$ (ii) $x-\frac{2}{x}$.

EXERCISE LXXVI (pp. 229-30).

1. $b-a$. **2.** $\frac{cd-ab}{a-c}$. **3.** $\frac{m^2-2mn+pn}{p-n}$. **4.** a.

5. $\frac{a+b+c}{a}$. **6.** $3m+2$. **7.** a. **8.** $2(a+b)$.

EXERCISE LXXVII (pp. 230-31).

1. 3. **2.** 0. **3.** 1. **4.** 2. **5.** $\frac{1}{3}$. **6.** $-\frac{1}{2}$. **7.** $\frac{7}{8}$.

8. $\frac{ab-cd}{c+d-a-b}$. **9.** 0. **10.** 0. **11.** $\frac{a^2-ab+b^2}{b-a}$. **12.** $\frac{-mn}{m+n+p}$.

EXERCISE LXXVIII (pp. 232-33).

1. 4. **2.** 7. **3.** 9. **4.** 5. **5.** $\frac{5}{6}$. **6.** 9. **7.** 19. **8.** $\frac{2}{5}$. **9.** 44.
10. $-3\frac{1}{2}$. **11.** $-\frac{5}{6}$. **12.** 20. **13.** $\frac{24}{65}$. **14.** $10\frac{40}{53}$. **15.** 7. **16.** 5.

17. 4. **18.** $a+b$. **19.** $\frac{4a_1a_2a_3}{a_1a_2+a_2a_3+a_3a_1}$. **20.** $10a-24b$.

21. $a-b$. **22.** $\frac{l+m+n-lm^2-mn^2-nl^2}{1-lm-mn-ln}$.

23. $\frac{b^2c+bc^2+c^2a+ca^2+a^2b+ab^2}{-2(bc+ca+ab)}$. **24.** $a+b$. **25.** $2(a+c)$.

EXERCISE LXXIX (pp. 235-36).

1. 12. **2.** 5. **3.** 1. **4.** $\frac{bc-ad}{b+c-a-d}$. **5.** $a-2b$. **6.** 4. **7** 6.

8. 1. **9.** $\frac{14}{21}$. **10.** $\frac{513}{731}$. **11.** 72. **12.** 10. **13.** 2. **14.** $1\frac{1}{2}$.
15. 3. **16.** $\frac{109}{138}$. **17.** 3. **18.** $5\frac{3}{8}$. **19.** $2\frac{2}{3}$. **20.** $\frac{2}{11}$. **21.** $\dfrac{p(b-pd)}{c(p^2-1)}$.

EXERCISE LXXX. (pp. 240-41).

1. -8. **2.** $4\frac{1}{2}$. **3.** $-\frac{59}{39}$. **4.** $\frac{184}{73}$. **5.** $-\frac{621}{229}$. **6.** $\frac{4}{69}$. **7.** $\frac{52}{123}$.
8. $-\frac{1}{2}m(b+d)$. **9.** $-4\frac{1}{3}$. **10.** $-5\frac{1}{2}$. **11.** $\dfrac{l^2+m^2}{l+m}$. **12.** 1. **13.** 16.
14. 1. **15.** $-\frac{24}{7}$. **16.** 7. **17.** $\frac{4}{5}$. **18.** $\frac{1}{16}$. **19.** $-\frac{73}{48}$. **20.** $\frac{2}{7}$.
21. 2. **22.** 2. **23.** $\dfrac{a^2+b^2}{a+b}$. **24.** $-\dfrac{a^2-3ab+b^2}{a+b}$.
25. $\dfrac{a^2b-a^2c+ab^2-b^2c}{ac-2ab+bc}$. **26.** $\dfrac{2mn-m^2p-nlp}{2lmp-m^2-ln}$.
26. $\dfrac{a^2-bc}{b+c-2a}$. **27.** $2(c-a-b)$.

EXERCISE LXXXI (pp. 243-44).

1. 2. **2.** 3. **3.** 13. **4.** $-\frac{64}{15}$. **5.** $\dfrac{m(n-m)}{2m-n+c}$. **6.** $-\frac{5}{8}$.
7. $\dfrac{ab(m+n)}{ma+nb}$. **8.** $4\frac{1}{2}$. **9.** $\frac{1}{2}(a+b+3)$. **10.** 13. **11.** $-1\frac{1}{2}$.
12. $\frac{45}{37}$. **13.** $-b$. **14.** $-\frac{5}{3}$. **15.** $2\frac{1}{2}$ or 0. **16.** $-\frac{40}{7}$. **17.** $-\frac{19}{2}$.
18. $\dfrac{pn(n-p)+mn(m-n)-pm(p-m)}{2n(p-m)}$. **19.** $\dfrac{abc-abd-cad-ca^2}{bad-ba^2-b^2c+b^2d}$.

EXERCISE LXXXII (pp. 246-48).

1. $-a$. **2.** -6. **3.** $\dfrac{b-a}{2}$. **4.** -9. **5.** $-3\frac{1}{5}$. **6.** $-3\frac{1}{2}$. **7.** $-a$.
8. $a+b+c$. **9.** $a^3+b^3+c^3$. **10.** $bc+ca+ab$. **11.** $a^2+b^2+c^2$
$+2ab+2ac+2bc$. **12.** $-a-b-c$. **13.** $-\dfrac{a+b}{2}$. **14.** 4.
15. $3\frac{1}{2}$. **16.** $-\dfrac{bc+ca+ab}{abc}$. **17.** 0 or, $-\frac{1}{2}(a+b)$. **18.** 4. **19.** 2.
20. $-\frac{23}{13}$. **21.** $-\dfrac{2ab}{c}$. **22.** $-\frac{5}{6}$. **23.** $a+b$. **24.** $\dfrac{b(a+c)}{a-c}$
25. 1. **26.** $\dfrac{ab}{a+b}$. **27.** $4\frac{1}{2}$. **28.** 4. **29.** $\frac{11}{28}$.
30. $\dfrac{3pqr}{pq+qr-rp}$. **31.** $\dfrac{a(n^2+mn-a^2)}{m^3+m^2n-n^3-mn^2+na^2}$. **32.** $\dfrac{ab-cd}{a+b-c-d}$.

CHAPTER XIX.

PROBLEMS LEADING TO SIMPLE EQUATIONS.

1. The subject of problems was considered in Chapter XI. We shall here add more illustrative examples.

Ex. 1. A farmer bought equal numbers of two kinds of sheep, one at £3 each, the other at £4 each. Had he expended his money equally in the two kinds, he would have had two more sheep than he had. How many did he buy? (A. E. 1891).

Let x = number of each kind of sheep he purchased.

Then $2x$ = the total number of sheep bought.

Hence, the total sum spent on two kinds $= £3x + £4x = £7x$. If he spent his money equally on two kinds, then the number he would have in all $= \frac{£7x}{2} \div £3 + \frac{£7x}{2} \div £4$.

$$= \left(\frac{7x}{6} + \frac{7x}{8}\right).$$

By the question therefore $\frac{7x}{6} + \frac{7x}{8} = 2x + 2$.

Multiplying both sides by 24, $28x + 21x = 48x + 48$.

$$\therefore x = 48.$$

$\therefore$ The total number of sheep bought $= 2x = 96$.

Ex. 2. A person finds that if he invests his money in the 4 per cents at 98, his income will be Rs. 39 less than if he invests it in the 5 per cents at 112; what is the sum to be invested?

Let Rs. x = sum to be invested. Then his income from investment in the first kind = Rs. $\frac{4x}{98}$ and his income from investment in the second kind = Rs. $\frac{5x}{112}$.

By the question $\frac{5x}{112} - \frac{4x}{98} = 39$.

Multiplying both sides by 784, $35x - 32x = 784 \times 39$,

or $3x = 784 \times 39$, whence $x = \frac{784 \times 39}{3} = 10192$.

$\therefore$ The sum to be invested = Rs. **10192**.

Ex. 3. A man selling mangoes gives away half his stock and 2 more to one person, $\frac{1}{3}$ of the remainder and 8 more to a second, then $\frac{2}{3}$ of the remainder and 2 more to a third, and finds that he has only 2 left. How many mangoes had he at first?

Let x be the number of mangoes he had at first. He gives $\frac{x}{2}+2$ mangoes to the first man. He has $x-\left(\frac{x}{2}+2\right)$ *i. e.*, $\frac{x}{2}-2$ mangoes left. He gives $\frac{1}{3}\left(\frac{x}{2}-2\right)+8$ *i.e.*, $\frac{x}{6}+\frac{22}{3}$ mangoes to the second man. He has then $\left(\frac{x}{2}-2\right)-\left(\frac{x}{6}+\frac{22}{3}\right)$ *i. e*, $\frac{x}{3}-\frac{28}{3}$ mangoes left. He gives to the third man $\frac{2}{3}\left(\frac{x}{3}-\frac{28}{3}\right)+2$ *i. e.* $\frac{2x}{9}-\frac{38}{9}$ mangoes; he has $\left(\frac{x}{3}-\frac{28}{3}\right)-\left(\frac{2x}{9}-\frac{38}{9}\right)$ *i. e.* $\frac{x}{9}-\frac{46}{9}$ mangoes left.

$\therefore$ By the question $\frac{x}{9}-\frac{46}{9}=2$.

$\therefore$ $x-46=18$, or, $x=64$.

$\therefore$ He had 64 mangoes originally.

Ex. 4. A person walked out a certain distance at the rate of $3\frac{1}{2}$ miles an hour, and then ran part of the way back at the rate of 7 miles an hour, walking the remaining distance in 7 minutes. He was out 35 minutes. How far did he run? A. E. 1890.

Let x miles = the distance the man ran while returning.

In running back this distance at the rate of 7 miles an hour he took $\frac{x}{7}$ hour, or, $\frac{x}{7}\times 60$ minutes.

While returning, in 7 minutes he walked $\frac{3\frac{1}{2}}{60}\times 7$ miles *i.e.* $\frac{49}{120}$ miles.

$\therefore$ The total distance, he ran and walked while returning $=(x+\frac{49}{120})$ miles which is also the distance he walked out at the rate of $3\frac{1}{2}$ miles an hour. In walking out this distance he took

$$\frac{x+\frac{49}{120}}{3\frac{1}{2}} \text{ hrs.} =(x+\tfrac{49}{120})\tfrac{2}{7}\times 60 \text{ minutes.}$$

$\therefore$ The total time he took in walking out and returning back $=\frac{x}{7}\times 60+7+(x+\frac{49}{120})\frac{2\times 60}{7}$ minutes.

By the question we get

$$\frac{x\times 60}{7}+7+\left(x+\frac{49}{120}\right)\frac{2\times 60}{7}=35.$$

Multiplying by 7, $60x+49+120x+49=245$ or $180x=147$

$$\therefore x=\frac{147}{180}=\frac{49}{60}.$$

$\therefore$ The man ran $\frac{49}{60}$ mile.

Ex. 5. If 19 lbs of gold weigh 18 lbs in water, and 10 lbs of silver weigh 9 lbs in water, find the quantity of gold and silver in a mass of gold and silver weighing 106 lbs in air and 99 lbs in water. (B. M. 1888).

Let x = number of pounds of gold; then $106-x$ = number of pounds of silver in the mass.

Since 19 lbs of gold weigh 18 lbs in water, $\therefore$ x lbs of gold weighs $\frac{x}{19}\times 18$ lbs in water.

Again since 10 lbs of silver weigh 9 lbs in water. $\therefore$ $106-x$ lbs of silver weigh $\frac{(106-x)}{10}\times 9$ lbs in water.

$\therefore$ The total weight of the mass in water = the weight in water of gold in it + the weight in water of silver in it.

$$=\frac{18x}{19}+\frac{(106-x)\times 9}{10}.$$

By the question therefore $\frac{18x}{19}+\frac{(106-x)\times 9}{10}=99.$

Multiplying both sides by $\frac{190}{9}$, we have $20x+2014-19x=2090$, whence $x=76$.

Thus there are 76 lbs of gold in the mass and hence $(106-76)$ or 30 lbs of silver.

Ex. 6. A and B can do a piece of work in p days; B and C in q days; find in how many days can C and A do the work, supposing that A can do r times as much work as C in a given time.

Let w denote the proposed work.

$\therefore \frac{w}{p}$ = the work done by A and B in one day;

and $\frac{w}{q}$ = the work done by B and C in one day.

Let x days = the time in which C alone can do it.

$\therefore \frac{w}{x}$ = the work done by C in one day;

and $\frac{rw}{x}$ = the work done by A in one day.

Hence $\frac{w}{x}+\frac{rw}{x}$ or $\frac{(r+1)w}{x}$ = the work done by A and C in one day.

∴ The required number of days

$$=w\div\frac{(r+1)w}{x}=\frac{x}{r+1}.$$

Also $\frac{w}{p}-\frac{rw}{x}+\frac{w}{x}$ or $\frac{w}{p}-\frac{w}{x}(r-1)$ = work done by B and C in one day.

$$\therefore \frac{w}{p}-\frac{w}{x}(r-1)=\frac{w}{q}, \text{ whence } x=(r-1)\frac{pq}{q-p}.$$

Hence the required time in days $=\frac{x}{r+1}=\frac{r-1}{r+1}\left(\frac{pq}{q-p}\right)$.

Ex. 7. Two passengers have together 7 maunds of luggage and are charged for the excess above the weight allowed Rs. 5 and Rs. 7 respectively; had the luggage all belonged to one of them, he would have been charged. Rs. 13. How much luggage is each passenger allowed to carry free of charge? And how much luggage had each passenger?

Let x maunds = weight allowed free to each passenger.

∴ Two passengers get an allowance of $2x$ maunds and hence they have to pay for $(7-2x)$ maunds.

∴ The charge for $7-2x$ maunds = Rs. 12.

∴ The charge for 1 maund $=\frac{\text{Rs. } 12}{7-2x}$......(1).

Again if the luggage belonged to one person he would have to pay for $7-x$ maunds.

∴ The charge for $7-x$ maunds by the question = Rs. 13.

∴ The charge for 1 maund $=\frac{\text{Rs. } 13}{7-x}$......(2).

From (1) and (2) $\frac{12}{7-2x}=\frac{13}{7-x}$, or, $84-12x=91-26x$.

∴ $14x=7$ ∴ $x=\frac{1}{2}$

i.e., the weight of luggage allowed free $=\frac{1}{2}$ maund.

∴ Now charge for 1 maund of luggage from (1)

$$=\text{Rs}\frac{12}{7-2x}=\text{Rs.}\frac{12}{7-1}=\text{Rs. } 2.$$

Since one passenger had to pay Rs. 5 for the excess luggage he had $\frac{\text{Rs } 5}{\text{Rs. } 2}$ or $2\frac{1}{2}$ maunds *plus* $\frac{1}{2}$ a maund (allowed free) *i. e.*, 3 maunds.

∴ The other had $7-3$ or 4 maunds of luggage.

Ex. 8. Two vessels contain mixtures of wine and water; in one there is twice as much wine as water, and in the other three times as much water as wine. Find how much must be drawn off from each to fill a third vessel which holds 15 gallons in order that its contents may be half wine and half water. (B. E. 1890).

Let x=number of gallons to be drawn off from the first vessel.

Then $15-x$=number of gallons to be drawn off from the second vessel.

Now the quantity of wine drawn from the first vessel

$$=\frac{2x}{3} \text{ gallons.}$$

and the quantity of wine drawn from the second vessel

$$=\frac{(15-x)1}{4} \text{ gallons.}$$

The total quantity of wine in the third vessel

$$\left(\frac{2x}{3}+\frac{15-x}{4}\right) \text{ gallons.}$$

By the question $\frac{2x}{3}+\frac{15-x}{4}=\frac{15}{2}$,

$\therefore$ $8x+45-3x=90$, or $5x=45$ whence $x=9$.

Therefore 9 gallons must be drawn off from the 1st vessel and therefore 6 gallons from the second vessel.

Ex. 9. A garrison had sufficient provisions for 30 months but at the end of 4 months the number of troops was doubled and 3 months after it was re-inforced with 400 men more, on which accounts the provisions lasted only 15 months altogether. Required the number of men in the garrison before the augmentation took place. (B. M. 1872.)

Let x=the number of men in the garrison before the augmentation.

Then $30x$=the number of men for whom the provisions *would have been sufficient* for 1 month.....(1).

Now the provisions which were used by x men for 4 months= provisions of $4x$ *men* for 1 *month*...(2).

Again, after 4 months the number of troops was doubled *i. e.* became $2x$, and in 3 months they used provisions which=provisions of $6x$ *men for* 1 *month*..... (3).

Lastly, remaining part of the provisions lasted for $15-(4+3)$ *i.e.* 8 months when there were $2x+400$ men. Hence this remaining provisions=provisions for $8\times(2x+400)$ *men for one month*......(4).

Hence from (2), (3), (4) the provisions would have been sufficient for $4x+6x+8(2x+400)$ men for 1 month......(5).

From (1) and (5), $30x = 4x + 6x + 8(2x + 400)$

or $30x - 26x = 3200$ whence $x = 800$.

$\therefore$ There were 800 men in the garrison before the augmentation.

2. Time and distance.

Ex. 10. A who travels $3\frac{1}{2}$ miles an hour starts $2\frac{1}{2}$ hours before B who goes the same road at $4\frac{1}{2}$ miles an hour; where will he overtake A? (A. E. 1889).

Suppose the place where B overtakes A to be x miles from the starting point.

Then A travels for $\frac{x}{3\frac{1}{2}}$ hour before he is overtaken and B for $\frac{x}{4\frac{1}{2}}$ hours. But by the question A travels for $2\frac{1}{2}$ hours more than B, since B starts $2\frac{1}{2}$ hours after A.

$$\text{Hence } \frac{x}{3\frac{1}{2}} - \frac{x}{4\frac{1}{2}} = 2\tfrac{1}{2} \text{ or } \frac{2x}{7} - \frac{2x}{9} = \tfrac{5}{2}.$$

Multiplying both sides by 126, we get

$$36x - 28x = 315, \text{ or } 8x = 315.$$

$$\therefore x = \frac{315}{8} = 39\tfrac{3}{8}.$$

$\therefore$ The place where B overtakes A is $39\frac{3}{8}$ miles distant from the starting point.

3 Race.

Ex. 11. Two persons started at the same time from A. One rode on horse back, at the rate $7\frac{1}{2}$ miles an hour and arrived at B 30 minutes later than the other who travelled the same distance by train at the rate of 30 miles an hour. Find the distance between A and B

Let x miles = the distance between A and B.

Time taken by the first person to travel the distance $= \frac{x}{7\frac{1}{2}}$ hours.

Time taken by the other to travel the distance $= \frac{x}{30}$ hours. But the first arrived 30 minutes or $\frac{1}{2}$ an hour later.

$$\therefore \frac{x}{7\frac{1}{2}} - \frac{x}{30} = \tfrac{1}{2}, \text{ or } \frac{2x}{15} - \frac{x}{30} = \tfrac{1}{2}$$

$$\therefore 4x - x = 15, \text{ or } 3x = 15, \therefore x = 5$$

$\therefore$ The distance between A and $B = 5$ miles.

Ex. 12 A hare is 50 of her own leaps before a greyhound; and she takes 4 leaps for the greyhound's 3: but 2 of the greyhound's leaps are equal to 3 of the hare's; how many leaps must the greyhound take to catch the hare?

Let $3x$ = number of leaps the greyhound takes to overtake the hare.

Then $4x$ = the number of leaps the hare takes in the same time.

Hence the distance covered by $3x$ leaps of the greyhound = the distance covered by $(4x+50)$ leaps of the hare, for she was at the start, 50 of her own leaps before the greyhound.

Now 3 leaps of the hare = 2 leaps of the greyhound.

$\therefore$ $4x+50$ leaps of the hare

$=(4x+50)\times\frac{2}{3}$ leaps of the greyhound.

Hence $(4x+50)\times\frac{2}{3}=3x$ or $8x+100=9x$ whence $x=100$.

Hence the number of leaps the greyhound takes to overtake the hare $=3x=300$.

4. Percentage.

Ex. 13. A person bought an article and sold it at a profit of 6 per cent. Had he bought it at 4 per cent. less and sold at Re. 1-3 as. more his profit would have been 12 per cent. For how much did he buy it? (P. E. 1891).

Let Rs x = cost price of the article. Then the price he sold it to gain 6 per cent.

$$=\text{Rs.}\left\{x+\frac{6x}{100}\right\}=\text{Rs.}\frac{106x}{100}.$$

If he purchased it at 4 per cent. less the cost price would

$$=\text{Rs.}\left\{x-\frac{4x}{100}\right\}=\frac{96x}{100}.$$

If he sold for Rs. 1. 3 as. *i. e.* Rs. $\frac{19}{16}$ more, the selling price

would = Rs. $\left\{\frac{106x}{100}+\frac{19}{16}\right\}$.

Since by the question this price would bring a gain of 12 per cent.

$$\frac{106x}{100}+\frac{19}{16}=\frac{96x}{100}\left\{1+\frac{12}{100}\right\}.$$

$$\text{or } \frac{53x}{50}+\frac{19}{16}=\frac{24x}{25}\left\{1+\frac{3}{25}\right\}.$$

Multiplying both sides by 10000 we get

$$10600x+11875=10752x \text{ or } 152x=11875$$

$$\text{or } x=\frac{11875}{152}=\frac{625}{8}=78\frac{1}{8}.$$

$\therefore$ The cost price = Rs. $78\frac{1}{8}$ = Rs. 78. 2 as.

Ex. 14. Of the candidates in a certain examination 45 per cent passed. If there had been 30 more candidates of whom 19 failed, the number of successful candidates would have been 44·8 per cent. How many candidates were there? (C. U. 1890.)

Let x be the number of candidates. Then $\frac{45x}{100}$ candidates passed.

If there were 30 candidates more, the number of candidates would have been $x+30$, and total number of candidates passed in this case would be $=\frac{45x}{100}+30-19=\frac{45x}{100}+11$.

Since in the second case 44·8 per cent. of the candidates would have passed we have

$$\frac{45x}{100}+11=\frac{44 \cdot 8}{100}(x+30).$$

Multiplying by 1000, $450x+11000=448x+13440$,

or $2x=2440$ whence $x=1220$.

$\therefore$ The no. of candidates $=1220$.

5. Digits.

Ex. 15. A number consists of two digits of which the digit in the unit's place is double of the other: if the digits be inverted, the new number exceeds the original number by 18. Find the number. (C. U. 1896).

Let the digit in the unit's place $=2x$.

Then the digit in the *tens'* place $=x$.

Then the number $=x\times 10+2x=12x$.

If the digits be inverted, the new number will have $2x$ in its *tens'* place and x in its unit's place and hence $=2x\times 10+x=21x$.

By the question, $21x-12x=18$, or $9x=18$, $\therefore x=2$.

$\therefore$ The digits in the unit's place $=4$ and that in the *tens'* place $=2$ and the number $=24$.

6. Clock.

Ex. 16. At what time are the hands of a watch together between 5 and 6 o'clock?

Let $x=$ no. of minutes past 5 when the hands are together. Now the minute hand is at the 12 o'clock mark when it is 5 o'clock and hour hand is at the mark 5. In x minutes the minute hand moves through x divisions from 12 o'clock mark and by the time the hour hand moves through $\frac{x}{12}$ divisions, (the hour hand moves through 5

divisions when the minute hand moves through 60 divisions). hence its distance from 12 o'clock mark when it is x minutes past 5 = original distance from 12 o'clock mark to 5 o'clock mark *i.e.* 25 divisions *plus* $\frac{x}{12}$ divisions $= \left(25 + \frac{x}{12}\right)$ divisions.

But since the two hands are together the distance between them = 0.

$$\therefore\ x - \left(25 + \frac{x}{12}\right) = 0 \text{ or } \frac{11x}{12} = 25.$$

$$\therefore\ x = \frac{300}{11} = 27\frac{3}{11}.$$

Hence the two hands are together at $27\frac{3}{11}$ minutes past 5.

N B. The motion of two hands of a clock may be considered to be analogous to the motion of two men in a circular path. Two hands are at right angles when they are 15 divisions apart and opposite when they are 30 divisions apart.

Ex. 17. A man who went out between 5 and 6 and returned between 6 and 7, found that the hands of his watch had exactly changed places. When did he go out ?

Let x minutes past 5 be the time when he went out *i.e.*, the distance of the minute-hand from 12 o'clock mark then was x divisions. By the time the hour hand moved from 5 o'clock mark through $\frac{x}{12}$ divisions; hence its distance from 12 o'clock mark = 25 divisions $+ \frac{x}{12}$ divisions $= \left(25 + \frac{x}{12}\right)$ divisions.

When he returned between 6 and 7 the distance of the hour hand from 12 o'clock mark was x divisions = the distance of the minute hand from 12 o'clock mark at the time of going out. $\therefore$ the hour hand moved through $x - 30$ divisions from the 6 o'clock mark when he returned. In that time the minute hand moved through $(x - 30) \times 12$ divisions from 12 o'clock mark. $\therefore$ the distance of the minute hand when he returned was $(x - 30)12$ divisions from 12 o'clock mark. But when he went out, the hour hand was at the same place as on his return was occupied by the minute hand.

$$\therefore\ (x - 30)12 = 25 + \frac{x}{12} \text{ or } 144x - 4320 = 300 + x.$$

$$\therefore\ 143x = 4620 \text{ whence } x = \frac{4620}{143} = 32\frac{4}{13}$$

$\therefore$ He went out at $32\frac{4}{13}$ minutes past 5.

7. Arrangement into rectangles and squares.

(1) When a number of men is arranged into a **solid** rectangle with a men in one side and b men in the other side the total number of men $= ab$, for there are b lines of men, each line containing a men. If the arrangement be into a **solid** square with a men in each side (*a men in front*) the number of men $= a^2$ for the same reason.

(2) If the men be arranged into a **hollow** rectangle d **deep** with a men in one side and b men in the other, the number of men $= ab - (a-2d)(b-2d)$, there being a hollow rectangle which if solid would have contained $(a-2d)(b-2d)$ men. If the men be arranged into a **hollow square** d deep with a men in each side (a men in front) the number of men $= a^2 - (a-2d)^2$, for there would be a hollow square within, which might have contained $(a-2d)^2$ men.

Ex. 18. A number of troops being arranged into a solid rectangle with 4 men in one side less than in the other, it was found that there were 60 men over ; but if the number of men in the smaller side be increased by 3, the troops would have just sufficed to form the rectangle. Find the number.

Let $x =$ no. of men in one side of the rectangle.

Then $x-4 =$ no. ,, ,, the other ,, ,, ,, ,,

Then the total no. of men $= x(x-4)+60 = x^2 - 4x + 60$.

Again if the number of men in the smaller side be increased by 3, the total number of men

$$= x(x-4+3) = x(x-1) = x^2 - x.$$

Hence by the question, $x^2 - x = x^2 - 4x + 60$

$$\text{or } 3x = 60, \therefore x = 20.$$

$\therefore$ The no. of men $x^2 - x = 400 - 20 = 380$.

Ex. 19. An officer can arrange his men into a hollow square 8 deep and also into a hollow square 7 deep with 2 men more in the front. Find the number of men.

Let x be the number of men in the front of the first arrangement. Then $x+2$ is the number of men in the front of the second arrangement.

Since in the first arrangement the square is 8 deep, the total number of men $= x^2 - (x-16)^2 = 32x - 256$......(1)

Since in the second arrangement the square is 7 deep, the total number of men $= (x+2)^2 - (x+2-14)^2$

$$= (x+2)^2 - (x-12)^2 = 28x - 140 \text{......(2)}$$

Hence from (1) and (2), $32x - 256 = 28x - 140$.

$$\therefore 4x = 116 \text{ or } x = 29.$$

$\therefore$ The number of men $= 28 \times 29 - 140 = 672$.

EXERCISE LXXXIII.

1 A sum of £200 was contributed by A, B and C. A contributed £10 more than B and C £6 less than A. How much did each contribute?

2. A bag contains 258 coins some of which are rupees and the rest half-rupee pieces; they amount altogether to Rs. 179. How many coins were there of each kind?

3. A person had a sum of money to divide among A, B, C. He gave A one third of what he had and *Rs.* 20 more, B one third of what remained and *Rs.* 30 more, C one third of what was left and *Rs.* 20 more, and nothing was left to him. How much did A, B, and C get respectively?

4. A person bought a picture at a certain price and paid the same price for the frame: if the frame had cost £1 less and the picture 15*s.* more, the price of the frame would have been half that of the picture. Find the cost of the picture. (C. E. 1860.)

5. A workman was appointed for 28 days at *Rs.* 2. 8*as.* a day, but on days he was idle he was to pay *Re.* 1 a day instead of receiving anything. He received *Rs.* 52. 8*as.*; for how many days was he idle?

6. Two passengers are charged for excess of luggage *Rs.* 22 in all. Had the luggage all belonged to one person he would have been charged *Rs.* 23. They had in all 12 maunds of luggage. What is the charge for each maund of excess luggage?

7. A person distributes *Rs* 40 among 100 people, giving to some *as.* 4 each and to the rest *as.* 8 each. How many were there of each class?

8. A person bought a certain number of eggs; half of them at 2 a penny and half at 3 a penny. He sold them again at the rate of 5 for 2*d.* and lost a penny by the transaction. What was the number of eggs?

9. A boy spends his money in buying apples. Had he received 5 more for his money, they would have cost a half-penny each less; but if 3 less, half a penny each more. How much money did he spend?

10. A market woman sells 1000 oranges, some at a gain of 25 per cent. and the rest at a gain of 15 per cent. and thereby gains 18 per cent. on the whole. How many of each sort does she sell?

(B. M. 1897.)

11. If 19 lbs. of gold weigh 18 lbs. in water, and 20 lbs. of silver weigh 18 lbs. in water, find the quantity of gold and silver in a mass of gold and silver weighing 137 lbs. in air and weighing 126 lbs. in water.

12. A ship was sold at a loss of 10 p. c. If the ship was sold for Rs. 2000 more, the gain would have been 10 p.c.; what did the ship cost?

13. A sum of money was divided equally among 24 persons; had there been six more, each would have received a shilling less. Find the sum.

14. A person mixed 24 gallons of spirit at 9*s.* a gallon with 40 gallons at 11*s.* a gallon. Find what quantity of spirit at 13*s.* 9*d.* must he add, so that the whole may be worth 12*s.* a gallon.

15. The length of a field is twice its breadth; another field which is 40 yds. longer and 20 yds. broader contains 4880 square yds. more than the former; find the length and breadth of the field.

16. A and B start together from the same point on a walking match round a circular course. After half an hour A has walked three complete circuits and B four and a half. Assuming that each walks with uniform speed, find when B next overtakes A.

(P. E. 1892).

17. A number consists of two digits, of which the digit in the unit's place is thrice the digit in the tens' place; if 36 is added to the number, the digits are reversed. What is the number?

18. Three persons can severally do a work in 6, 8 and 10 days. Find the time in which they can perform the work if they work together.

19. What is the first time after 7 o'clock at which the two hands of a watch are (1) directly opposite, and (2) 10 degrees apart?

20. A cask A contains 24 gallons of wine and 36 gallons of water. Another cask B contains 18 gallons of wine and 6 gallons of water. How many gallons must be drawn from each cask so as to produce a mixture of 14 gallons of wine and 14 gallons of water?

21. A person rides a distance at the rate of 12 miles an hour and walks the distance back at the rate of 4 miles an hour. He takes 2 hours in all. Find the distance.

22. A person looks at a clock between 3 and 4 but mistaking the hour-hand for the minute-hand reads the time 56 minutes earlier than the real time. What is the time?

23 Of the candidates in a certain examination 35 per cent. failed in mathematics and 55 in English. 40 per cent. of the boys passed in both the subjects. Find the percentage of the candidates who failed in both the subjects.

24. A man wished to distribute a certain sum of money to a number of beggars. He found that if he would distribute 20*d.* to each, his money would fall short by 9*d.* and that if he would give 18*d.* to each he would have 9*d.* left. What had he at first and how many beggars were there?

25. A walks one mile per hour faster than B and three quarters of a mile per hour faster than C. To walk a certain distance B takes three quarters of an hour more than C and two hours more than A. Find the rates of walking of A, B and C.

26. A garrison had provisions for 60 months. At the end of 8 months it was doubled, and increased by 400 men 6 months afterwards, and in consequence the provisions were exhausted in 30 months from the first; find the original number of men in the garrison.

27. The sum of the ages of two men is 90. The age of one is $\frac{5}{7}$th that of the other. How long is it since one's age was three times that of the other?

28. A hare is 80 of her own leaps before a greyhound and takes 10 leaps for the greyhound's 8. But in 3 leaps the greyhound goes a distance for which the hare takes 4 leaps. How many leaps must each take before the hare is caught?

29. A cistern can be filled by one pipe in 32 minutes and emptied by another in 40 minutes: supposing it at first empty, find the time in which it will be filled when both the pipes are open.

30. How many maunds of rice at Rs. 5 per maund must be mixed with 1200 maunds at Rs. 6 per maund in order that there may be a gain of 20 per cent. by selling the whole at Rs. 6. 14 as. per maund? (C. E. 1875).

31. A and B have the same income; A saves $\frac{1}{5}$ of his income, but B by spending Rs. 900 more a year than A, at the end of 4 years is Rs. 2000 in debt; what is their income?

32. A merchant makes a mixture of 200 gallons of two kinds of wine, one at 10*s.* a gallon and the other at 13*s.* a gallon, and sells each gallon for 10*s.* 9*d* thereby gaining 3*d.* for each gallon. How many gallons of each kind must he take?

33. I bought 25 yards of cloth for Rs. 223. 8 as.; for a part I paid Rs. 8. 8 as. a yard and for the rest Rs. 9. 8 as. a yard. How many yards of each were there? (C. E. 1859).

34. Some engravings at 35*s.* each and some books at 16*s.* each were purchased by a man. The total cost was £33. 10*s* and the number of books was 10 more than the number of engravings. How many were there of each?

35. A's monthly income is 5/4 of B's and a third of A's income exceeds the difference of their incomes by Rs. 80. Find their respective incomes.

36. Two persons A and B start at the same time from the same place to travel round a circular course 72 miles in circumference. A travels 3 miles an hour and B $2\frac{1}{2}$ miles. After how many hours will they come together for the first time?

37. Two towns X and Y on a railway are 64 miles apart. Coals at X cost 18*s*. per ton and at Y 16*s*. per ton; they cost two pence per ton per mile to carry on the line. Find the distance from X of the place, at which it is immaterial to the consumer whether he buys coals from X or from Y. (A. E. 1896).

38. The number of months in the age of a man on his birth-day in the year 1875 was exactly half of the number denoting the year in which he was born. In what year was he born? (A.E.1898).

39. A's present age is 6/5 of B's present age; 24 years ago A's age was 4/3 of B's. Find their present ages.

40. A party of travellers coming to a hotel find that there are a too few bedrooms for each to have one. If they sleep two and two in a room, there are b empty rooms. How many rooms are left empty, if they sleep three in a room? (M. M. 1894).

41. Two passengers have together 7 mds. of luggage, and for the excess above the weight allowed free, one of them is charged Rs. 3 and the other Rs. 5. If all the luggage had belonged to one passenger, he would have been charged Rs. 11. What amount of luggage is each passenger allowed free of charge? (B. M. 1900).

42. A composition of copper and tin containing 140 cubic inches weighs 42 lb. 3 *oz*. How many ounces of each are there if a cubic inch of copper weighs $5\frac{1}{4}$ *oz*. and a cubic inch of tin, $4\frac{1}{4}$ *oz*.? (M. M. 1891).

43. Two vessels contain mixtures of wine and water; in one there is twice as much wine as water and in the other three times as much water as wine. Find how much must be drawn off from each to fill a third vessel which can hold 15 gallons, in order that its contents may be half wine and half water.

44. An officer, on forming his troops into a solid square, found that there were 100 men over; he then tried to form them into a column with 10 men more in front, and 5 men less in depth than before and found that 100 men were wanting. Find the number of troops.

45. A greyhound finding a hare at the distance of 20 yards from him pursues her. He makes 4 leaps for the hare's 3, but he covers as much space in 3 leaps as the hare does in 4. How many leaps must the greyhound take to catch the hare? (Greyhound's one leap covers 1 foot).

46. An officer can form his men into a hollow square 20 deep and also into a solid column, with 10 men more in front and 32 men in depth. Find the number of men.

47. A bag contains 160 coins consisting of half crowns, shillings, six pences and four pences, and the values of the sums of money represented by each denomination are the same; how many of each are there?

48. *A* can do a piece of work in 20 days, but after he has been upon it for 8 days, *B* is sent to help him and they finish it together in 4 days. In what time could *B* have done the whole?

49. How much must be taken from each of two bars of metal, the first containing 28 *oz.* of silver and 12 of tin and the second containing 16 *oz.* of silver and 24 of tin to form a bar of 20 *oz.* containing 10 *oz.* of silver and 10 *oz.* of tin?

50. A train 234 ft. long runs at the rate of 24 miles an hour: and another 294 ft. long runs on a parallel rail (i) in the opposite directions (ii) in the same direction, at the rate of 30 miles an hour. How long will they take to pass each other?

51. A person rowed down a river a distance of 18 miles in $1\frac{1}{2}$ hours with the stream and rowed back again in $4\frac{1}{2}$ hours. Find the rate of the stream per hour.

52. When are the hands of a watch (1) at right angles (2) in the same straight line (3) 10 divisions apart, (*a*) between 2 and 3 o'clock (*b*) between 9 and 10 o'clock?

53. Find a number such that whether divided into two equal parts or into three equal parts, the product of the parts shall be the same. (B. M. 1894.)

54. A person has a number of rupees which he tries to arrange in the form of a square. On the first attempt he has 116 over; when he increases the side of the square by 3 rupees, he wants 25 rupees to complete the square. How many rupees has he? (B. M. 1875).

55. A ship left Bombay on a voyage of 3 weeks, with provisions for that time at the rate of 1 seer a day for each man. At the end of a week a storm arose which washed 4 men overboard and so damaged the vessel that the speed was reduced by half and each man could be allowed only $\frac{3}{4}$ of a seer per diem. What was the original number of the crew?

56. A person bought an article and sold it at a profit of 6 per cent. Had he bought at 4 per cent. less and sold at *Re.* 1. 3*a.* more, his profit would have been 12 per cent. For how much did he buy it? (P. E. 1891).

57. If the cost of maintaining a family be *Rs.* 50 a month, when rice is 12 seers a rupee, and is *Rs.* 48 when rice is 14 seers a rupee (the other expenses remaining the same), what will be the cost when rice is 16 seers per rupee?

58. A regiment was drawn into a solid square; when, sometime after 116 men were removed from it, it was again drawn up into a solid square with 2 men less in the front. Find the original number of men in the regiment.

59. An officer can form his men into a hollow square 30 deep, also into a hollow square 20 deep, the front in the latter formation containing 10 men more. Find the number of men.

60. I went out between 3 and 4 o'clock and returning between 7 and 8 o'clock found that the hands of my watch have exactly changed places. When did I go out?

CHAPTER XX.

SIMULTANEOUS EQUATIONS.

1. Two or more equations, which are satisfied by the same values of the unknown quantities they contain are called **simultaneous equations.**

They are simple when the unknown quantities occur in them in the first power only.

Thus the equations $x-y=2$ and $x+y=8$ which will be *simultaneously* satisfied when $x=5$ and $y=3$ are simultaneous equations of the first degree in x and y.

2. An equation, like $x-y=2$, containing two unknowns, is satisfied by infinite pairs of values of x and y, *e.g.*:

$$\left.\begin{matrix} x=0 \\ y=-2 \end{matrix}\right\} \quad \left.\begin{matrix} x=1 \\ y=-1 \end{matrix}\right\} \quad \left.\begin{matrix} x=2 \\ y=0 \end{matrix}\right\} \quad \left.\begin{matrix} x=3 \\ y=1 \end{matrix}\right\}, \text{ etc.}$$

Similarly another equation like $x+y=4$ is also satisfied by an infinite pairs of values of x and y, *e.g.*:

$$\left.\begin{matrix} x=0 \\ y=4 \end{matrix}\right\} \quad \left.\begin{matrix} x=1 \\ y=3 \end{matrix}\right\} \quad \left.\begin{matrix} x=2 \\ y=2 \end{matrix}\right\} \quad \left.\begin{matrix} x=3 \\ y=1 \end{matrix}\right\}, \text{ etc.}$$

If now the second equation is to be *simultaneously* true for the same values of x and y as the first, we find that both the equations are satisfied only for one pair of values *viz.* $x=3, y=1$.

Hence to solve for two unknowns we must have two equations in them. Similarly if there be three unknown quantities, there must be three equations in order that they may be solved for.

Thus the equations $x+y+z=6$ and $2x+3y+4z=20$ are true for several sets of values x, y and z, *e.g.*:

$$\left.\begin{matrix} x=0 \\ y=4 \\ z=2 \end{matrix}\right\} \quad \left.\begin{matrix} x=1 \\ y=2 \\ z=3 \end{matrix}\right\} \quad \left.\begin{matrix} x=2 \\ y=0 \\ z=4 \end{matrix}\right\} \quad \left.\begin{matrix} x=3 \\ y=-2 \\ z=5 \end{matrix}\right\}, \text{ etc. and hence cannot be definitely solved.}$$

But if there be another equation $x-y+z=2$, only one set of values of x, y, z will satisfy the three equations *viz.*, $x=1$, $y=2$, $z=3$.

Generally to solve simultaneous equations we must have as many equations as there are unknown quantities.

3. Consider the two equations, $x-y=2$ and $2x-2y=4$ They are satisfied by infinite pairs of values of x and y, *e. g.*:

$x=0, y=-2$; $x=1, y=-1$; $x=2, y=0$; $x=3, y=1$; etc.

Here the equation $2x-2y=4$ is derived from the equation $x-y=2$ by multiplying it by 2. The two equations are therefore one and the same equation.

Hence *to solve equations in two unknowns, there must be two equations in the unknowns, and these equations must be independent of each other.*

Again the equations $x+y+z=6$, $2x+3y+4z=20$ and $x+2y+3z=14$ are satisfied by several sets of values of x, y and z, *e.g.* $x=0$, $y=4$, $z=2$; $x=1$, $y=2$, $z=3$. Here the equation $x+2y+3z=14$ is derivable from the first two equations by subtraction and virtually there are two equations. *Thus to solve equations in 3 unknowns there must be three independent equations.*

Generally to solve simultaneous equations in several unknowns there must be **as many** equations as there are unknown quantities and these equations must be **independent**, that is, no one must be derived from any one or more of the others.

Simultaneous equations : two unknowns.

4. We give three methods of solving simultaneous equations containing two unknowns. *All simultaneous equations in two unknowns must be brought to the standard from $ax+by=c$, before they are solved.*

5. **First method.** The following examples illustrate the method.

Ex. 1. Solve $3x-7y=4$......(1), $2x+5y=22$......(2)

From (1) $-7y=4-3x$.

$$\therefore\ y=\frac{4-3x}{-7}=\frac{3x-4}{7}\ ......(3)$$

From (2) $5y=22-2x$.

$$\therefore\ y=\frac{22-2x}{5}\ ......(4)$$

$\therefore$ from (3) and (4) $\dfrac{3x-4}{7}=\dfrac{22-2x}{5}$.

Multiplying by 35, $15x-20=154-14x$;

transposing, $29x=174$. $\therefore\ x=6$.

From (3) by substitution, $y=\dfrac{3\times6-4}{7}=2$.

$\therefore$ we have $x=6$, $y=2$.

Verification. Substituting these values of x and y, the two sides of *each equation* will be found to be equal.

Ex. 2. Solve $a_1x+b_1y=c_1$......(1), $a_2x+b_2y=c_2$......(2).

From (1) $b_1y=c_1-a_1x$, $\therefore\ y=\dfrac{c_1-a_1x}{b_1}$......(3)

From (2) $b_2y=c_2-a_2x$, $\therefore\ y=\dfrac{c_2-a_2x}{b_2}$......(4)

From (3) and (4), $\dfrac{c_1-a_1x}{b_1}=\dfrac{c_2-a_2x}{b_2}$,

or $b_2c_1-b_2a_1x=b_1c_2-b_1a_2x$,

or $x(b_1a_2-b_2a_1)=b_1c_2-b_2c_1$,

$$\therefore\ x=\frac{b_1c_2-b_2c_1}{b_1a_2-b_2a_1}.$$

Substituting this value of x in (3), $y=\dfrac{c_1-a_1\dfrac{b_1c_2-b_2c_1}{b_1a_2-b_2a_1}}{b_1}$

Simplifying $y=\dfrac{a_2c_1-a_1c_2}{b_1a_2-b_2a_1}$.

From the above we deduce the first method :

From each of the two equations find the value of one unknown quantity in terms of the other and then equate the two values thus obtained.

EXERCISE LXXXIV.

Solve by the first method

1. $4x+y=11$, $5x-2y=4$.
2. $3x-y=4$, $4x+3y=1$.
3. $13x-7y=23$, $8x+2y=52$.
4. $5x-3y=16$, $4x+7y=41$.
5. $\dfrac{x}{3}-\dfrac{3y}{4}=\dfrac{1}{2}$, $9x+11y=76$.
6. $3x-2y+17=0$, $\frac{5}{3}x+3y=7$.
7. $3x+4y=7a-b$, $5x-7y=-2(a-6b)$.
8. $x+y=a+b$, $ax+by=a^2+b^2$.

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6 Second method. The following examples illustrate the method.

Ex. 1. $3x-7y=4$......(1), $2x+5y=22$......(2)

From (1) we get $7y=3x-4$, or $y=\dfrac{3x-4}{7}$......(3)

Substituting this value of y in (2) we get,

$$2x+\frac{5(3x-4)}{7}=22.$$

Multiplying both sides by 7, $14x+15x-20=154$

$\therefore\ 29x=174,\ \therefore\ x=6.$

Substituting in (3) we get $y=\dfrac{3\times 6-4}{7}=2.$

Hence $x=6,\ y=2.$

Ex. 2. Solve $a_1x+b_1y=c_1\ldots(1),\ a_2x+b_2y=c_2\ldots(3)$

From (1) $b_1y=c_1-a_1x,\quad \therefore\ y=\dfrac{c_1-a_1x}{b_1}\ldots\ldots(3)$

Substituting this value of y in (2),

$$a_2x+b_2\frac{c_1-a_1x}{b_1}=c_2.$$

$$\therefore\ b_1a_2x+b_2c_1-b_2a_1x=b_1c_2.$$

$$\text{or } x(b_1a_2-b_2a_1)=b_1c_2-b_2c_1.$$

$$\therefore\ x=\frac{b_1c_2-b_2c_1}{b_1a_2-b_2a_1}.$$

Substituting this value of x in (3) and simplifying,

$$y=\frac{a_2c_1-a_1c_2}{b_1a_2-b_2a_1}.$$

From the above we deduce the second method :

Find the value of one of the unknown quantities in terms of the other from either equation, and substitute this value in the other equation.

EXERCISE LXXXV.

Solve by the second method

1. $7x-8y=5,\quad 4y-3x=-1.$
2. $5x+2y=45,\quad 3y-2x=1.$
3. $5x+2y=17,\quad x-\frac{5}{6}y=\frac{13}{6}.$
4. $4x+11y=6,\quad \frac{1}{2}x-y+4=0.$
5. $5x-7y+9=0,\quad 6x+11y=0,$
6. $13x-7y=47,\quad \frac{5}{2}x+\frac{8}{3}y+3=0.$
7. $x+my=n,\ mx-ny=l.$
8. $\dfrac{x}{m+n}+\dfrac{y}{m-n}=2m,\quad \dfrac{x-y}{4mn}=1.$

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7. Third method. The following examples illustrate the method.

Ex. 1. Solve $6x+5y=46\ldots(1),\ 8x-7y=34\ldots(2)$

Multiplying (1) by 4, $24x+20y=184\ldots(3)$

Multiplying (2) by 3, $24x-21y=102\ldots(4)$

Subtracting (4) from (3) x is eliminated and we get

$$41y=82, \therefore y=2.$$

*Substituting this value of y in either of the two equations, equation (1), for example, $6x+10=46$ or $6x=36$, $\therefore x=6$.

Hence $x=6, y=2$,

We might determine x by multiplying (1) by 7 and (2) by 5 and adding the results.

Ex. 2. Slove $a_1x+b_1y=c_1$...(1) $a_2x+b_2y=c_2$......(2)

Multiplying equation (1) by a_2 and equation (2) by a_1 we get

$a_1a_2x+b_1a_2y=a_2c_1$......(3), $a_2a_1x+b_2a_1y=c_2a_1$......(4)

Subtracting (4) from (3) so as to eliminate x,

$y(b_1a_2-b_2a_1)=a_2c_1-c_2a_1$...(A)

$$\therefore y=\frac{a_2c_1-c_2a_1}{b_1a_2-b_2a_1}.$$

Substituting this value of y in (1), $a_1x+b_1\dfrac{a_2c_1-c_2a_1}{b_1a_2-b_2a_1}=c_1$,

$$\text{or } a_1x=c_1-b_1\frac{a_2c_1-c_2a_1}{b_1a_2-b_2a_1}=\frac{b_1c_2a_1-c_1b_2a_1}{b_1a_2-b_2a_1}.$$

$$\therefore x=\frac{b_1c_2-c_1b_2}{b_1a_2-b_2a_1}.$$

From the above we deduce the third method :—

Multiply the equations by such numbers as will make the co-efficients of one of the unknown quantities the same (at least numerically), in both the resulting equations. Then add or subtract as necessary, to eliminate this unknown quantity and form an equation in the other unknown.

[This method is most generally used.]

EXERCISE LXXXVI.

Slove by the third method

1. $16x+5y=133$, $6x+19y=67$.

2. $6x-17y=-27$, $4x+31y=109$.

3. $15x+57y=890$, $18x-23y=-280$.

4. $5y-2x=26$, $12x-y=18$. **5.** $6x+2y=23$, $5y-x=17$.

6. $90x-14y=200$, $56y-50x=13$.

7. $104x-19y=3856$, $19x+104y=2424$.

8. $71x-45y=1431$, $37x-23y=757$.

9. $bx+ay=2ab$, $ax-by=a^2-b^2$.

Solve by the third method

10. $\frac{x}{a}+\frac{y}{c}=a+b, \frac{x}{b}+\frac{y}{a}=a+c.$

11. $\left.\begin{array}{l}(a+b)x+(a-b)y=2a, \\ (a-b)x+(a+b)y=2b\end{array}\right\}$ A. E. 1891.

12. $\left.\begin{array}{l}(a+b)x+(a-b)y=2ac, \\ (b+c)x+(b-c)y=2bc\end{array}\right\}$ A. E. 1895.

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8. We add illustrative examples in two unknowns

Ex. 1. Solve $41x+37y=197$.........(1)

$$37x+41y=193\text{.........(2)}$$

Here we have the peculiarity that the co-efficients of x and y in the two equations are interchanged.

Adding (1) and (2),

$$78x+78y=390,$$

$$\text{or } x+y=5\text{..... (3)}$$

Subtracting (2) from (1),

$$4x-4y=4,$$

$$\text{or } x-y=1\text{......(4)}$$

From (3) and (4) by addition $x=3$, and by subtraction $y=2$.

Ex. 2. Solve $4(5x-7)-16(x-2y)=23(y+1)$,

$$(x-3)(y+15)=(x+10)(y+7)-106.$$

From the first equation

$$20x-28-16x+32y=23y+23.$$

$$\therefore 4x+9y=51\text{......(1)}$$

From the second equation,

$$xy+15x-3y-45=xy+7x+10y+70-106,$$

$$\therefore 8x-13y=9\text{.........(2)}$$

Multiplying (1) by 2,

$$8x+18y=102\text{......(3)}$$

Subtracting (2) from (3),

$$31y=93 \text{ or } y=3.$$

Hence from (1) $4x+27=51$, or, $4x=24$.

$$\therefore x=6.$$

Thus $x=6, y=3$.

Ex. 3. Solve

$$4(3x-2y)+7(5x+y)=62\ldots\ldots\ldots(1)$$
$$2(3x-2y)-3(5x+y)+8=0\ldots\ldots(2)$$

Here we need not reduce the equations to the standard form but proceed thus :

Put $3x-2y=u$, $5x+y=v$.

Hence (1) and (2) respectively become

$$4u+7v=62\ldots\ldots\ldots(3)$$
$$2u-3v=-8\ldots\ldots\ldots(4)$$

To slove (3) and (4), multiply (4) by 2,

then $4u-6v=-16\ldots\ldots(5)$

Subtracting (5) from (3), $13v=78$,

$\therefore\ v=6.$

Hence from (3) $4u+42=62$,

$\therefore\ 4u=20$, or, $u=5$.

Thus we have $3x-2y=5$,

$5x+y=6$.

Solving these we get $x=\frac{17}{13}$, $y=-\frac{7}{13}$.

Ex. 4. Solve

$$\frac{x-y}{4}+\frac{y+2}{7}=2y-8\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{11x+9y}{6}-\frac{3x+4y-1}{2}=1\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Multiplying (1) by 28,

$$7(x-y)+4(y+2)=28(2y-8)$$

or simplifying, $7x-59y=-232\cdots\cdots(3)$

Multiplying (2) by 6,

$$11x+9y-3(3x+4y-1)=6$$

or simplifying, $2x-3y=3\ldots\ldots\ldots\ldots\ldots(4)$.

To solve (3) and (4) multiply them respectively by 2 and 7, hence

$$14x-118y=-464,\ 14x-21y=21.$$

$\therefore$ subtracting $-97y=-485$ or $y=5$.

Substituting this value of y in (4),

$2x-15=3$, $\therefore\ 2x=18$ or $x=9$.

Hence $x=9$, $y=5$.

Ex. 5. Solve

$$\frac{x-3y+1}{2x+y-2}-\tfrac{5}{9}=\frac{3x-5}{9}-\frac{x-1}{3}\ldots\ldots(1).$$

$$\frac{x-y+4}{3y-5x+1}+\frac{x+1}{5}=\frac{2x-3}{10}\ldots\ldots\ldots(2).$$

From (1) $\dfrac{x-3y+1}{2x+y-2}=\dfrac{3x-5}{9}-\dfrac{x-1}{3}+\tfrac{5}{9}.$

$$=\frac{3x-5-3x+3+5}{9}=\tfrac{3}{9}=\tfrac{1}{3}.$$

$\therefore\ 3x-9y+3=2x+y-2,$

or $x-10y=-5\ldots\ldots(3)$

From (2) $\dfrac{x-y+4}{3y-5x+1}=\dfrac{2x-3}{10}-\dfrac{x+1}{5}$

$$=\frac{2x-3-2x-2}{10}=-\tfrac{5}{10}=-\tfrac{1}{2}.$$

$\therefore\ 2x-2y+8=-3y+5x-1,$

or $3x-y=9\ldots\ldots\ldots(4)$

Multiplying (3) by 3, $3x-30y=-15\ldots\ldots(5)$

Subtracting (5) from (4),

$29y=24$ or $y=\tfrac{24}{29}.$

Substituting in (4), $3x-\tfrac{24}{29}=9.$

$$\therefore\quad 3x=9+\frac{24}{29}=\frac{285}{29}\ \text{ or }\ x=\frac{95}{29}.$$

Thus $x=\dfrac{95}{29},\ y=\dfrac{24}{29}.$

Ex. 6. Solve

$$\left.\begin{array}{l}\dfrac{1}{5x}+\dfrac{y}{9}=5\ldots\ldots(1)\\[2ex]\dfrac{1}{3x}+\dfrac{y}{2}=14\ldots(2)\end{array}\right\}\ \text{C. E. 1870.}$$

Here we may regard $\dfrac{1}{x}$, y as unknown quantities and solve the equations in these.

Multiplying (1) by 45, $9.\dfrac{1}{x}+5.y=225\ldots(3)$

Multiplying (2) by 6, $2.\dfrac{1}{x}+3.y=84\ldots(4)$

To solve (3) and (4) multiply them respectively by 3 and 5; hence

$$27\cdot\frac{1}{x}+15.y=675,\ 10.\frac{1}{x}+15.y=420.$$

$\therefore$ by subtraction $17\cdot\frac{1}{x}=255$, hence $\frac{1}{x}=\frac{255}{17}=15$ or $x=\frac{1}{15}$.

Substituting in (i), $3+\frac{y}{9}=5$ or $\frac{y}{9}=2$, whence $y=18$.

Thus $x=\frac{1}{15}$, $y=18$.

Ex. 7. Solve $\frac{9}{x}-\frac{4}{y}=1\ldots(1)$, $\frac{18}{x}+\frac{20}{y}=16\ldots\ldots(2)$.

Here we suppose $\frac{1}{x}$ and $\frac{1}{y}$ to be the unknown quantities and solve the equations in them.

Multiplying (1) by 2 we get, $\frac{18}{x}-\frac{8}{y}=2\ldots\ldots(3)$.

Subtracting (3) from (2) we get

$$\frac{28}{y}=14.\qquad \therefore\ 14y=28,\qquad \therefore\ y=2.$$

Substituting in (1) this value of y we get

$$\frac{9}{x}-\frac{4}{2}=1:\text{ or }\frac{9}{x}-2=1;\text{ or }\frac{9}{x}=3;$$

$$\text{or } 3x=9.\qquad \therefore\ x=3.\quad \text{Thus } x=3,\ y=2.$$

Ex. 8. Solve $\frac{m}{x}+\frac{n}{y}=a\ldots(1)$, $\frac{n}{x}+\frac{m}{y}=b\ldots(2)$

Here we proceed by regarding $\frac{1}{x}$, $\frac{1}{y}$ as the unknown quantities.

Multiplying (1) by m and (2) by n we get

$$\frac{m^2}{x}+\frac{mn}{y}=am\ldots\ldots\ldots(3),\ \frac{n^2}{x}+\frac{mn}{y}=nb\ldots\ldots\ldots\ldots\ldots(4)$$

Subtracting (4) from (3) we get $\frac{m^2-n^2}{x}=am-nb$,

$$\therefore\ (am-nb)x=m^2-n^2,\quad \therefore\ x=\frac{m^2-n^2}{am-nb}.$$

Again, multiplying (1) by n and (2) by m, we get

$$\frac{mn}{x}+\frac{n^2}{y}=an\ldots(5),\ \frac{mn}{x}+\frac{m^2}{y}=bm\ldots(6)$$

Subtracting (6) from (5), $\frac{n^2-m^2}{y}=an-bm$.

$\therefore y\,(an-bm)=n^2-m^2$ or $y=\frac{n^2-m^2}{an-bm}$.

Ex. 9. Solve

$14x+13y=35xy\ldots(1)$ $\qquad$ $21x+19y=56xy\ldots(2)$

Dividing (1) by xy, $\frac{14}{y}+\frac{13}{x}=35\ldots(3)$

Dividing (2) by xy, $\frac{21}{y}+\frac{19}{x}=56\ldots(4)$

Equations (3) and (4) are of the type of examples 7 and 8. Solving these,

$x=-\frac{1}{7},\ y=\frac{1}{9}$.

Ex. 10. Solve

$$\frac{3}{x+2y+1}-\frac{11}{3x-y-2}=4\ldots(1)$$

$$\frac{5}{x+2y+1}+\frac{12}{y-3x+2}=13\ldots(2)$$

Put $x+2y+1=u$, $3x-y-2=v$; then the equations (1) and (2) respectively become

$\frac{3}{u}-\frac{11}{v}=4$, $\frac{5}{u}-\frac{12}{v}=13$.

Solving these in the usual way,

$u=\frac{1}{5},\ v=1$.

$\therefore\ x+2y+1=\frac{1}{5}\ldots(3)$, $3x-y-2=1\ldots\ldots(4)$

From (3) $5x+10y+5=1$ or $5x+10y=-4\ldots(5)$

Also from (4) $3x-y-2=1$ or $3x-y=3\ldots\ldots(6)$

From (5) and (6) we find $x=\frac{26}{35}$, $y=-\frac{27}{35}$.

EXERCISE LXXXVII.

Solve

1. $15x+23y=61$,
   $23x+15y=53$.

2. $69x-49y=182\frac{1}{2}$,
   $49x-69y=12\frac{1}{2}$.

3. $(x+7)(y-3)+7=(y+3)(x-1)+5$, $\}$ C. E. 1888.
   $5x-11y+35=0$.

4. $6(2x-3y)-5(x+y)=11$,
   $2(2x-3y)+3(x+y)=27$

5. $\frac{x-2y}{\text{[illegible]}}+\frac{3x+y}{\text{[illegible]}}=8$, $\frac{x-2y}{\text{[illegible]}}+\frac{3x+y}{\text{[illegible]}}=5$.

Solve

6. $\dfrac{5x-3y}{4}+11=\dfrac{9x+2y}{10}+\dfrac{y}{3},\quad \dfrac{8}{3}-\dfrac{x-2y}{12}=\dfrac{x}{6}+\dfrac{y}{9}.$

7. $\left.\begin{aligned}&\dfrac{2x+y}{5}-3=\dfrac{3x-5y}{2},\\ &\dfrac{x}{2}+\dfrac{y}{3}=\dfrac{y}{2}-\dfrac{x}{4}+8.\end{aligned}\right\}$ P. E. 1888.

8. $\left.\begin{aligned}&\dfrac{7+x}{5}-\dfrac{2x-y}{4}=3y-5,\\ &\dfrac{5y-7}{2}+\dfrac{4x-3}{6}=18-5x.\end{aligned}\right\}$ C. E. 1880.

9. $\left.\begin{aligned}&\dfrac{x-2}{2}-\dfrac{x+y}{14}=\dfrac{x-y-1}{8}-\dfrac{y+12}{4},\\ &\dfrac{x+7}{3}+\dfrac{y-5}{10}=1-x-\dfrac{5(y+1)}{7}\end{aligned}\right\}$ C. E. 1882.

10. $\left.\begin{aligned}&2x-\dfrac{2y-1}{3}=3\tfrac{5}{24}+\dfrac{3x-2y}{4},\\ &4y-\dfrac{5-2x}{4}=6-\dfrac{3-2y}{5}\end{aligned}\right\}$ C. E. 1873.

11. $\dfrac{23}{10}x-\dfrac{7y}{15}-\dfrac{28}{9}+\dfrac{x+y}{15}=\dfrac{x}{6}+\dfrac{11y}{90}+3\dfrac{11}{45},$

$4x-\dfrac{16}{3}=\dfrac{x}{6}-\dfrac{2y}{3}+7\tfrac{1}{2}.$

12. $\dfrac{3x-4y}{2}+7=2x+5y-11=\dfrac{2y-7x}{3}+8.$

13. $\dfrac{2x+5y+4}{13}=\dfrac{5x+3y+1}{14}=\dfrac{3x+4y+5}{15}.$

14. $\dfrac{3x+7-4y}{5}=\dfrac{5}{6}+\dfrac{2x}{15}-\dfrac{3y}{10},$

$\dfrac{x-1}{3}-\dfrac{3x}{20}+\dfrac{y}{2}=\dfrac{x-y}{15}+\dfrac{11}{10}+\dfrac{y}{6}.$

15. $\dfrac{\cdot 02y+\cdot 05}{\cdot 15}=\dfrac{\cdot 035x-\cdot 05}{3},$

$4{\cdot}8x+{\cdot}64y-\dfrac{\cdot 72x-\cdot 1}{\cdot 5}=1{\cdot}6x+\dfrac{5{\cdot}2+\cdot 01y}{\cdot 25}.$

16. $ax-by=a-b,\quad \dfrac{x}{2a}+\dfrac{y}{2b}=\dfrac{1}{a+b}.$

Solve

17. $x+\frac{ay}{a-b}=b=\frac{ax}{a+b}+y.$ (M. M. 1898).

18. $\frac{x-y}{a}+\frac{x+y}{b}=c,\quad \frac{x-y}{b}-\frac{x+y}{a}=c.$ (M. M. 1895).

19. $\frac{x}{a}+\frac{y}{b-a}=5m,\quad \frac{x}{b}+\frac{y}{a-b}=7m.$ (M. M. 1891).

20. $(a+b)x+by=ax+(a+b)y$
$=a^3-b^3.$ (B. M. 1896).

21. $\frac{(a-b)x+(a+b)y}{a^2-b^2}=\frac{ab}{a-b}=\frac{ab(x-y)-(a^2y-b^2x)}{2ab^2}.$ (B. M. 1902).

22. $2x+\frac{3}{y}=4,\quad 3x+\frac{2}{y}=5.$ (A. E. 1898).

23. $ax+\frac{b}{y}=2,\quad \frac{3b}{y}-2ax=1.$

24. $\frac{3}{2x}+\frac{2}{3y}=5,\quad \frac{2}{x}+\frac{3}{y}=13.$ (M. E. 1881).

25. $\frac{4}{x}+\frac{10}{y}=2,\quad \frac{3}{x}+\frac{2}{y}=\frac{19}{20}.$ (C. E. 1879).

26. $\frac{1}{ax}+\frac{1}{by}=1,\quad \frac{1}{3ax}+\frac{1}{6by}=\frac{2}{3}.$

27. $\frac{a+b}{x}-5b=\frac{a-b}{y}-a,\quad \frac{a}{x}-2a=\frac{b}{y}-3b.$ (A. E. 1894).

28. $\frac{a}{x}+\frac{b}{y}=m,\quad \frac{a}{y}+\frac{b}{x}=n.$

29. $\frac{1}{x}+\frac{1}{y}=\frac{5}{6},\quad 3x+2y=2xy.$ (A. E. 1899).

30. $ax+by=cxy,\quad mx-ny=dxy.$

31. $\frac{x+y}{xy}=5,$
$\frac{x-y}{xy}=9.$

32. $\frac{2}{3x-5y+1}+\frac{7}{x+2y-3}=29,$
$\frac{5}{3x-5y+1}-\frac{6}{x+2y-3}=2.$

33. $\frac{1}{3x+y}+\frac{4}{2x-y}=14,$
$\frac{7}{3x+y}-\frac{3}{2x-y}=5.$

34. $\frac{2x-6y+5}{x-3y+2}+\frac{6x-8y+3}{3x-4y+1}=4\tfrac{1}{2},$
$\frac{3x-9y+5}{x-3y+2}+\frac{9x-12y+7}{3x-4y+1}=6\tfrac{1}{3}.$

35. $\frac{3x-5}{8}+\frac{x+y-1}{2x-y+4}=x-\frac{5x+3}{8},\quad \frac{y-1}{6}+\frac{3x-4y+5}{x+y+1}=\frac{y+2}{6}.$

9. Method of Cross Multiplication.

Given $ax+by+cz=0$ ... ... (1)

$a'x+b'y+c'z=0$ ... ... (2)

To prove that $\dfrac{x}{bc'-b'c}=\dfrac{y}{ca'-c'a}=\dfrac{z}{ab'-a'b}$.

Multiply (1) by c' and (2) by c, then

$$c'ax+c'by+cc'z=0, \qquad ca'x+cb'y+cc'z=0\ ;$$

hence subtracting $x(c'a-ca')+y(bc'-b'c)=0$.

$\therefore\ x(ca'-c'a)=y\,(bc'-b'c)$ or $\dfrac{x}{bc'-b'c}=\dfrac{y}{ca'-c'a}$ ....(3)

Again, multiply (1) by a' and (2) by a, then

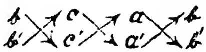
$$aa'x+ba'y+ca'z=0. \qquad aa'x+ab'y+ac'z=0\ ;$$

hence subtracting $y(a'b-ab')+z(ca'-c'a)=0$.

$\therefore\ y(ab'-a'b)=z(ca'-c'a)$, or $\dfrac{y}{ca'-c'a}=\dfrac{z}{ab'-a'b}$ ....(4)

$\dfrac{x}{bc'-b'c}=\dfrac{y}{ca'-c'a}=\dfrac{z}{ab'-a'b}$ ....(5)

The above result (5) called **the rule of cross multiplication** is very important, and the student can easily remember it thus. Write down the co-efficients x, y, z in order as below, beginning and ending with those of y and form products as indicated by arrows. Then subtracting each ascending product



from the corresponding descending product we get the three expressions $bc'-b'c$, $ca'-c'a$, $ab'-a'b$ which are the denominators of x, y, z respectively in (5).

If we make $z=1$ in equations, (1), (2) and (5) we get solutions of the simultaneous equations in two unknowns $ax+by+c=0$ and $a'x+b'y+c'=0$ from

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{1}{ab'-a'b}.$$

This gives $x=\dfrac{bc'-b'c}{ab'-a'b}$, $y=\dfrac{ca'-c'a}{ab'-a'b}$.

Ex. Solve $4x+5y-3=0$,

$7x-2y+6=0$

By the rule of cross-multiplication,

$$\frac{x}{5\times 6-(-2)(-3)}=\frac{y}{(-3)\times 7-4\times 6}=\frac{1}{4(-2)-5\times 7}$$

$$\text{or } \frac{x}{24}=\frac{y}{-45}=\frac{1}{-43}. \qquad \therefore\ x=-\frac{24}{43},\ y=\frac{45}{43}$$

This method of cross-multiplication is highly useful in Mathematics and it readily gives the solutions of simultaneous equations in two unknowns.

Simultaneous equations : Three unknowns.

10. Solution by the method of cross-multiplication, of simultaneous equations in three unknowns two of which are of the form ax+by+cz=0.

Solve $a_1x+b_1y+c_1z=0$ ....(1)
$a_2x+b_2y+c_2z=0$ ....(2)
$a_3x+b_3y+c_3z=d$ ....(3)

From (1) and (2) by the method of cross-multiplication

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-a_1c_2}=\frac{z}{a_1b_2-b_1a_2}$$

Supposing each of these fractions $=k$, we have

$$x=(b_1c_2-b_2c_1)\,k\ ;\ y=(c_1a_2-a_1c_2)\,k\ ;\ z=(a_1b_2-b_1a_2)\,k \ldots(A)$$

Substituting in (3) we get

$$a_3(b_1c_2-b_2c_1)k+b_3(c_1a_2-a_1c_2)k+c_3(a_1b_2-b_1a_2)k=d$$

$$\text{or } k\{a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-a_1c_2)+c_3(a_1b_2-b_1a_2)\}=d$$

$$\therefore k=\frac{d}{a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-a_1c_2)+c_3(a_1b_2-b_1a_2)}$$

Substituting in (A) this value of k,

$$x=\frac{(b_1c_2-b_2c_1)d}{a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-a_1c_2)+c_3(a_1b_2-b_1a_2)}$$

$$\text{Similarly } y=\frac{(c_1a_2-a_1c_2)d}{a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-a_1c_2)+c_3(a_1b_2-b_1a_2)}.$$

$$z=\frac{(a_1b_2-b_1a_2)d}{a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-a_1c_2)+c_3(a_1b_2-b_1a_2)}$$

Ex. 1. Solve $6x-9y+4z=0$ ... ... (1)
$8x+5y-6z=0$ ... ... (2)
$3x+4y+5z=26$ ... ... (3)

From (1) and (2) by the rule of cross-multiplication,

$$\frac{x}{(-9)(-6)-4\times 5}=\frac{y}{4\times 8-6(-6)}=\frac{z}{6\times 5-(-9)\times 8},$$

$$\text{or } \frac{x}{54-20}=\frac{y}{32+36}=\frac{z}{30+72};$$

$$\therefore \frac{x}{34}=\frac{y}{68}=\frac{z}{102} \text{ or } \frac{x}{1}=\frac{y}{2}=\frac{z}{3}=k \text{ (suppose).}$$

$\therefore x=k, y=2k, z=3k$.........(A).

Substituting in (3), these values of x, y and z,

$3k+8k+15k=26$ or $26k=26$.

$\therefore k=1$; hence substituting in (A) this value of k,

$x=1, y=2, z=3$.

Ex. 2. Solve

$x+y+z=0$ ... ... (1)

$bcx+cay+abz=0$ ... ... (2)

$ax+by+cz+(b-c)(c-a)(a-b)=0$ ... (3) (C. E. 1896.)

From (1) and (2) by the rule of cross-multiplication,

$$\frac{x}{ab-ca}=\frac{y}{bc-ab}=\frac{z}{ca-bc}=k \text{ (suppose).}$$

$$\left.\begin{aligned} \therefore x&=k\ (ab-ca)=ka(b-c)\\ y&=k\ (bc-ab)=kb(c-a)\\ z&=k\ (ca-bc)=kc(a-b)\end{aligned}\right\}\text{.........(A)}$$

Substituting in (3)

$$k\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}+(b-c)(c-a)(a-b)=0.$$

$\therefore k(b-c)(c-a)(a-b)=(b-c)(c-a)(a-b)$, $\therefore k=1$.

Hence from (A), $x=a(b-c)$, $y=b(c-a)$, $z=c(a-b)$.

Ex. 3 Solve $6x+y=5z$......(1), $7y+z=6x$...(2)

$$\text{and } \frac{6}{x}+\frac{4}{y}+\frac{8}{z}=3\cdots(3)$$

From (1) and (2) we get $\left.\begin{aligned}6x+y-5z=0\\ 6x-7y-z=0\end{aligned}\right\}$

Hence $$\frac{x}{1\times(-1)-(-5)\times(-7)}=\frac{y}{(-5)\times6-6\times(-1)}$$
$$=\frac{z}{6\times(-7)-1\times6},$$

or $$\frac{x}{-1-35}=\frac{y}{-30+6}=\frac{z}{-42-6};$$

or $$\frac{x}{-36}=\frac{y}{-24}=\frac{z}{-48}.$$

$\therefore \frac{x}{3}=\frac{y}{2}=\frac{z}{4}$ (multiplying each fraction by -12).

Suppose each of these fractions $=k$; then we have

$x=3k, y=2k, z=4k$.........(A)

Substituting these values of x, y, z in (3) we have

$$\frac{6}{3k}+\frac{4}{2k}+\frac{8}{4k}=3 \text{ or } \frac{2}{k}+\frac{2}{k}+\frac{2}{k}=3$$

$$\therefore \frac{6}{k}=3 \text{ or } 3k=6, \quad \therefore k=2.$$

Hence from (**A**) $\qquad x=6,\ y=4,\ z=8.$

EXERCISE LXXXVIII.

Solve by the method of cross-multiplication :—

1. $\left.\begin{array}{l} 3x+2y-8=0, \\ 4x-3y-5=0. \end{array}\right\}$ **2.** $\left.\begin{array}{l} 5x+6y+3=0, \\ 2x-8y+22=0. \end{array}\right\}$

3. $\left.\begin{array}{l} 4x+5y=2, \\ 3x-2y=36. \end{array}\right\}$ **4.** $\left.\begin{array}{l} -7x+8y+3=0, \\ 5x-6y=1. \end{array}\right\}$

5. $11x=3y-11,\ 16x-3y=-1.$ **6.** $5x-6y=0,\ 10x-9y=15.$

7. $\frac{1}{4}(x+y)+\frac{1}{2}(x-y)=4,\ 5x-7y=0.$

8. $y(x+3)=x(y+4),\ 5x+4=4y+3.$

9. $\frac{3x+2y}{19}=2(x-y),\ \frac{x-y}{3}+2y=x+6\frac{1}{2}.$

10. $(x+4)(y+5)=(x-2)(y+8)+48,$
$2x+7=3y+2.$

11. $x-2y+z=0,\ 9x-8y+3z=0,\ 2x+3y+5z=30.$

12. $x-2y+z=0,\ 5z-3x-4y=0,\ 7x+8y+9z=98.$

13. $x-2(3y-2z)=0,\ 2y+3(x-z)=0,\ 5x+7y+9z=67.$

14. $x+2y-4z=0,\ x-y+z=0,$
$\frac{4}{x}+\frac{10}{y}+\frac{6}{z}=6.$

15. $3x=4y=6z,\ 2x-3y+z=$ [illegible].

16. $ax=by=cz,\ x(b-c)+y(c-a)+z(a-b)=(a-b)(b-c)(a-c).$

17. $x+y+z=0,\ ax+by+cz=0,\ a^2x+b^2y+c^2z+(b-c)(c-a)$
$(a-b)=0.$ (C. E. 1904).

18. $x+y+z=0,\ (a+b)x+(a+c)y+(b+c)z=0,\ abx+acy+bcz=1$

19. $ax+by+cz=0,$
$(b+c)x+(c+a)y+(a+b)z=0,$
$a^2x+b^2y+c^2z+(b-c)(c-a)(a-b)=0,$

20. $ax+by+cz=0,$
$a^2x+b^2y+c^2z=0,$
$x+y+z=(b-c)(c-a)(a-b).$

Solve by the method of cross-multiplication :—

21. $x+y+z=0$,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0,$$

$$ax+by+cz+(a-b)(b-c)(c-a)=0.$$

22. $ax+by+cz=0,\ bcx+cay+abz=0,$

$$\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}=ab+bc+ca.$$

23. $x+y+z=0,\ (a-b)x+(b-c)y+(c-a)z=0,$

$$\frac{b+c-2a}{y}+\frac{a+c-2b}{z}-\frac{a+b-2c}{x}=1.$$

24. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0,\ \frac{2}{x}+\frac{8}{y}-\frac{6}{z}=0,\ x+y+z=6.$

25. $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0,\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0,$

$$(b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z=2a+2b+2c.$$

26. $\frac{3x-2y}{y-z}=\frac{y+2z}{z-3x}=\frac{z+x}{3x-y+1}=3.$

27. $\frac{x-2y+z}{2x+y}=\frac{3x+y-z}{y-2z}=\frac{2x-3y+5z}{z-3x+3}=2\frac{1}{2}.$

---

11. Solution of equations in three unknowns of the from

$$a_1x+b_1y+c_1z=d_1 \ldots\ldots(1)$$
$$a_2x+b_2y+c_2z=d_2 \ldots\ldots(2)$$
$$a_3x+b_3y+c_3z=d_3 \ldots\ldots(3)$$

First method :—

(1) Eliminate as in art. 7 one of the unknown quantities from any two pairs of the given equations and get two equations in the other two unknowns.

(2) Solve the two equations in the two unknowns thus obtained by any method, preferably by the method of cross-multiplication.

(3) Then substituting the values of these two unknowns in one of the three original equations, we get the value of the remaining unknown.

Thus, multiplying (1) by a_2 and (2) by a_1 we get

$$a_1a_2x+b_1a_2y+c_1a_2z=d_1a_2,\ a_1a_2x+a_1b_2y+a_1c_2z=a_1d_2.$$

Subtracting, $(b_1a_2-a_1b_2)y+(c_1a_2-a_1c_2)z=d_1a_2-a_1d_2 \ldots\ldots\ldots(4)$

Similarly multiplying (1) by a_3 and (3) by a_1, and then subtracting we get,

$$(b_1a_3 - a_1b_3)y + (c_1a_3 - a_1c_3)z = d_1a_3 - a_1d_3 \ldots(5)$$

From (4) and (5) by the method of elimination or cross-multiplication we get y and z. Substituting these values of y and z in (1), (2) or (3) we get the value of x.

Second method :—

(1) Eliminate as in art. 7 the constants from any two pairs of the given equations and get two equations with zero constants (*i.e.* of the form ax+by+cz=0) and then by the method of cross-mnltiplication find the values of the unknowns from these two and any one of the original equtions.

Thus multiplying (1) by d_2 and (2) by d_1 and subtracting we get $(a_1d_2 - d_1a_2)x + (b_1d_2 - b_2d_1)y + (c_1d_2 - c_2d_1)z = 0 \ldots(A)$

Similarly multiplying the equation (1) by d_3 and (3) by d_1 and subtracting we get

$$(a_1d_3 - a_3d_1)x + (b_1d_3 - b_3d_1)y + (c_1d_3 - c_3d_1)z = 0 \ldots(B).$$

The equations (A) and (B) with one of the given equations may now be solved by the method of cross-multiplication as in art. 16.

Ex. 1. Solve $5x + y + z = 17$ ....(1)

$$2x + 3y + 4z = 29 \qquad \ldots\ldots\ldots(2)$$

$$3x - 4y + 2z = 2 \qquad \ldots\ldots\ldots(3)$$

Multiplying (1) by 2, and (2) by 5, we have

$$10x + 2y + 2z = 34,$$

and $10x + 15y + 20z = 145$.

Hence, by subtraction, $13y + 18z = 111$ ...(4)

Again, multiplying (2) by 3, and (3) by 2, we have

$$6x + 9y + 12z = 87,$$

and $6x - 8y + 4z = 4$.

Hence, by subtraction, $17y + 8z = 83$ ......(5)

Now from (4) and (5), we have

$$13y + 18z - 111 = 0$$

and $17y + 8z - 83 = 0$.

Hence $$\frac{y}{18(-83) - 8(-111)} = \frac{z}{17(-111) - 13(-83)} = \frac{1}{13.8 - 17.18}$$

or $$\frac{y}{-1494 + 888} = \frac{z}{-1887 + 1079} = \frac{1}{104 - 306}$$

or $$\frac{y}{-606} = \frac{z}{-808} = \frac{1}{-202}$$

Therefore $y=\dfrac{-606}{-202}=3$, and $z=\dfrac{-808}{-202}=4$.

Substituting the values of y and z in (1), we have

$$5x+3+4=17,$$

whence $5x=10$ and $\therefore x=2$.

Thus we have $x=2, y=3, z=4$.

Ex. 2. Solve $2x-3y+4z=6$......(1)

$$4x-2y+z=5......(2)$$

$$6x-4y+2z=7......(3)$$

Multiplying (1) by 5 and (2) by 6, we have

$$10x-15y+20z=30,$$

and $24x-12y+6z=30$.

Hence, by subtraction, $14x+3y-14z=0$...(4)

Again multiplying (1) by 7 and (3) by 6, we have

$$14x-21y+28z=42,$$

and $36x-24y+12z=42$.

Hence, by subtraction, $22x-3y-16z=0$......(5)

Therefore from (4) and (5) by cross-multiplication,

$$\frac{x}{-48-42}=\frac{y}{-308+224}=\frac{z}{-42-66},$$

or, $\dfrac{x}{-90}=\dfrac{y}{-84}=\dfrac{z}{-108}$, or, $\dfrac{x}{15}=\dfrac{y}{14}=\dfrac{z}{18}$.

Supposing each of these fractions $=k$, we have

$$x=15k,\ y=14k,\ z=18k.$$

Hence from (1), $\quad k(30-42+72)=6$,

or, $60k=6$, $\therefore k=\frac{1}{10}$.

$\therefore x=\frac{3}{2}, y=\frac{7}{5}, z=\frac{9}{5}$.

Ex. 3. Solve $yz+2zx+5xy=20xyz$,

$$7yz+3zx+4xy=25xyz,$$

$$3yz+zx+6xy=23xyz.$$

Dividing the given equations by xyz,

$$\frac{1}{x}+\frac{2}{y}+\frac{5}{z}=20,$$

$$\frac{7}{x}+\frac{3}{y}+\frac{4}{z}=25,$$

$$\frac{3}{x}+\frac{1}{y}+\frac{6}{z}=23.$$

Solving these in $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, as in ex. 1 or 2,

we have $\frac{1}{x}=1$, $\frac{1}{y}=2$, $\frac{1}{z}=3$.

$\therefore\ x=1,\ y=\frac{1}{2},\ z=\frac{1}{3}$.

12. Simultaneous equations : more than three unknowns.

Methods similar to that applied in case of equations with three unknowns may be applied to equations with more than three unknown quantities. Thus if there be four unknown quantities say x, y, z, u, from each of any three pairs of the four equations necessary to solve for them, we eliminate one unknown quantity, say, x and thus get three equations in three unknowns y, z, u which we can solve as shown above.

EXERCISE LXXXIX.

Solve

1. $2x-5y+6z=13,\ 6y+2z-6x=-4,\ 3y+4z-x=24.$
2. $x+y-z=2,\ 2x+3y-3z=3,\ 4x+3y-5z=9.$
3. $2x-8y+5z=5,\ 3x+y+z=14,\ 2x+5y-3z=4.$
4. $14z-7y-5x=25,\ 2x+7y-6z=1,\ 5x+6y-2z=20.$
5. $x+12y-9z=7,\ 4x-y+4z=50,\ 9x+8y-3z=97.$
6. $8x+4y+5z=69,\ 5x+3y+z=34,\ 5x+2y+4z=46.$
7. $\frac{7}{12}x-\frac{1}{2}y-\frac{1}{4}z=\frac{1}{12},\ \frac{1}{2}x+y-\frac{7}{8}z=3\frac{1}{4},\ \frac{1}{6}x+\frac{1}{3}y+\frac{1}{2}z=5\frac{5}{6}.$
8. $\frac{1}{2}(x+y)+\frac{1}{3}(y+z)-\frac{1}{5}(z+x)=6,\ 2x+3y-3z=0.$

   $2x+3y=2z+6.$
9. $\frac{2y-3}{7}=\frac{3z}{4}-x,\ \frac{4x}{5}+\frac{4z}{8}=\frac{y}{2}+\frac{11}{10},\ x+y+z=11.$
10. $\frac{5+2y-2\frac{1}{3}z}{5}=\frac{4x+2y-z}{6},\ 11x+5y-6z=5x+7y-4z-8,$

    $\frac{10x+4y-5z}{10}+\frac{8x-2y+z}{3}=\frac{x+y+z}{4}.$
11. $\frac{3x+y-6z}{5}+\frac{4x+y-7z}{7}=\frac{5x-6y+z}{2},$

    $\frac{4x-3y+2z}{4}+\frac{2x+y-2z}{6}=3,\ \frac{x+5y-10z}{3}=\frac{3x-2y+3z}{8}.$
12. $\frac{2x-y+3z-1}{4}=\frac{x+2y-5z-1}{5}=\frac{3x+y-7z}{6}=\frac{x+y+z}{8}.$
13. $x+y+z=a+b+c,\ bx+cy+az=cx+ay+bz$

    $=bc+ca+ab.$

Solve

14. $\frac{4}{x}-\frac{5}{y}+\frac{6}{z}=3,\ \frac{3}{x}-\frac{2}{y}+\frac{5}{z}=2,\ \frac{7}{x}-\frac{6}{y}+\frac{4}{z}=4.$

15. $\frac{1}{x}-\frac{4}{15y}+\frac{1}{3z}=\frac{38}{15},\ \frac{1}{6x}+\frac{1}{4y}+\frac{1}{z}=\frac{61}{12},\ \frac{1}{5x}-\frac{1}{8y}+\frac{1}{z}=\frac{161}{40}.$

16. $\frac{2}{x}+\frac{3}{y}=5,\ \frac{5}{y}+\frac{3}{z}=6,\ \frac{4}{z}+\frac{1}{x}=2.$

17. $2x-3z+2u=3,$
$3y-2z+u=8,$
$5y-3u=2,$
$4x+2z=22.$

18. $2u-3x=3,$
$5y-6x=2,$
$4u-3z=9,$
$4z-3u=2.$

19. $2x+3z-2v=1,$
$4y-5u+3v=3,$
$5u-6y-8x=0,$
$x+y+z=6,$
$3y-4z+x=-5.$

20. $2x+3y+4z+5u-6v=13,$
$4x-8z+12v-6y=16,$
$12y-10u+6v=13,$
$9y-10x=6,$
$y-x=\frac{5}{6}.$

21. $\frac{1}{x}+\frac{1}{y}=5,\ \frac{2}{y}+\frac{3}{z}=18,\ \frac{5}{u}-\frac{4}{z}=4,$
$\frac{6}{u}-\frac{7}{v}=-11,\ \frac{8}{v}-\frac{10}{x}=20.$

22. $x+3y=5xy,\ 4y+5z=6yz,\ 2x+3z=2zx.$

23. $3xyz=2yz+3zx+4xy,\ 5yz+3zx-8xy=\frac{3}{2}xyz,$
$7xy-6xz+3yz=\frac{5}{4}xyz.$

24. $xyz=a(yz-zx-xy)=b(zx-xy-yz)=c(xy-yz-zx).$

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13. Miscellaneous Examples.

Ex. 1. Solve $y+z=15$......(1)
$z+x=14$......(2)
$x+y=11$......(3)

Equations of this class may be solved thus:

Adding up, $2x+2y+2z=40$ or $x+y+z=20$......(4),

Subtracting (1) from (4), $x=5$

Subtracting (2) from (4), $y=6$

Subtracting (3) from (4), $z=9$

Thus $x=5, y=6, z=9.$

Ex. 2. Solve

$$\frac{1}{x}+\frac{1}{y}=5\ \text{(i)},\ \frac{1}{y}+\frac{1}{z}=7\ \text{(ii)},\ \frac{1}{z}+\frac{1}{x}=6\ \text{(iii)}.$$

These are equations in $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ of the type of ex. 1.

Adding the three equations we get

$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=18, \quad \therefore \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=9 \ldots(1)$$

Subtracting (ii) from (I) we get $\frac{1}{x}=2$, $\therefore x=\frac{1}{2}$

Subtracting (iii) from (I) we get $\frac{1}{y}=3$, $\therefore y=\frac{1}{3}$

Subtracting (i) from (1) we get $\frac{1}{z}=4$, $\therefore z=\frac{1}{4}$.

Ex. 3. Solve

$$\frac{xy}{x+y}=c \text{ (i)}, \quad \frac{yz}{y+z}=a \text{ (ii)}, \quad \frac{zx}{z+x}=b \text{ (iii)}.$$

From (i) we have $\frac{x+y}{xy}=\frac{1}{c}$, or, $\frac{1}{x}+\frac{1}{y}=\frac{1}{c}$ ......(1)

" (ii) " " $\frac{y+z}{yz}=\frac{1}{a}$, or, $\frac{1}{y}+\frac{1}{z}=\frac{1}{a}$......(2)

" (iii) " " $\frac{z+x}{zx}=\frac{1}{b}$, or, $\frac{1}{x}+\frac{1}{z}=\frac{1}{b}$......(3)

Equations (1). (2), (3) are of the type of ex. 1, the unknown quantities being $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$.

Adding (1), (2) and (3) we have,

$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

$$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ldots\ldots\ldots(4)$$

Subtracting (2) from (4),

$$\frac{1}{x}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{1}{a}$$

$$=\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)=\frac{ac+ab-bc}{2abc}, \quad \therefore x=\frac{2abc}{ac+ab-bc}$$

Subtracting (3) from (4),

$$\frac{1}{y}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)-\frac{1}{b}$$

$$=\frac{1}{2}\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)=\frac{ab+bc-ca}{2abc}, \quad \therefore y=\frac{2abc}{ab+bc-ca}$$

Subtracting (1) from (4),

$$\frac{1}{z}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{1}{c}$$

$$=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)=\frac{bc+ac-ab}{2abc}, \therefore z=\frac{2abc}{bc+ac-ab}.$$

Ex. 4. Solve.

$$2x-3y+2z=12\ldots\ldots(1)$$
$$2y-3z+2x=2\ldots\ldots(2)$$
$$2z-3x+2y=-3\ldots..(3)$$

Adding (1), (2) and (3) we get

$$x+y+z=11, \therefore 2x+2y+2z=22\ldots\ldots(4)$$

Subtracting (3) from (4) we get $5x=25$, $\therefore x=5$.
Subtracting (1) from (4) we get $5y=10$, $\therefore y=2$.
Subtracting (2) from (4) we get $5z=20$, $\therefore z=4$.

Ex. 5. Solve

$$x-ay+a^2z=a^3\cdots(1)$$
$$x-by+b^2z=b^3\cdots(2)$$
$$x-cy+c^2z=c^3\ldots(3)$$

Subtracitng (2) from (1)

$$-(a-b)y+(a^2-b^2)z=a^3-b^3,$$

or $-y+(a+b)z=a^2+ab+b^2\ldots\ldots(3)$

Subtracting (3) from (2)

$$-(b-c)y+(b^2-c^2)z=b^3-c^3,$$

or $-y+(b+c)z=b^2+bc+c^2\ldots\ldots(4)$

Subtracting (4) from (3),

$$(a-c)z=a^2-c^2+ab-bc$$
$$=(a-c)(a+c)+b(a-c)$$
$$=(a-c)(a+b+c).$$
$$\therefore z=a+b+c.$$

Substituting the value of z in (3)

$$-y+(a+b)(a+b+c)=a^2+ab+b^2,$$

whence on simplifying $y=bc+ca+ab$.

Substituting the value of z and y in (1)

$$x-a(bc+ca+ab)+a^2(a+b+c)=a^3,$$

whence on simplifying $x=abc$.

Thus $x=abc$, $y=bc+ca+ab$, $z=a+b+c$.

Ex. 6. Solve $x+y+z=a+b+c$ ......(1)

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3 \quad \text{......(2)}$$

$$ax+by+cz=a^2+b^2+c^2 \quad \text{......(3)}$$ C. E. 1907

From (1) $\quad (x-a)+(y-b)+(z-c)=0$ ......(4)

From (2) $\left(\frac{x}{a}-1\right)+\left(\frac{y}{b}-1\right)+\left(\frac{z}{c}-1\right)=0$

$$\text{or } \frac{x-a}{a}+\frac{y-b}{b}+\frac{z-c}{c}=0 \quad \text{...(5)}$$

From (4) and (5) by cross-multiplication,

$$\frac{x-a}{\frac{1}{c}-\frac{1}{b}}=\frac{y-b}{\frac{1}{a}-\frac{1}{c}}=\frac{z-c}{\frac{1}{b}-\frac{1}{a}},$$

$$\text{or } \frac{x-a}{a(b-c)}=\frac{y-b}{b(c-a)}=\frac{z-c}{c(a-b)}=k \text{ (suppose)}$$

$\therefore\ x-a=ka(b-c),\ y-b=kb(c-a),\ z-c=kc(a-b)$ ......(6)

Now equation (3) may be written

$$a(x-a)+b(x-b)+c(z-c)=0.$$

Hence substituting from (6)

$$k\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}=0, \text{ whence } k=0$$

$\therefore$ from (6) $x-a=0$ or $x=a$,

$y-b=0$ or $y=b$,

$z-c=0$ or $z=c$.

Ex. 7. Solve

$$x+y+z=a+2b+3c \quad \text{.........(1)}$$

$$bx+cy+az=ab+2bc+3ac \quad \text{...(2)}$$

$$6ax(b-c)+3by(c-a)+2cz(a-b)=6(a-b)(a-c)(b-c) \text{...(3)}$$

From (1), $(x-a)+(y-2b)+(z-3c)=0$ ....(4)

From (2), $b(x-a)+c(y-2b)+a(z-3c)=0$ ......(5)

By cross-multiplication $\dfrac{x-a}{a-c}=\dfrac{y-2b}{b-a}=\dfrac{z-3c}{c-b}$,

Suppose each fraction $=k$, then

$$(x-a)=(a-c)k,\ y-2b=(b-a)k,\ (z-3c)=(c-b)k \text{......}(A)$$

Again, equation (3) may be written as

$$6ax(b-c)+3by(c-a)+2cz(a-b)=$$
$$6a^2(b-c)+6b^2(c-a)+6c^2(a-b).$$

Transposing and re-arranging,

$$6a(b-c)(x-a)+3b(c-a)(y-2b)+2c(a-b)(z-3c)=0\ldots(6)$$

Substituting the values of $(x-a)$, $(y-2b)$ and $(z-3c)$ obtained above (A), in (6) we get

$$6a(b-c)(a-c)k+3b(c-a)(b-a)k+2c(a-b)(c-b)k=0.$$

$\therefore\ k=0$, being common to each term on the left.

Putting $k=0$ in (A) we get

$x-a=0$, $\quad\therefore\ x=a$.
$y-2b=0$, $\quad\therefore\ y=2b$.
$z-3c=0$, $\quad\therefore\ z=3c$.

Ex. 8. Solve $yz=20\ldots(1)$
$zx=10\cdots(2)$
$xy=\ 8\ldots(3)$

Multiplying (1), (2), (3) we have

$$x^2y^2z^2=1600 \text{ or } xyz=40\ldots(4)$$

Dividing (4) in turn by (1), (2), (3).

$$x=2,\ y=4,\ z=5$$

Ex. 9. Solve

$$xy+10(x+y)=56\ldots\ldots(1)$$
$$yz+10(y+z)=32\ldots\ldots(2)$$
$$zx+10(z+x)=43\ldots\ldots(3)$$

Adding 100 to both sides of (1), (2), and (3) we get

$$(x+10)(y+10)=156\ldots\ldots(4)$$
$$(y+10)(z+10)=132\ldots\ldots(5)$$
$$(z+10)(x+10)=143\ldots\ldots(6)$$

These are equations of the type of ex. 8.

Multiplying (4), (5), and (6)

$$(x+10)^2(y+10)^2(z+10)^2=156\times132\times143$$
$$=13\times12\times11\times12\times11\times13=11^2.\ 12^2.\ 13^2.$$

$$\therefore\ (x+10)(y+10)(z+10)=11\times12\times13\ldots\ldots(7)$$

Dividing (7) by (4) we get $z+10=11$, $\therefore\ z=1$

" " " (5) " " $x+10=13$, $\therefore\ x=3$

" " " (6) " " $y+10=12$, $\therefore\ y=2$.

Ex. 10. Are the following equations consistent?

$$4x-5y+6z=3\ldots\ldots(1)$$
$$8x+7y-3z=2\ldots\ldots(2)$$
$$7x+8y+9z=1\ldots\ldots(3)$$
$$5x+2y-8z=2\ldots\ldots(4)$$

Here there are 4 equations connecting 3 unknown quantities. Three of the eqations are sufficient to determine the unknown

quantities, and if the roots so obtained satisfy the fourth equation, then the given equations are consistent.

Now from (1), (2) and (3) we get as in art. 11, $x=\frac{20}{43}$, $y=-\frac{11}{43}$, $z=-\frac{1}{43}$.

Hence for these values, $5x+2y-8z=\frac{100}{43}-\frac{22}{43}+\frac{8}{43}=\frac{86}{43}=2$.

Thus (4) is satisfied or the given equations form a consistent system.

14. A simple equation or an equation of the first degree in one unknown cannot have more than one root.

Every equation of the first degree in x can be reduced to the form $ax=c$. If possible, let it have two roots α, β. Then we must have $a\alpha=c$, $a\beta=c$; whence by subtraction $a\alpha-a\beta=0$ or $a(\alpha-\beta)=0$. But since a in *not* zero, we have $\alpha-\beta=0$ or $\alpha=\beta$, *i.e.* the two roots are one and the same.

Again *simultaneous equations of the first degree in two or more unknowns cannot have more than one set of solutions.*

For, let there be two equations in x and y, *viz.* $a_1x+b_1y=c_1$ and $a_2x+b_2y=c_2$. Then eliminating x we get (See ex. 2, art. 7) a *simple equation* (A) to determine y and hence y has *one* value. Similarly x must have *one* value.

EXERCISE XC.

Solve

1. $x+y=12$, $y+z=16$, $z+x=6$.
2. $3x+5y=8$, $5y+7z=10$, $7z+3x=4$.
3. $ax+by=2c$, $by+cz=2a$, $cz+ax=2b$.
4. $\frac{3}{x}+\frac{5}{y}=3$, $\frac{5}{y}+\frac{2}{z}=7$, $\frac{2}{z}+\frac{3}{x}=6$.
5. $\frac{a}{x}+\frac{b}{y}=2$, $\frac{b}{y}+\frac{c}{z}=2$, $\frac{c}{z}+\frac{a}{x}=2$.
6. $xy=4(x+y)$, $xz=6(x+z)$, $yz=8(y+z)$.
7. $x+y=5xy$, $y+z=3yz$, $z+x=4zx$.
8. $\frac{x+y}{xy}=\frac{y+z}{yz}=\frac{z+x}{xz}=\frac{2}{3}$.
9. $y+z-3x+8=0$, $z+x-3y+4=0$, $x+y-3z+6=0$.
10. $3x-4y+3z=12$, $3y-4z+3x=10$, $3z-4x+3y=8$.
11. $x+ay+bcz=a^2$, $x+by+caz=b^2$, $x+cy+abz=c^2$
12. $x+y+z=a+b+c$, $ax+by+cz=ab+bc+ca$,
 $bx+cy+az=a^2+b^2+c^2$.

Solve

13. $x+y+z=a+b+c,$
$bx+cy+az=bc+ca+ab,$
$(b-c)x+(c-a)y+(a-b)z=0.$

14. $ax+by+cz=a+b,\ bx+cy+az=b+c,\ cx+ay+bz=c+a.$ (M. M. 1879).

15. $x+y+z=a+b+c,$
$ax+by+cz=a^2+b^2+c^2,$
$a^2x+b^2y+c^2z=a^3+b^3+c^3.$

16 $x+y+z=a^2+b^2+c^2,$
$ax+by+cz=a^3+b^3+c^3,$
$ax(b^2-c^2)+by(c^2-a^2)+cz(a^2-b^2)$
$+(a-b)(b-c)(c-a)(ab+bc+ca)=0.$

17. $x+y+z=a+b+c,$
$bcx+cay+abz=3abc,$
$(a^2-bc)x+(b^2-ca)y+(c^2-ab)z=a^3+b^3+c^3-3abc.$

18. $ax+cy+bz=cx+by+az=bx+ay+cz=a^3+b^3+c^3-3abc.$

19 $c^2x+b^2y+a^2z=3,\ c^3x+b^3y+a^3z=a+b+c,$
$\frac{x}{c}+\frac{y}{b}+\frac{z}{a}=\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}.$

20. $\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=3,\ \frac{1}{a^2x}+\frac{1}{b^2y}+\frac{1}{c^2z}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$
$\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}=\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}.$

21. $xy=12,\ yz=20,\ zx=15.$

22. $xy+x+y=29,$
$yz+y+z=41,$
$zx+z+x=34.$

Are the following equations consistent ?

23. $2x+3y=8,\ 3x-2y+1=0$ and $6x+5y=16.$

24. $x+2y+3z=14,\ 3x+4y-2z=5,$
$\frac{17x+3}{10}+\frac{y}{5}-\frac{z}{3}=\frac{2}{5},\ 4x-y+2z=8.$

25 $2x+3y=5,\ 3y+4z=7,\ 2z+x=3,\ 2x+3y+5z=11.$

26. For what value of k are the following equations consistent ?
$ax+by=a+b,\ by+cz=b+c,\ cz+ax=c+a,\ kx+by+cz=a+b+c.$

15. Method of Indeterminate Multipliers.

Let us consider the equations

$$a_1x+b_1y+c_1=0\ldots(1),\quad a_2x+b_2y+c_2=0\ldots\ldots(2)$$

Multiply (2) by l; then $a_2lx+b_2ly+c_2l=0\ldots\ldots(3)$

Adding (1) and (3) we get,

$$(a_1+a_2l)x+(b_1+b_2l)y+(c_1+c_2l)=0\ldots\ldots(4)$$

Let us take l so that $b_1+b_2l=0$,

or $l=-\dfrac{b_1}{b_2}$. Then from (4) we get

$$(a_1+a_2l)x+(c_1+c_2l)=0,$$

$$\text{or,}\quad x=-\frac{(c_1+c_2l)}{a_1+a_2l}$$

$$=-\frac{c_1-\dfrac{c_2b_1}{b_2}}{a_1-\dfrac{a_2b_1}{b_2}},\ \text{substituting for } l.$$

$$=-\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1}=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}.$$

Similarly taking l such that $a_1+a_2l=0$, we get $y=\dfrac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$.

The method may be extended to more than two variables. Consider the equations :

$$a_1x+b_1y+c_1z=d_1\ldots\ldots\ldots(1)$$
$$a_2x+b_2y+c_2z=d_2\ldots\ldots\ldots(2)$$
$$a_3x+b_3y+c_3z=d_3\ldots\ldots\ldots(3)$$

Multiply (1) by l, then, $a_1lx+b_1ly+c_1lz=d_1l\ldots\ldots\ldots(4)$

Multiply (2) by m, then, $a_2mx+b_2my+c_2mz=d_2m\ldots(5)$

Adding (4), (5) and (3) we get

$$(a_1l+a_2m+a_3)x+(b_1l+b_2m+b_3)y+(c_1l+c_2m+c_3)z$$
$$=d_1l+d_2m+d_3\ldots\ldots\ldots(6)$$

Choose l and m so as to satisfy the equations

$$b_1l+b_2m+b_3=0\ldots\ldots(7)$$
$$c_1l+c_2m+c_3=0\ldots\ldots(8),$$

so that equation (6) reduces to $(a_1l+a_2m+a_3)x=d_1l+d_2m+d_3$

$$\text{or } x=\frac{d_1l+d_2m+d_3}{a_1l+a_2m+a_3}\ldots\ldots\ldots\ldots(9)$$

Now from (7) and (8) we get by cross-muitiplication

$$\frac{l}{b_2c_3-b_3c_2}=\frac{m}{b_3c_1-b_1c_3}=\frac{1}{b_1c_2-b_2c_1},$$

$$\text{or } l=\frac{b_2c_3-b_3c_2}{b_1c_2-b_2c_1},\; m=\frac{b_3c_1-b_1c_3}{b_1c_2-b_2c_1}.$$

Hence from (9)

$$x=\frac{d_1\dfrac{b_2c_3-b_3c_2}{b_1c_2-b_2c_1}+d_2\dfrac{b_3c_1-b_1c_3}{b_1c_2-b_2c_1}+d_3}{a_1\dfrac{b_2c_3-b_3c_2}{b_1c_2-b_2c_1}+a_2\dfrac{b_3c_1-b_1c_3}{b_1c_2-b_2c_1}+a_3}$$

$$=\frac{d_1(b_2c_3-b_3c_2)+d_2(b_3c_1-b_1c_3)+d_3(b_1c_2-b_2c_1)}{a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)}.$$

The values of y and z can be similarly found.

Note. It appears from the above that we can *always* multiply equations (1), (2), (3) by suitable constants so that the products being added together the resulting equation may have the co-efficients of any two of the unknown qnantities zero, and hence this resulting equation immediately gives the value of the third unknown. In general the proper constants are not evident on inspection; hence the above process is necessary.

Ex. 1. Determine x only from tho equations

$$\left.\begin{array}{l}2x+3y+4z=38\ldots(1)\\ 3x-2y+5z=26\ldots(2)\\ 4x+6y-3z=21\ldots(3)\end{array}\right\}\quad \text{(C. E. 1901.)}$$

Multiply (1) by l, (2) by m and add the results to (3), then $x(2l+3m+4)+y(3l-2m+6)+z(4l+5m-3)$

$$=38l+26m+21\ldots(4)$$

Choose l, m so as to satisfy the equations

$$3l-2m+6=0\ldots\ldots(5)$$

$$4l+5m-3=0\ldots\ldots(6)$$

Then (4) reduces to $x\ (2l+3m+4)=38l+26m+21$

$$\text{or } x=\frac{38l+26m+21}{2l+3m+4}\ldots\ldots(7)$$

From (5) and (6) by cross-multiplication,

$$l=-\tfrac{24}{23},\; m=\tfrac{33}{23}.$$

Substituting these values of l, m in (7)

$$x=\frac{-38\times\frac{24}{23}+26\times\frac{33}{23}+21}{-2\times\dfrac{24}{23}+\dfrac{3\times33}{23}+4}$$

$$=\tfrac{429}{143}=3.$$

Ex. 2. Are the following equations independent ?

$$3x-2y+4z=5 \ldots\ldots(1)$$
$$2x+y-5z=9 \ldots\ldots(2)$$
$$12x-y-7z=37 \ldots\ldots(3)$$

Here we are to see whether any one of the equations is deducible from the other two.

Multiplying (1) by l and (2) by m and adding,

$$x(3l+2m)+y(-2l+m)+z(4l-5m)=5l+9m \ldots\ldots(4)$$

Now choose l, m so that coefficients of x, y in (4) may be the same as those in (3), that is,

$$\text{let } \left.\begin{array}{r} 3l+2m=12 \\ -2l+m=-1 \end{array}\right\} \text{ whence } l=2,\ m=3.$$

With these values of l, m, (4) becomes

$$12x-y-7z=37, \text{ the same as (3).}$$

Hence equation (3) can be obtained by multiplying (1) by 2 and (2) by 3, and adding the products.

Thus the equations are not independent.

N.B. The student is advised to apply the method of undetermined multipliers in solving the examples 1 to 6, Exercise LXXXVI and the examples 1 to 6, Exercise LXXXIX.

16. Problems Leading to Simultaneous Equations.

The method of procedure in case of problems leading to simultaneous equations is the same as in the case of problems leading to simple equations and is illustrated below.

Ex. 1. Find two numbers such that, three times the first added to four times the second gives 180 ; and twice the second subtracted from 5 times the first gives 40.

Let x and y be the numbers.

Then by the question we have the equations

$$3x+4y=180 \ldots\ldots\ldots(1), \qquad 5x-2y=40 \ldots\ldots\ldots(2)$$

To solve the equations, multiply (2) 2,

$$\text{then } 10x-4y=80 \ldots\ldots\ldots(3)$$

Adding (1) and 3, $13x=260$, $\therefore\ x=20$.

Hence from (1), by substitution, $3\times 20+4y=180$,

$$\text{or } 4y=120, \quad \therefore\ y=30$$

Therefore the numbers are 20 and 30.

Ex. 2. A nnmber consits of 3 digits whose sum is 14. The middle digit is equal to the sum of the other two ; and the number will be increased by 99 if the digits be reversed. Find the number.

Let x, y, z be the digits in the hundreds', tens' and unit's place respectively.

Then by the first two conditions of the problem,

$x+y+z=14$.........(1) $\quad x+z=y$.........(2)

Now the given number $=100x+10y+z$, and the number obtained by reversing the digits $=100z+10y+x$.

Hence by the third condition of the problem

$$(100z+10y+x)-(100x+10y+z)=99,$$
$$\text{or } 99z-99x=99,$$
$$\text{or } z-x=1 \text{.........(3)}$$

From (2) we get, $x+z-y=0$ .....(4)

Snbtracting (4) from (1), we have

$$2y=14, \qquad \therefore y=7.$$

Substituting this value of y in (2), we have

$$x+z=7 \text{..... ...(5)}$$

From (3) and (5) $z=4$, $\quad x=3$.

Hence the number is 374.

Ex. 3. A sum of money is divided equally among a certain number of persons. If there had been 8 more each would have received *Re.* 1. 5*as.* 4*p.* less than he did; if there had been 10 fewer each would have received *Rs.* 2. 10*as.* 8*p.* more than he did. Find the number of persons and the sum of money divided.

Let x be the number of persons, and y the number of rupees received by each. Then the sum divided $=Rs.\ xy$.

Since *Re.* 1. 5*as.* 4*p.* $=Rs.\ 1\frac{1}{3}=Rs.\ \frac{4}{3}$ and *Rs.* 2. 10*as.* 8*p.* $=Rs.\ 2\frac{2}{3}=Rs.\ \frac{8}{3}$, we have from the conditions

$$xy=(x+8)(y-\tfrac{4}{3}) \quad \text{...............(1)}$$
$$xy=(x-10)(y+\tfrac{8}{3}) \quad \text{...............(2)}$$

From (1) $\quad xy=xy+8y-\frac{4}{3}x-\frac{32}{3}$

$$\text{or } \tfrac{4}{3}x-8y=-\tfrac{32}{3} \quad \text{...............(3)}$$

From (2) $\quad xy=xy-10y+\frac{8}{3}x-\frac{80}{3}$,

$$\text{or } \tfrac{8}{3}x-10y=\tfrac{80}{3} \text{...............(4)}$$

Solving (3) and (4) we get $x=40$, $y=8$.

Thus the number of persons $=40$, each receiving *Rs.* 8, and the sum of money $=Rs.\ (40\times 8)=Rs.\ 320$.

Ex. 4. In a quarter of a mile race, A gives B a start of 22 yards and beats him by 2 seconds, and in a 300 yard-race he gives him a start of 2 seconds and beats him by $10\frac{1}{2}$ yards. Find the rates of each. (M. M. 1888).

Let x *yds.* per second = rate of A

y " " " = " " B

In the first race the time taken by A to run 440 *yds.* is 2 seconds less than the time taken by B to run $440-22$ or 418 *yds.*

Hence by the question $\frac{418}{y}-\frac{440}{x}=2$;

$$\text{or } \frac{209}{y}-\frac{220}{x}=1\ldots\ldots(1)$$

In the second race the time taken by A to run 300 *yds.* is less by 2 seconds, than the time taken by B to run $(300-10\frac{1}{3})$ or $\frac{869}{3}$ *yds.*

$$\text{Hence } \frac{869}{3y}-\frac{300}{x}=2\ldots\ldots(2)$$

Multiplying (1) by 15 and (2) by 11, we get,

$$\frac{3135}{y}-\frac{3300}{x}=15\ldots\ldots\ldots(3)$$

$$\frac{9559}{3y}-\frac{3300}{x}=22\ldots\ldots\ldots(4)$$

Subtracting (3) from (4), we have,

$$\frac{154}{3y}=7, \text{ or, } y=\frac{154}{3\times7}=\frac{22}{3}=7\tfrac{1}{3}.$$

Substituting in (1) this value of y,

$$\frac{209\times3}{22}-\frac{220}{x}=1.$$

$$\text{or, } \frac{57}{2}-\frac{220}{x}=1, \text{ or } \frac{220}{x}=\frac{55}{2} \quad \therefore\ x=\frac{220\times2}{55}=8.$$

Thus A's rate$=8$ yds., and B's rate$=7\frac{1}{3}$ yds. per second

Ex. 5. A boat goes up-stream 30 miles and down-stream 44 miles in 10 hours; it also goes up-stream 40 miles and down-stream 55 miles in 13 hours: find the rate of the stream and of the boat. (C. E. 1880).

Let the rate of the boat in still water$=x$ miles per hour, and the rate of the stream$=y$ miles per hour.

Therefore the boat goes $x+y$ miles per hour down the stream and $x-y$ miles per hour up the stream.

Hence by the question,

$$\frac{30}{x-y}+\frac{44}{x+y}=10\ldots\ldots\ldots(1)$$

$$\frac{40}{x-y}+\frac{55}{x+y}=13\ldots\ldots\ldots(2)$$

Multiplying (1) by 4 we have $\frac{120}{x-y}+\frac{176}{x+6}=40\ldots\ldots(3).$

Multiplying (2) by 3 we have $\frac{120}{x-y}+\frac{165}{x+y}=39$......(4)

Subtracting (4) from (3), $\frac{11}{x+y}=1$, or, $x+y=11$......(5)

Substituting this value of $x+y$ in (1), we have

$$\frac{30}{x-y}=10-4=6, \qquad \therefore\ x-y=5\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

From (5) and (6) $x=8, y=3$.

Thus the rates of the boat and of the stream are respectively 8 miles and 3 miles per hour.

Ex. 6. *A*, *B* and *C* sit down to play every one with a certain number of shillings. *A* loses to *B* and to *C* as many shillings as each of them has. Next *B* loses to *A* and to *C* as many as each of them now has. Lastly *C* loses to *A* and to *B* as many as each of them now has. After all every one of them has 16 shillings. How much had each originally?

Let x, y, z respectively be the number of shillings, *A*, *B*, *C* had originally.

After the first game *A* has $x-y-z$ shillings, *B* has $2y$ shillings and *C* has $2z$ shillings.

After the second game *A* has $2(x-y-z)$ shillings, *B* has $2y-(x-y-z)-2z$, or $-x+3y-z$ shillings, and *C* has $4z$ shillings.

After the last game *A* has $4(x-y-z)$ shillings, *B* has $2(-x+3y-z)$ shillings and *C* has $4z-2(x-y-z)-(-x+3y-z)$ or $(-x-y+7z)$ shillings.

Hence by the question

$$4(x-y-z)=16, \text{ or, } x-y-z=4\ldots\ldots\ldots(1)$$
$$2(-x+3y-z)=16, \text{ or, } -x+3y-z=8\ldots\ldots(2)$$
$$-x-y+7z=16\ldots\ldots\ldots(3)$$

Adding (1) and (2), we get, $2y-2z=12$ or, $y-z=6$...(4)

Adding (1) and (3), we get $-2y+6z=20$, or, $y-3z=-10$...(5)

By subtraction, from (4) and (5) $2z=16$, $\therefore z=8$.

Hence from (4) $y=14$.

Therefore from (1) $x=26$.

Hence *A*, *B*, *C* respectively had 26, 14, 8 shillings.

EXERCISE XCI.

1. The sum and difference of two numbers are respectively 110 and 10. Find the numbers.

2. Find two numbers such that twice the first added to 3 times the second gives 133 and 4 times the first diminished by 5 times the second gives 13.

3. Two persons possess Rs. 300, and 3 times the first person's share together with 4 times the second person's share amount to Rs. 1000. Find the money each has.

4. If 7 cows and 9 sheep are together worth Rs. 325, and 5 cows and 11 sheep are together worth Rs. 255, find the price of a sheep and a cow.

5. $\frac{2}{3}$ of a number added to $\frac{4}{7}$ of another number gives 32; and $\frac{5}{6}$ of the first exceeds $\frac{2}{5}$ of the second by 1. Find the numbers.

6. What fraction is that which becomes 1 when 2 is added to its numerator, and $\frac{1}{2}$ when 11 is added to its denominator?

7. What fraction is that which, if 5 be added to each of the numerator and denominator, equals $\frac{2}{3}$, and if 5 be subtracted from each of them equals $\frac{3}{7}$?

8. 3 lbs of tea and 4 lbs of sugar cost Rs. 2-12 as.; but if tea were to fall $16\frac{1}{4}$ p.c. and sugar rise by 50 p.c., they would cost Rs. 2-10 as. Find the price of tea and sugar per lb.

9. A number consists of 2 digits and is equal to $6\frac{1}{4}$ times the sum of its digits; if 18 be subtracted from the number the digits are reversed: find the number.

10. Reverse the digits of a number and it will become $\frac{4}{7}$ths of what it was before; also the difference between the two digits is 1; find the number. (C. E. 1883.)

11. Find the number of 3 digits which is the same when reversed, and the sum of whose digits is 16 and the difference 2. (C. E. 1883.)

12. A number consists of 3 digits of which the sum is 18, and the sum of the extreme digits is double the middle one. If 321 be subtracted from the number the difference is $\frac{2}{3}$ of the original number. Find the number.

13. A train travels a certain distance at a uniform rate; had the speed been 5 miles an hour more, the journey would have occupied 1 hour less, and had it been 9 miles per hour less, the journey would have occupied 3 hours more. Find the distance and the speed.

14. In a half-mile race A gives B, 22 yds. start and wins by 6 seconds. In a three-quarter mile race he gives him 20 seconds start but is beaten by 29 yds. 1 ft. In what time can each of them run a mile? (M. E. 1892.)

15. A challenged B to ride a bicycle race of 1040 yds, he first gave B a start of 120 yds., and lost by 5 seconds: he then gave B 5 seconds' start and won by 120 feet. How long does each take to ride the distance? (C. E. 1881).

16. A person walks from A to B a distance of $7\frac{1}{2}$ miles in 2 hours $17\frac{1}{2}$ minutes and returns in 2 hours 20 minutes, his rates of

walking up-hill, down-hill and on a level road being 3, $3\frac{1}{2}$ and $3\frac{1}{4}$ miles per hour respectively. Find the length of the level road between A and B. (B. E. 1884).

17. Find the sides of a rectangle whose area is unaltered, if its length be increased by 4 feet, while its breadth is diminished by 3 feet and which loses 80 square feet if its length be increased by 16 feet, while its breadth is diminished by 10 feet.

18. There are three numbers such that the product of the first and the second is $3\frac{1}{3}$ times their sum; the product of the second and the third is 6 times their sum and the product of the first and the third is $3\frac{3}{4}$ times their sum. Find the numbers.

19. A party of travellers coming to a hotel find that there are 6 too few bed rooms for each to have one. If they sleep two in a room, there are 9 empty rooms. How many rooms are left empty, if they sleep three in a room?

20. *A*, *B*, *C* are employed on a piece of work. After 3 days. *A* is discharged, one third of the work being done. After four days more *B* is discharged, another third of the work being done. *C* then finishes the work in 5 days. Find in how many days each could separately do the work.

21. A bag contains six-pences, shillings and half-crowns; the three sums of money expressed by the different coins are the same; if there are 102 coins in the bag, find the number of six-pences, shillings and half-crowns.

22. The sum of the ages of a man and his wife is 6 times the sum of the ages of their children. Two years ago the sum of their ages was 10 times the sum of the ages of the children and 6 years hence the sum of their ages will be three times the sum of the ages of their children. How many children have they? (B. E. 1881).

23. A tradesman sells two articles together for 46 rupees, making 10 P.C. profit on one and 20 P.C. on the other. If he had sold each article at 15 P.C. profit the result would have been the same. At what price does he sell each article?

24. Divide *Rs*. 46 among *A*, *B*, *C* such that $\frac{1}{5}$ of *A's* share with $\frac{1}{4}$ of *B's* share equals *Rs*. 6, and $\frac{1}{2}$ of *B*'s share with $\frac{1}{10}$ of *C*'s share equals *Rs*. 10.

25. In a mixed fraction the difference between the denominator and the numerator is $\frac{1}{2}$ the integral part; twice the integral part equals the sum of the numerator and the denominator increased by 6; and if 45 be added to the numerator and 1 be subtracted from the denominator, the fractional part becomes equal to the integral part. Find the fraction.

26. *A* and *B* can do a piece of work in 15 days, *B* and *C* in 20 days, *C* and *A* in 25 days. In what time can each do the work separately?

27. One pound of tea and three pounds of sugar cost 6 shillings, but if sugar were to rise 50 P.C. and tea 10 P.C. in price, they would cost 7 shillings. Find the price of tea and sugar (B E. 1866).

28. For 35 shillings I can buy 20 lbs of sugar, 3 lbs. of coffee and 6 lbs. of tea, or I can buy 15 lbs. of sugar, 4 lbs. of coffee and 8 lbs. of tea, while if I had 16 shillings more, I could buy 10 lbs. of each. What is the price of 1 lb of each article ?

29. A and B start from two stations 32 miles apart. If they walk towards each other they meet in 4 hours ; but if they walk in the same direction they meet in 16 hours. Find the rate of walking of each.

30. A train running from A to B meets an accident 50 miles from A, after which it moves with $\frac{3}{5}$th of its original velocity and arrives at B 3 hours late. Had the accident happened 50 miles further on, it would have been only 2 hours late. Find the distance from A to B and the original speed of the train. (M M. 1857).

31. A boat's crew rowed $3\frac{1}{2}$ miles down a river and up again in 100 minutes. Had the stream been half as strong again, they would have taken $31\frac{1}{4}$ minutes longer. Find the rate of the stream. (B. M. 1860).

32. A person invests Rs. 2740, partly in 3 P.C. stock at 102 and partly in 4 P.C. stock at 96 ; if his total income from the investments is *Rs.* 110 annually, what sum does he invest in each ?

33. Two trains, 90 *ft.* and 86 *ft.* long, move on parallel lines with uniform speeds. They are observed to pass each other in $1\frac{1}{2}$ sec. when moving in opposite directions and in 12 sec. when moving in the same direction. Find the rate of each train.

34. Two cyclists ride from A to B, a distance of 55 miles, and the first arrives 30 minutes before the second. They then ride from B to A, the first giving the second a start of 4 miles and yet arriving 6 minutes before him. Find the rate of each cyclist in miles per hour. (B. M. 1901).

35. A gentleman went out for a walk ; and after having been out 12 minutes, was overtaken by his servant who had run from the house at thrice his master's pace. The master then bade the servant run back at the same rate to the house and bring his cigars, while he walked on at his former pace. If the master was one mile from the house when overtaken the second time at what rate did he walk ? (M. M. 1873).

36. A number consists of two digits. When the number is divided by the sum of the digits, the quotient is 7. The sum of the reciprocals of the digits is 9 times the reciprocal of the product of the digits. Find the number. (M. M. 1887).

37. A certain number consists of two digits whose sum is 8, another number is obtained by reversing the digits. If the product of these two is 1855, find the number. (B. M. 1877).

38. A number consists of 3 digits whose sum is 10. The middle digit is equal to the sum of the other two : and the number will be increased by 99 if the digits be reversed. Find the number. (B. M. 1888).

39. A father's age is four times that of his eldest son and five times that of his younger son ; when the eldest son has lived to three times his present age, the father's age will exceed twice that of his younger son by three years. Find their present ages. (B. M. 1882).

40. The gross income of a certain man was £40 more in the second of two particular years than in the first, but in consequence of the income tax rising from 4*d*. in the pound to 6*d*. in the pound in the second year his net income after paying the tax was unaltered. Find his income in each year. (B. M. 1891).

CHAPTER XXI.

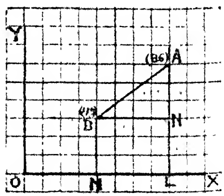
STRAIGHT LINE GRAPHS.

1. In Chapter III we considered the subject of plotting points in a system of rectangular co-ordinates and shortly explained the use of the squared paper. We shall now solve by calculation some questions considered there graphically.

Ex. 1. Calculate the distance between the points (8, 6) and (4, 3).

Let the point (8, 6) be A and the point (4, 3) be B as in the figure. Draw AL, BM perp. to the axis of x and BN perp. to AL.

Now BN = ML = OL − OM, also AN = AL − NL = AL − BM.



Scale ·1″ = 1. fig. 12.

Hence from the rt.-angled triangle ABN, we have

$$AB^2 = BN^2 + AN^2$$
$$= (OL - OM)^2 + (AL - BM)^2 \qquad \dots \quad (1)$$
$$= (8-4)^2 + (6-3)^2$$
$$= 4^2 + 3^2 = 25.$$
$$\therefore AB = 5.$$

Generally, if A be the point (x_1, y_1) and B the point (x_2, y_2) we have from (1)

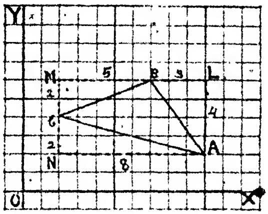
$AB^2=(x_1-x_2)^2+(y_1-y_2)^2$, a formula giving the square of the distance between two points whose co-ordinates are given.

Note. In the above diagram the student will mark that the distance of A from B measured *parallel to the axis of x* (*i.e.* BN) is the *difference of the abscissæ* of the points; so the distance of A from B measured *parallel to the axis of y* (*i.e.* AN) is the *difference of the ordinates* of the points.

Ex. 2. Calculate the area of the triangle whose vertices are (10, 2), (7, 6), and (2, 4).

Let the points be A, B, C in order. Through A, B, C draw lines parallel to one or other of the axes to form the rect. ALMN. Then we have

$\triangle ABC=$rect. ALMN $-\triangle ALB-\triangle BMC-\triangle CNA$............(1).



Scale $\cdot 1'' = 1$. fig. 13.

Now AL = distance of B from A measured parallel to the axis of y.
= diff. of the ordinates of B and A.
$=6-2=4$.

BL = distance of A from B measured parallel to the axis of x.
= diff. of the abscissæ of A and B.
$=10-7=3$.

Similarly other lengths are as marked in the diagram.

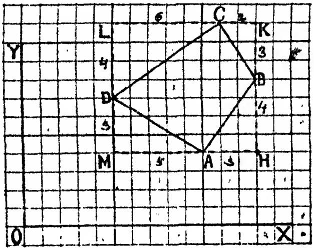
Hence from (1)

$\triangle ABC = AL.LM-\frac{1}{2}.AL.LB-\frac{1}{2}BM.MC-\frac{1}{2}CN.NA.$
$=4.8-\frac{1}{2}.\ 4.\ 3-\frac{1}{2}.\ 5.\ 2-\frac{1}{2}.\ 2.\ 8.$
$=32-6-5-8.$
$=13$ square units.
$=\cdot 13$ square inch.

Generally if A, B, C be the points (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively, we have, proceeding as above and simplifying,

$$\Delta ABC = \tfrac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}.$$

Ex. 3. *Find the area of the quadrilateral formed by the points (10, 4), (13, 8), (11, 11) and (5, 7).*



Scale $\cdot 1'' = 1$. fig. 14.

Let the points be A, B, C, D in order. Draw parallels through these points to one or other of the axes to from the rectangle $HKLM$. Then the quadrilateral $ABCD$

$$= \text{rect. } HKLM - \Delta AHB - \Delta BKC - \Delta CLD - \Delta DMA$$
$$= HK.KL - \tfrac{1}{2}.AH.BH - \tfrac{1}{2}.BK.KC - \tfrac{1}{2}.CL.LD - \tfrac{1}{2}.DM.MA$$
$$= 7.8 - \tfrac{1}{2}.3.4 - \tfrac{1}{2}.3.2 - \tfrac{1}{2}.6.4. - \tfrac{1}{2}.3.5$$
$$= 56 - 6 - 3 - 12 - 7\tfrac{1}{2} = 27\cdot 5 \text{ square units} = \cdot 275 \text{ square inch}$$

EXERCISE XCII.

1. Find the distance of the origin from the following points :—

(i) (16, 12) (ii) (15, 8) (iii) (99, 20) (iv) (60, 11)

2. Find the distance between the following pairs of points :—

(i) (3, 2) and (15, 7) (ii) (10, 7) and (45, 19).
(iii) (9, 25) and (33, 32) (iv) (19, 23) and (82, 39).

3. Show by a diagram that the distances between the following pairs of points are equal :—

(i) (b, o), (o, a) (ii) (a, o), (o, b) (iii) (o, o), (a, b).

4. Prove that the co-ordinnates of the middle point of a straight line is half the sum of the co-ordinates of its extremities.

5 Calculate the lengths of the sides and areas of the triangles of which the vertices are

(i) $(-3, 2), (3, 10), (7, 2)$ (ii) $(o, o), (8, 10), (8, -6)$

(iii) $(6, 6), (12, 6), (12, 12)$ (iv) $(2, 14), (-10, 14), (1, -7)$.

6. Plot the locus of points which are equidistant from the origin and the point $(6, -6)$.

7. Plot the locus of points which are equidistant from the points $(2, 3)$ and $(10, 7)$.

8. Draw and calculate the areas of quadrilaterals whose vertices are

(i) $(5, 3), (11, 7), (6, 8)$ and $(3, 9)$.

(ii) $(10, 10), (5, 5), (-5, 8), (2, 15)$.

9. Plot the points $(5, 5)$, $(10, 5)$, $(15, 10)$, $(10, 15)$ and $(5, 10)$ and find the area of the figure formed by joining the points in order.

10. Prove that the line joining the points $(7, 3)$ and $(11, 11)$ subtends a right angle at the point $(5, 5)$.

2. Consider the expression $3x-12$ containing a variable quantity x. Denoting $3x-12$ by y, we have the equation

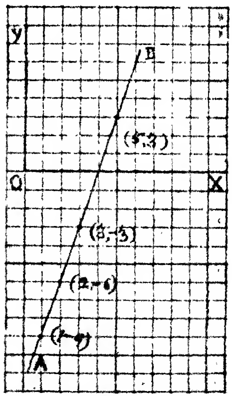
$$y = 3x - 12 \quad \text{.........................(1)}$$

By giving to x a series of numerical values, we get a series of corresponding values of y or $3x-12$ from (1). Thus if

$x = \;\; 1, \;\; 2, \;\; 3, \;\; 4, \;\; 5 \ldots\ldots\ldots\ldots$

$y = -9, -6, -3, \;\; 0, \;\; 3 \ldots\ldots\ldots\ldots$

Now we may suppose each pair of these corresponding values of x and y to be the abscissa and ordinate respectively of a point in a plane with reference to two axes in the plane, and plot the points $(1, -9)$, $(2, -6)$, $(3, -3)$, $(4, 0)$, $(5, 3)$............



Scale $\cdot 1'' = 1$. fig. 15.

as in the diagram. It appears from the figure that these points lie on the straight line AB. Similarly

any values of x and y which satisfy the equation (1) are the abscissa and ordinate respectively of some point which lies on AB (produced if necessary).

Again, the abscissa and ordinate of any point on AB (produced both ways) being put for x and y respectively in (1), the equation will be satisfied. *Thus we find from the figure that (6,6) is a point on AB, and by putting $x=6, y=6$ the equation (1) is satisfied.*

We thus get the geometrical interpretation of the algebraic equation (1) *viz.*, the equation $y=3x-12$ represents geometrically the straight line AB which is called **the graph of the equation** and the equation is called the equation of the graph AB.

3. Function. An expression which contains a variable quantity x is called a function of x and is usually denoted by the symbol $f(x)$. Thus $2x+3$, $3x^2-4x+9$ are functions of x of the first and second degrees respectively.

A function of the first degree is called a *linear function*, and a function of the second degree is called a *quadratic function.*

4. Graph of a function. Let $f(x)$ be a function of x. Denoting the function by y we have the equation $y=f(x)$, which for a series of values of x gives a corresponding series of values of y. Take *each* pair of corresponding values of x and y to be the co-ordinates of a point in a plane, and plot a series of points in the plane from these co-ordinates. The line, straight or curved, through the points thus plotted is called the *graph of the function* $f(x)$ or the *graph of* the *equation* $y=f(x)$, the points being joined by means of a continuous curve.

In art. 2 we have found that the straight line AB is the graph of the function $3x-12$ or of the equation $y=3x-12$.

5. Graph as a locus. If the abscissa and ordinate of a moving point are denoted (as is always done) by x and y respectively and if these quantities satisfy the equation $y=f(x)$, then it is evident that the moving point must trace our the graph of the function $f(x)$. Hence a graph may be defined thus :—

The graph of an equation $y=f(x)$ is the locus of a point which moves so that its co-ordinates x and y always satisfy the equation.

To test whether a point lies on the graph of an equation or not, we are to substitute the co-ordinates of the point in the equation *i.e.* put $x=$abscissa and $y=$ordinate of the point ; and then see whether the equation is or is not satisfied. In the former case the point is on the graph and in the latter case it is not.

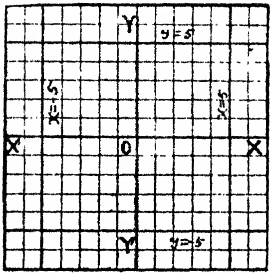
Thus the point (10, 18) is on the graph of $y=3x-12$, for $x=10$, $y=18$ evidently satisfies it.

6. The following graphs should be thoroughly understood by the student.

(1) *The graph of $x=0$ is the axis of y*, for every point on the axis of y has its x or abscissa equal to zero.

(2) *The graph of $y=0$ is the axis of x*, for every point on the axis of x has its y or ordinate equal to zero.

(3) *The graph of $x=a$ where a is a constant is a straight line parallel to the axis of y at a distance of a units from it.* The graphs of $x=5$, $x=-5$ are drawn in the figure.



Scale $\cdot 1'' = 1$. fig. 16.

(4) *The graph of $y=b$ where b is a constant is a straight line parallel to the axis of x at a distance of b units from it.* The graphs of $y=5$ and $y=-5$ are drawn in the figure.

7. We shall now consider graphs of equations of the first degree. In the following examples the scale used will be $\cdot 1'' = 1$.

Ex. 1. Draw the graphs of (i) $y=x$
(ii) $y=-x$

(i) To draw the graph of $y=x$, tabulate the corresponding values of x and y as follows:—

$x=0,\ 1,\ 2,\ 3,\ldots\ldots\ldots -1,\ -2,\ -3,\ldots\ldots\ldots$

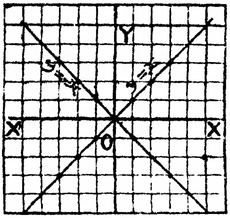
$y=0,\ 1,\ 2,\ 3,\ldots\ldots\ldots -1,\ -2,\ -3,\ldots\ldots\ldots$

Hence the graph passes through the origin (0, 0) and the following points :—

$(1, 1), (2, 2), (3, 3)\ldots\ldots(-1, -1), (-2, -2), (-3, -3)\ldots\ldots$

Plotting these points we find that the graph consists of the straight line in the figure 17, bisecting the angle XOY.

(ii) To draw the graph of $y=-x$, tabulate the corresponding values of x and y as follows :—



Scale. $\cdot 1''=1$. fig. 17.

$x=0,\quad 1,\quad 2,\quad 3\ldots\ldots\ldots-1,\quad -2,\quad -3$
$y=0,\quad -1,\quad -2,\quad -3\ldots\ldots\ldots\quad 1,\quad 2,\quad 3$

Hence the graph passes through the origin (0, 0) and the following points :—

$(1,-1), (2,-2), (3,-3)\ldots\ldots\ldots(-1, 1), (-2, 2), (-3, 3)\ldots\ldots\ldots$

Plotting the points we find that the garph consists of the straight line in the figure bisecting the angle YOX'.

Ex. 2. Draw the graphs of

(i) $y=-2x$ (ii) $y=3x-9$

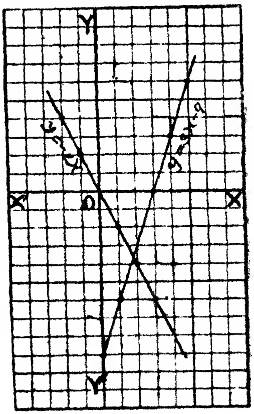
(i) To draw the graph of $y=-2x$, we tabulate the corresponding values of x and y as follows :—

$x=0,\quad 1,\quad 2,\quad 3\ldots-1,\quad -2,\quad -3$

$y=0,\quad -2,\quad -4,\quad -6\ldots\quad 2,\quad 4,\quad 6$

Hence the graph passes through the origin and the following points :— $(1, -2)$, $(2, -4)$, $(3, -6)\ldots(-1, 2)$, $(-2, 4)$, $(-3, 6)\ldots\ldots$

Plotting the points we find the grahp to be a straight line (fig. 18).



Scale. $\cdot 1''=1$. fig. 18.

(ii) To draw the graph of $y=3x-9$, we tabulate the corresponding values of x and y as below :—

$x=\quad 0,\quad 1,\quad 2,\ 3,\quad 4\ldots\ldots-\ 1,\ -\ 2\ldots$

$y=-9,\quad -6,\quad -3,\ 0,\quad 3\ldots\ldots-12,\ -15\ldots$

Plotting the points $(0, -9)$, $(1, -6)$, $(2, -3)$, $(3, 0)$, $(4, 3)\ldots$ we get the graph to be a straight line (fig. 18).

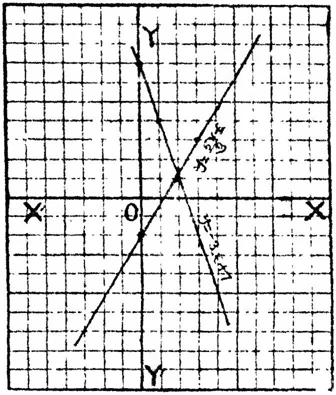
Ex. 3. Draw the graphs of (i) $-3x+7$ (ii) $(5x-6)/3$

(i) Here we are to draw the graph of $y=-3x+7$. We tabulate the co-responding values of x and y thus :—

$x=0, 1, 2, \quad 3 \ldots\ldots -1, -2. -3, \ldots$

$y=7, 4, 1, -2, \ldots\ldots 10, \quad 13, \quad 16, \ldots$

Plotting the points (0, 7), (1, 4), (2, 1)...(−1, 10), (−2, 13)...... we get the graph to be a straight line (fig. 19).



Scale. ·1″=1. fig· 19.

(ii) Here we are to draw the graph of $y=\dfrac{5x-6}{3}$. We tabulate the corresponding values of x and y thus :—

$x= \quad 0, 3, 6, \ldots -3, - \ 6, \ldots$

$y= -2, 3, 8, \ldots -7, -12, \ldots$

Plotting the above points we find the graph to be a straight line (fig. 19.)

8. The most general form of an equation of the first degree in x and y is $Ax+By+C=0 \ldots\ldots (1)$,

where A, B, C are constant quantities.

From (1) we get $By=-Ax-C$

or $y=-\dfrac{A}{B}x-\dfrac{C}{B}$.

This we way write as $y=mx+c \ldots\ldots (2)$

where m is put for $-\dfrac{A}{B}$, and c for $-\dfrac{C}{B}$

Thus an equation of the form (1) may be reduced to the form (2). Now for numerical values of m and c the graph of $y=mx+c$ can always be drawn as in the preceding article. From the cases considered there and others, the student is likely to come to the conclusion that the graph of $y=mx+c$ is always a straight line.

Hence we infer that *the graph of the equation* $Ax+By+C=0$ *for all values of* A, B, C *is a straight line.*

The result is formally proved in the next article.

9. *To prove that the graph of the equation of the first degree in x and y is always a straight line.*

Let the equation be $Ax+By+C=0$.

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be three points on the graph of the equation.

Then $x=x_1$, $y=y_1$, will satisfy the equation.

$$\text{Hence } Ax_1+By_1+C=0 \ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{Similarly } Ax_2+By_2+C=0 \ldots\ldots\ldots\ldots\ldots(2)$$

$$Ax_3+By_3+C=0 \ldots\ldots\ldots\ldots\ldots(3)$$

From (2) and (3) by cross multiplication,

$$\frac{A}{y_2-y_3}=\frac{B}{x_3-x_2}=\frac{C}{x_2y_3-x_3y_2}=k \text{ (suppose).}$$

$\therefore A=k(y_2-y_3)$, $B=k(x_3-x_2)$, $C=k(x_2y_3-x_3y_2)$

Substituting these values of A, B, C in (1) and dividing out by k we have

$$x_1(y_2-y_3)+y_1(x_3-x_2)+(x_2y_3-x_3y_2)=0$$

or re-arranging, $x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0$.

But the expression on the left is double the area of the triangle whose vertices are the points $(x_1 y_1)$, (x_2, y_2) and (x_3, y_3), see ex. 2, art. 1. Hence the area of the triangle formed by joining any three points on the graph is zero or the graph is a straight line.

10. A straight line is determined by two points. Hence, since the graph of an equation of the first degree in the co-ordinates is a straight line, such a graph is obtained by plotting two convenient points on it and producing indefinitely the straight line joining them. The two points thus plotted should be taken as far apart as possible, as otherwise a small error in plotting them would displace the line considerably. Also it is advisable to plot a third point on the graph to test the accuracy of the other two points.

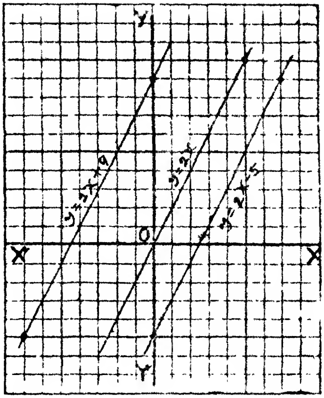
Sometimes it is useful in drawing the graph to determine it by joining the points where it cuts the co-ordinate axes. *By putting*

$y=0$ in the equation of the graph, the value of x gives the intercept on the axis of x, and by putting $x=0$ in the equation, the value of y gives the intercept on the axis of y.

Ex. 1. Draw the graphs of (i) $y=2x$ (ii) $y=2x+9$ (iii) $y=2x-5$

The equations being of the first degree in x and y, their graphs are *straight lines.*

(i) Here $x=0$ gives $y=0$, also $x=5$ gives $y=10$; hence the graph passes through the origin (0, 0) and the point (5, 10), as shown in the figure (fig. 20).



Scale. $\cdot 1'' = 1$. fig. 20

Obs. In general, an equation of the form $y=mx$ where m is a constant represents a straight line through the origin. Hence in such a case we are to plot one convenient point only. Thus in the equation $y=\frac{7}{9}x$, a convenient point on the graph is (9, 7); hence the straight line through the origin and this point is the required graph.

(iii) In drawing the graph of $y=2x+9$ we may take $x=0$, $y=9$ as one point and $x=-7$, $y=-5$ as another point. Hence joining these points (0, 9) and $(-7, -5)$ by a straight line we get the graph (fig. 20).

(iii) In drawing the graph of $y=2x-5$ one convenient point is $x=0, y=-5$ and another point is $x=7, y=9$.

Hence joining these points $(0, -5)$ and (7, 9) by a straight line we get the graph (fig. 20).

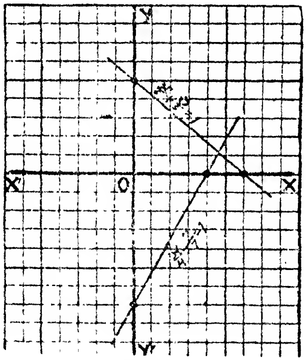
Note. It appears from the figure that the graphs of the equations $y=2x$, $y=2x+9$ and $y=2x-5$ are *parallel* straight lines. Observing that the equations differ in their constant parts only, we infer that *equations of the first degree which differ or can be made to differ in their constant terms only represent parallel straight lines.*

The proof of this result lies in the fact that two such graphs intercept a constant length on each common ordinate. (see fig. 20).

Ex. 2. Graph (i) $\frac{x}{6}+\frac{y}{5}=1$ (ii) $\frac{x}{4}-\frac{y}{7}=1$.

The equations being of the first degree, their graphs are str. lines.

(i) Here putting $x=0, y=5$ and again by putting $y=0$, $x=6$. Hence the points (0, 5) and (6, 0) are on the graph or in other words the graph cuts off an intercept 5 from the axis of y and an intercept 6 from the axis of x Hence it is as shown in fig. 21.



Scale. $\cdot 1''=1$, fig. 21.

(ii) Here putting $x=0, y=-7$ or the graph cuts off an intercept -7 from the axis of y. Again by putting $y=0$, $x=4$ or the graph cuts off an intercept 4 from the axis of x. Hence the graph is as in fig. 21.

Note. Generally the graph of $\frac{x}{a}+\frac{y}{b}=1$ is a straight line cutting off intercepts a and b from the axes of x and y respectively.

Ex. 3. Draw the graphs of (i) $4x-5y+12=0$

(ii) $5x+4y+24=0$

The equations being of the first degree, the graphs are st. lines.

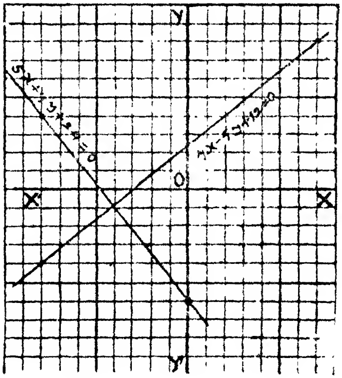
(i) Here $5y=4x+12$ or $y=\frac{4x+12}{5}$ . . . . . (1)

By trial we find from (1) a few pairs of corresponding values of x and y which are *integral.*

Thus putting $x=7, 2, -3, -8, \ldots\ldots\ldots$

we get $y=8, 4, 0, -4, \ldots\ldots\ldots$

Plotting any two of the points, say, (7, 8) and (−8, −4) and joining them by a straight line, we get the graph as in fig. 22.)



Scale. ·1″=1. fig. 22.

(ii) Here $4y=-(5x+24)$

or $y=-\frac{5x+24}{4}$ ....(2)

By trial we find from (2) a few corresponding values of x and y which are integral.

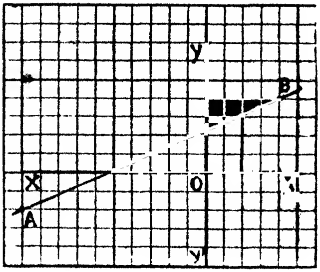
Thus, putting $x=\;\; 4, \;\; 0, -4, -8\ldots\ldots$

we get $y=-11, -6, -1, \;\; 4\ldots\ldots$

Plotting any two of the points, say, (0, −6) and (−8, 4) and joining them by a straight line, we get the graph as in fig. 22.

Note. It appears from the figure that the two graphs of the equations $4x-5y+12=0$ and $5x+4y+24=0$ are *perpendicular* straight lines. It may be observed that the coefficients of x and y in one equation may be obtained *by interchanging the coefficients of x and y in the other and changing the sign of one of them.* In all such equations the graphs are always perpendicular straight lines. Observe that nothing is said as to the constant terms in the equations, so they may be anything.

Ex. 4. Draw the graph of the function $\frac{3x+16}{7}$. Read off the value of the function when $x=2$, and find the value of x when the function is -1.



Scale. $\cdot 1'' + 1$. fig. 23.

We are to draw the graph of the equation $y=\frac{3x+16}{7}$.

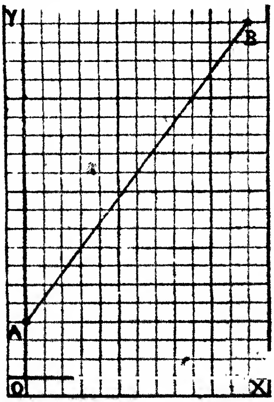
Puting $x=-10, y=-2$; also putting $x=4, y=4$.

Plotting these points A, B we get the graph to be the straight line AB (fig. 23).

When $x=2$ we find from the figure the corresponding value of y or the function $=3$ to the nearest integer.

Also when y or the function $=-1$, the corresponding value of $x=-8$ to the nearest integer.

Ex. 5. Draw the graph of the function $16x+9$.



[Scale $\cdot 4''=1$ along x axis and $\cdot 1''=3$ units along y axis. fig. 24.]

Here we are to draw the graph of the equation $y=16x+9$. The following are pairs of corresponding values of x and y :—

$$x=0,\quad 1,\quad 2,\quad 3,\quad 4,\ldots\ldots\ldots$$
$$y=9,\ 25,\ 41,\ 57,\ 73,\ldots\ldots\ldots$$

From this we find that as x increases, y increases very rapidly Hence unless the unit is chosen very small (which is practically inconvenient) the figure will be unweildy. To obviate this difficulty we take different units for measuring abscissa and ordinate. Let us take 4 sides of the small squares (*i.e.* ·4″) to represent the unit in measuring abscissæ and each side of the squares (*i.e.* ·1″) to represent 3 units in measuring ordinates. On these scales we plot the points (0, 9) and (3, 57). Remembering that in measuring abscissæ 3 is represented by 12 sides of the squares and in measuring ordinates 9 and 57 are denoted by 3 and 19 sides, we get the points to be A and B (fig. 24). Hence the graph is the straight line AB.

EXERCISE XCIII.

Draw the graphs of

1. $x=2$. **2.** $x=-7$. **3.** $x=12$. **4.** $x=-9$.

5. $y=6$. **6.** $y=-10$. **7.** $y=-4$. **8.** $y=8$.

9. $y=4x$. **10.** $y=-3x$. **11.** $y=7x$. **12.** $y=-5x$.

13. $y=\frac{3}{5}x$. **14.** $y=-\frac{4}{9}x$. **15.** $y=-\frac{2}{3}x$. **16.** $y=\frac{3}{4}x$.

17. $y=x+1$. **18.** $y=-x+2$. **19.** $y=x+3$.

20. $y=4x+12$. **21.** $y=5x-9$. **22.** $y=3x+15$.

23. $y=-3x+21$. **24.** $y=-4x+20$. **25.** $y=-5x+30$.

26. $3y=4x+18$. **27.** $2y=5x-20$. **28.** $4y=-5x+28$.

29. The abscissa of a point is invariable and equal to 7, graph the locus of the point.

30. The ordinate of a point is invariable and equal to -10, graph the locus of the pont.

31. The abscissa of a point is equal to the ordinate, draw the graph of its path.

32. The abscissa of a point is equal to 3 times the ordinate, draw the graph of its path.

Draw the graphs of the following functions :—

33. $x-5$. **34.** $3x-13$. **35.** $5x-19$.

36. $\dfrac{3x-4}{2}$. **37.** $\dfrac{5x-3}{3}$. **38.** $\dfrac{5-7x}{4}$.

39. $(8-x)/3$. **40.** $(3x+15)/5$. **41.** $(12-4x)/3$

42. Draw the graph of the function $\frac{5x+13}{7}$ and find its value when $x=2$.

43. Draw the graph of the function $\frac{3x-13}{4}$ and read off its value when $x=-1$. Find also for what value of x the function $=1$.

44. Draw the graph of the function $\frac{10-4x}{4}$ and find for what value of x the function $=-3$.

Draw the graphs of the following equations:

45. $3x-4y+5=0$. **46.** $7x+9y-24=0$. **47.** $4x-5y=6$.

48. $5x+3y=17$. **49.** $2x-10y+9=0$. **50.** $3x-4y=25$.

51. $\frac{x}{4}+\frac{y}{9}=1$. **52.** $\frac{x}{11}-\frac{y}{5}=1$.

53. $\frac{x}{12}+\frac{y}{18}=2$. **54.** $\frac{y}{8}-\frac{x}{7}=1$.

55. Find the area enclosed by the lines represented by $x=7$, $x=-9$, $y=3$, $y=-6$.

---

11. The following examples illustrate the notes to examples of the preceding article.

Ex. 1. Find the equation of the straight line through the points (2, 3) and (6, -7), and prove that the line passes through the point (-4, 18).

The equation $y=mx+c$...(1). for the suitable values of m and c represents any straight line. Here we are to make the line pass through the points (2,3) and 6, -7) and these conditions determine m and c.

Because (2, 3) is a point on (1) it follows that $x=2$; $y=3$ will satisfy (1).

$$\therefore\ 3=2m+c\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Again, because (6, -7) is a point on (1), it follows that $x=6$ $y=-7$ will satisfy (1)

$$\therefore\ -7=6m+c\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Subtracting (2) from (3) $4m=-10$ or $m=-\frac{5}{2}$.

$\therefore$ from, (2) $c=3-2m=3+5=8$.

$\therefore$ the equation of the line is $y=-\frac{5}{2}x+8$

or $5x+2y=16$

This equation is evidently satisfied by making $x=-4$ and $y=18$, or, the line passes through the point $(-4, 18)$.

Ex. 2. Find the equation to the straight line which cuts off intercepts 4 and -9 from the axes of x and y respectively.

The equation $y=mx+c$...(1), where m and c are constants, represents any straight line. Because it makes the intercept 4 on the axis of x, it is evident that the point (4, 0) is on (1);

hence $0=4m+c$..................(2)

Again, because the line makes an intercept -9 on the anxis of y it is evident that the point $(0, -9)$ is on (1);

hence $-9=c$ $\therefore$ from (2) $m=\frac{9}{4}$.

Thus the equation of the line is $y=\frac{9}{4}x-9$,

$$\text{or } \frac{x}{4}-\frac{y}{9}=1.$$

Ex. 3. Find the equation of the straight line through the point $(-4, 5)$ parallel to the straight line whose equation is $7x-2y+3=0$.

Equations of parallel straight lines differ in the constant terms. Hence the equation of any straight line (see note, ex. 1 art. 10) parallel to the straight line represented by $7x-2y+3=0$ is

$7x-2y+C=0$........................(1)

If this line passes through the point $(-4, 5)$ it follows that $x=-4, y=5$ will satisfy (1)

Hence $-28-10+C=0$ or $C=38$.

$\therefore$ the required equation is $7x-2y+38=0$.

Ex. 4. Find the equation of the straight line through the point $(3, -4)$ perpendicular to the straight line whose equation is $2x+3y-5=0$.

The equation of any straight line (see note, ex. 3 art. 10) perpendicular to the straight line represented by the equation $2x+3y-5=0$ is

$3x-2y+C=0$..............(1)

If this line passes through the point $(3, -4)$, it follows that $x=3, y=-4$ will satisfy (1).

Hence $9+8+C=0$ or $C=-17$

$\therefore$ the required equation is $3x-2y-17=0$.

12. Gradient, slope. Let OA be the graph of the equation $y=\frac{4}{5}x$ and OB that of the equation $y=-\frac{2}{3}x$. Then (supposing the axis of x to be horizonal and the axis of y vertical) in moving along OA there is a *rise* of 4 units for every 5 units traversed horizontally and in moving along OB there is a *fall* of 2 units for every 3 units traversed horizontally. The number $\frac{4}{5}$ is called the *gradient* or *slope* of OA, and the number $-\frac{2}{3}$ is called the *gradient* or *slope* of OB.

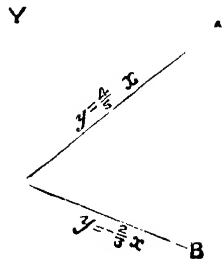


Fig. 25.

Thus the gradient of the straight line whose equation is $y=mx$ is m. Again since parallel straight lines have the same gradient, it follows that the gradient of the straight line whose equation is $y=mx+c$ is also m.

Ex. Find the gradient of the straight line whose equation is $3x+4y-7=0$

Here $3x+4y-7=0$

$$\therefore\ 4y=-3x+7 \text{ or } y=-\tfrac{3}{4}x+\tfrac{7}{4}$$

$\therefore$ the gradient of the straight line $=-\frac{3}{4}$.

EXERCISE XCIV.

Find the equation of the straight lines through each of the following pairs of points :—

1. $(5, 6), (-5, -6)$. **2.** $(3, 4), (5, -6)$.

3. $(0, 0), (2, 3)$. **4.** $(6, -4), (-7, 8)$.

Find the equation of the straight lines cutting off the following intercepts :—

5. 4 from the axis of x and 5 from the axis of y.

6. -7 from the axis of x and 9 from the axis of y.

7. 12 from the axis of x and -8 from the axis of y.

8. Find the equation of the straight line through the point $(5, -8)$ parallel to the straight line given by the equation $7x-11y+12=0$.

9. Find the equation of the straight line through the point $(-6, 2)$ parallel to the straight line given by the equation $6x-y+13=0$.

10. Find the equation of the straight line through the point $(3, -4)$ perpendicular to the straight line whose equation is $3x-7y+2=0$.

11. Find the equation of the straight line through the point $(3, 5)$ perpendicular to the straight line whose equation is $9x-y+5=0$.

12. Show that the three points $(6, -2)$, $(-4, 8)$, $(10, -6)$ lie on a straight line and find the equation of the straight line.

13. Show without drawing which of the points $(1, 5)$, $(2, 7)$, $(5, 3)$, $(-2, -1)$ lie on the straight line $y=2x+3$

14. Find the slopes of the following straight lines :—

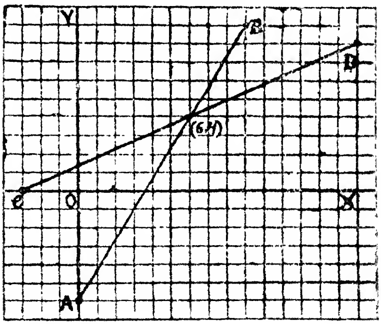
(i) $7x+12y-9=0$ (ii) $3x-12y+7=0$.
(iii) $x-7y+1=0$ (iv) $5-3x+11y=0$.

15. Find the equation of the straight line through the point $(3, -4)$ having a slope$=\frac{4}{3}$.

13. Graphical solutions of equations. We shall consider simultaneous equations of the first degree in two variables. The method is illustrated by the following examples.

Ex. 1. Find graphically the solutions of the equations.

$$5x-3y-18=0......(1)$$
$$4x-9y+12=0......(2)$$



Scale $\cdot 1''=1$. Fig. 26.

The graphs of the equations (1) add (2) are straight lines.

From (1) we have $3y=5x-18$ or $y=\frac{5x-18}{3}$. Hence by putting $x=0$, we get $y=-6$; also by putting $x=9$, we get $y=9$. Plotting these points $(0, -6)$, $(9, 9)$ which are A and B in the figure we get the graph of (1) to be the straight line AB.

From (2) $9y=4x+12$ or $y=\frac{4x+12}{9}$. Hence by putting $x=-3$ we get $y=0$. also by putting $x=15$, we get $y=8$. Plotting these points $(-3, 0)$, $(15, 8)$ which are C and D in the figure, we get the graph of (2) to be the straight line CD.

Now the co-ordinates of all points in AB satisfy the equation (1) ane the co-ordinâtes of all points in CD satisfy the equation (2). Hence the co-ordinates of the point *common* to AB and CD satisfy both the equations and are therefore their solutions.

Thus from the figure we find that $x=6$, $y=4$ are the required solutions.

Hence to obtain the solutions of two simultaneous equations of the first degree in two variables, we are to draw the graphs of those equations (straight lines) and read off the co-ordinates of their point of intersection.

Ex. 2. Solve the equations graphically

$123x+231y=360$.........(1) $241x-319y=375$.........(2)

To draw the graph of (1), put $y=0$, then $x=\frac{360}{123}=2.93$ nearly.

Again, putting $x=0$, we we get $y=\frac{360}{231}=1.56$ nearly.

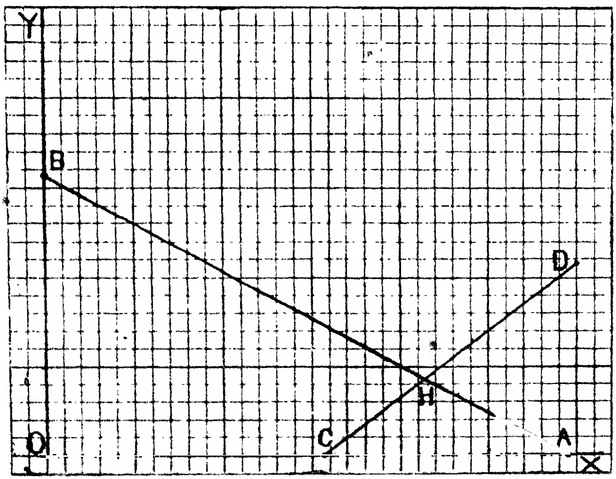


Fig. 27. Scale 1" = 1.

Hence the graph is the straight line joining A (2·93, 0), and B (0, 1·56).

To draw the graph of (2), put $y=0$, then

$$x=\frac{375}{241}=1\cdot55 \text{ nearly.}$$

Again, putting $x=3$, we get

$$y=\frac{348}{310}=1\cdot09 \text{ nearly.}$$

Hence the graph is the straight line joining C (1·55, 0) and D (3, 1·09).

Draw the two graphs on the scale $1''=1$ and let them intersect at H whose co-ordinates are 2·15 and ·42

$\therefore x=2\cdot15,\ y=\cdot42$ (approximately).

14. Inconsistency and independence of equations.

The graphs of two equations are parallel straight lines, if they differ (or can be made to differ) in their constant terms only. Hence two such equations (since their graphs have no common point) have no solution.

Thus the equations $2x+3y-4=0$, $4x+6y+7=0$, (the latter being really $2x+3y+\frac{7}{2}=0$) are inconsistent and have no solution.

Similarly three equations of the first degree are consistent if they are satisfied by the same values of x and y. *i.e.* if the straight lines represented by them are concurrent.

Again, the graphs of two equations are coincident straight lines, if they are not independent *i.e.* if one is deducible from the other. Hence two such equations (since their graphs have all points common) have an infinite number of solutions.

Thus, the equations $3x-4y+7=0$, $6x-8y+14=0$ are not independent (the second being deducible from the first by multiplication by 2) and being equivalent to *one* equation connecting two variables have an infinite number of solutions.

15. Graphical solutions of problems.

Ex. 1. The monthly wages of a servant are *Rs.* 3, construct a graph serving as a ready-reckoner to show the wages for any number of days of the month.

Let the wages for x days be y annas. Now the daily wages are $\frac{48}{30}$ or $\frac{8}{5}$ annas, and for x days they are $\frac{8}{5}x$ annas. Hence $y=\frac{8}{5}x$ is the relation between x and y.

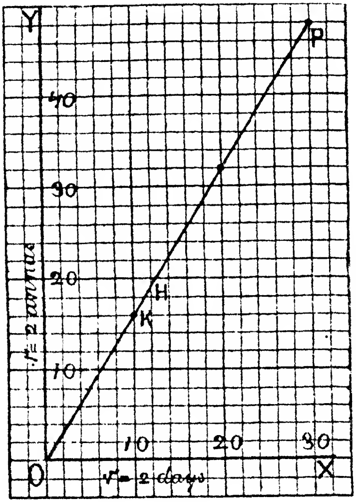


Fig. 28.

Let the distances along OX represent the number of days ($\cdot 1''$ representing 2 days) and the distances along OY represent the number of annas ($1''$ representing 2 annas). Then the graph of $y=\frac{8}{5}x$ (which is a straight line) may be obtained by joining O to the point P (30, 48). The abscissa and ordinate of any point on OP give respectively the number of days and the corresponding wages in annas.

Thus the abscissa of the point H represents 2×6 or 12 days and the corresponding ordinate represents $2\times 9\frac{1}{2}$ or 19 annas nearly. Hence the wages for 12 days are 19 annas.

Again, the ordinate of K represents 2×8 or 16 annas, and the corresponding abscissa represents 2×5 or 10 days. Hence for one rupee the servant can be engaged for 10 days.

Ex. 2. The cost of 15 articles is *12as. 6p.*, draw a graph to obtain the prices of any number of articles up to 12.

Let x articles cost y annas. Now the cost of each article is $(\frac{25}{2} \div 15)$ or $\frac{5}{6}$ annas and therefore of x articles $\frac{5}{6}x$ annas. Hence $y = \frac{5}{6}x$ is the relation connecting x and y.

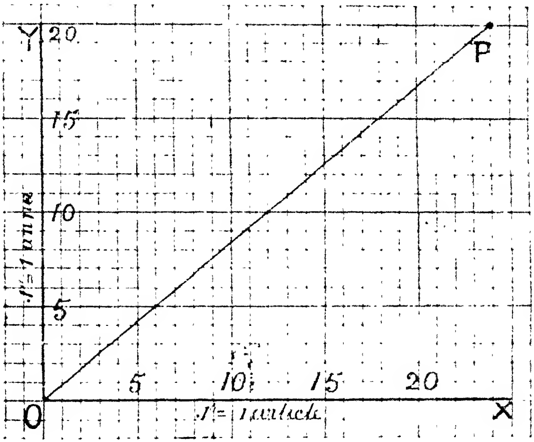


Fig 29.

Let ·1″ along OX represent one article and ·1″ along OY represent one anna. The graph of $y = \frac{5}{6}x$ which is a straight line may be obtained by joining O to the point P (24, 20).

From the figure we find $y = 12$ nearly when $x = 14$ or the price of 14 articles is 12 annas.

Again, we find $x = 11$ nearly, when $y = 9$, or, for 9 annas we can buy 11 articles.

Ex. 3. Given that one rupee $= 1{\cdot}3$ shillings, construct a graph giving at sight the number of rupees in a number of shillings and *vice versa*.

If x rupees $= y$ shillings, then $y = \frac{13}{10}x$ is the relation between x and y.

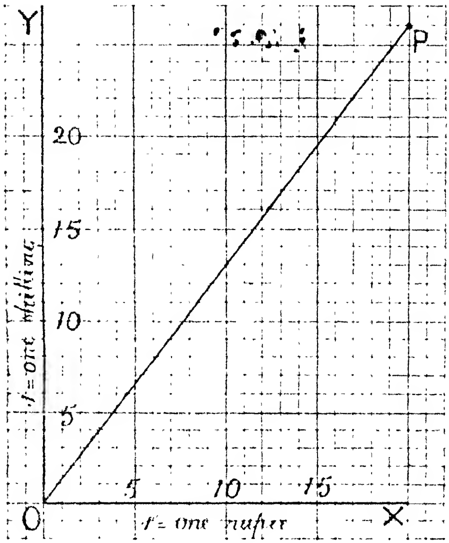


Fig. 30.

The graph of $y = \frac{13}{10}x$ is a straight line, obtained by joining O to the point P (20, 26). This is the required graph.

Thus when $x = 7$, $y = 9$ nearly *i.e.* in 7 rupees there are 9 shillings nearly. Also when $y = 12$, $x = 9\frac{1}{4}$ nearly showing that in 12 shillings there are $9\frac{1}{4}$ rupees nearly.

Ex. 4. A gives B a start of 2 miles. If A walks at the rate of 4 miles an hour and B at the rate of 3 miles, find when and at what distance A will overtake B.

Let A walk y miles in x hours, then $y=4x$ is the relation between x and y. Taking $\cdot5''$ along OX to represent one hour and $\cdot1''$ along OY to represent one mile, we draw the graph of $y=4x$ (which is a straight line) by joining O (0, 0) to the point P (3, 12). Then OP is the *motion-graph* of A.

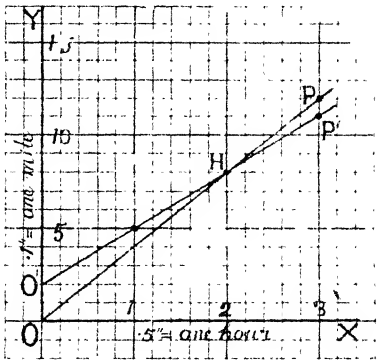


Fig. 31.

Let B walk y miles (from the beginning of his motion) in x hours (from the instant A starts). Then $y=3x+2$ is the relation between x and y. The graph of $y=3x+2$ (which is a straight line) is obtained by joining O' (0, 2) and P' (3, 11). This is the *motion graph* of B.

The two graphs intersect at H, of which the abscissa is 2 and ordinate is 8. Hence A and B meet after A has walked for 2 hours a distance of 8 miles.

Ex. 5. At what time between 3 and 4 o'clock are the two hands of a watch together?

Let the minute hand go over y minute-spaces in x minutes after 3 o'clock; then evidently $y=x$ is the relation between x and y. Taking $\cdot1''$ along OX to represent one minute and $\cdot1''$ along OY to represent one minute space, we draw the graph of $y=x$ by

joining the point O (0, 0) to the point P (25, 25). Then OP is the *motion-graph* of the minute-hand.

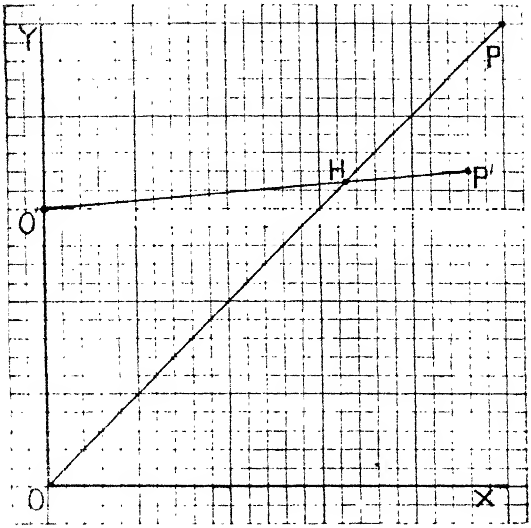


Fig. 32.

Let the hour-hand be y minute-spaces off from the 12 o'clock mark, in x minutes past 3. Then since in x minutes the hour-hand moves over $\frac{x}{12}$ minute-spaces, we have $y=\frac{x}{12}+15$, the relation between x and y. The graph of this is obtained by joining O' (0, 15) to P' (24, 17.) Then $O'P'$ is the motion-graph of the hour hand.

The two graphs intersect at H which corresponds to the position when the two hands meet. The abscissa of H being 16·4 nearly, the time required is 16 4 minutes past 3.

Note. If we want the time at which the hands of a watch are, say, 6 minute-spaces apart, we determine it from the abscissa of which the corresponding ordinate in the two graphs differ by 6. Thus from the fig. we find that the ordinates differ by 6 when the abscissa is 10 or 23. Hence the hands are 6 minute-spaces apart at 10 minutes past 3 and again at 23 minutes past 3.

Ex. 6. Draw a smooth curve to represent the variations in temperature from the data :

TIME.	6 A.M.	8 A.M.	10 A.M	12 NOON	2 P.M.	4 P.M.	6 P.M.	8 P.M.
Temp.	54	56	60	68	73	64	62	[illegible]

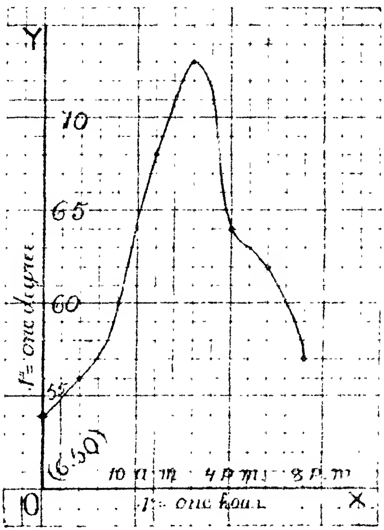


Fig. 33.

Let '1″ along the x-axis represent one hour and '1″ along the y axis represent one degree of temperature. Draw lines through the point (6, 50) parallel to the axes and plot the above points with reference to them. Join the points by a smooth curve as shown in the figure.

Note. In this example only a limited number of points is given and the problem of drawing the curve through them is indefinite. In such cases we are to draw the simplest curve passing evenly through the points.

EXERCISE XCV.

Solve the following equations graphically

1. $2x - 3y + 5 = 0$, $4x + 3y - 17 = 0$

2. $5x - 7y - 29 = 0$, $3x + 2y + 5 = 0$

3. $4x + 5y - 13 = 0$, $6x - y = 11$.

4. $9x + 2y + 12 = 0$, $5x - 7y + 31 = 0$.

5. $35x - 13y = 19$, $26x + 7y = 23$.

6. Show graphically that the graphs of the following equations meet at a point :—

$$5x - 3y + 26 = 0, \quad 2x + 3y + 2 = 0, \quad 7x - 5y + 38 = 0.$$

7 Prove that the following equations represent concurrent straight lines :

$$6x + 5y = 21, \quad 3x = y, \quad 9x - 7y + 12 = 0.$$

8. Prove that the graphs of the following equations pass through the point $(2, -5)$:

$$4x - 7y = 43, \quad y - 2x + 9 = 0, \quad 3x - 4y = 26.$$

9. Draw the triangle whose sides are the straight lines given by the equations $5x + 4y = 53$, $x + 3y = 15$, $3x - 2y = 1$. Find the co-ordinates of its angular points, the lengths of its sides and medians.

10. If oranges sell at Rs. 6 per hundred, draw a graph to read off the prices of any number of oranges up to 50. Find how many oranges can be bought for R1. 4 as.

11. Construct a ready reckoner to read off the price of rice at 5 pice per seer.

12. Construct a graph to convert readily degrees Fahrenheit into degrees Centigrade and *vice versa*.

[If $x°$ in centigrade scale be the same temperature as $y°$ in the Fahrenheit scale, then $\frac{x}{100} = \frac{y-32}{212-32}$ or $9x = 5y - 160$, is the relation between x and y.]

13. Draw a graph for converting at sight inches into metres and *vice versa*, it being known that 1 metre = 39.2 inches.

14. Given that 1 inch = 2·54 centimetres, draw a graph to convert inches into centimetres and *vice versa*.

Find the value of 5·6 centimetres in inches and 3·4 inches in centimetres, from the graph.

15. Given that 1 kilometre = ·62 mile, draw a graph to convert kilometres into miles and *vice versa*.

Read off from the graph the value of 7·3 kilometres in miles and the value of 2·7 miles in kilometres.

16. Draw a graph to convert speeds in miles per hour into speeds in feet per second.

17. Two men start, one from P to Q and the other from Q to P: the former walks at the rate of 5 miles an hour and the other at the rate of 3 miles. Find when they will meet, the distance between P and Q being 60 miles.

18. The establishment charges of a hostel are constant and other charges vary with the number of boarders. For 20 boarders the total expenses are Rs. 215 and for 25 boarders they are Rs 265. Draw a graph to represent the expenses for any number of boys.

19. Five boys get 65, 52, 49, 33, 23 out of a maximum of 70 marks. Draw a graph to raise the maximum to 100 and read off the corresponding marks of the candidates.

20. The sum of R. 200 is put to simple interest at 3 p. c. per annum, draw a graph to find the amount at the end of any number of years up to 10

21. The first thousand copies of a book cost Rs. 10 to print and for every thousand more the cost is Rs. 3 only, draw a graph to read off the cost of any number of thousands up to 10.

22. At what time the two hands of a watch are together between 4 and 5 o'clock? When are they 12 minute-spaces apart?

23 The top and bottom marks for a paper in an examination are 215 and 85 respectively. Show by a graph how to reduce these so that the top counts 174 and the bottom 34.

Read off from the graph the reduced mark to the nearest integer corresponding to the old mark 157 and the original mark corresponding to the reduced mark 121.

[Let x old marks correspond to y reduced marks, then $\dfrac{x-85}{215-85}=\dfrac{y-34}{174-34}$ is the relation between x and y.]

24. The number of years y, that a male aged x years may be expected to live is given thus:

x	0	4	8	12	16	20	24	28	32	36
y	41	51	49	46	43	39	36	34	31	28

Illustrate graphically and calculate the expectation of life of males aged, 9, 14, 22, 26, 34.

25. Draw a smooth curve to represent the variations in the height of a barometer from the data

TIME.	2 A.M.	3 A.M.	4 A.M	5 A.M.	6 A.M.	7 A.M.	8. A.M.	9 A.M
Height	29·6	29·9	30·1	29·8	30	29·5	29·6	29·3

MISCELLANEOUS EXERCISE PAPERS III.

PAPER I.

1. Find the H. C. F. of $x^4+x^3-11x^2-9x+18$ and $x^4-10x^3+35x^2-50x+24$.

2. Show that $\dfrac{x^2}{2x^2+yz}+\dfrac{y^2}{2y^2+zx}+\dfrac{z^2}{2z^2+xy}-1=0$,

if $x+y+z=0$.

3. Solve : (i) $\frac{1}{2}\left(x-\dfrac{a}{3}\right)-\frac{1}{3}\left(x-\dfrac{a}{4}\right)+\frac{1}{4}\left(x-\dfrac{a}{5}\right)=0$.

(ii) $\dfrac{2}{2x-5}+\dfrac{1}{x-3}=\dfrac{6}{3x-1}$

(iii) $\dfrac{1}{x+2}+\dfrac{3}{x+3}+\dfrac{5}{x+5}=\dfrac{9}{x+4}$

4. Simplify :—

$$\frac{y-z}{a^2-(y+z)a+yz}+\frac{z-x}{a^2-(z+x)a+zx}+\frac{x-y}{a^2-(x+y)a+xy}.$$

5. If $2s=a+b+c$, show that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

6. Extract the square root of $a^2+\dfrac{1}{a^2}-4\left(a+\dfrac{1}{a}\right)+6$.

7. A bag contains rupees, eight-anna pieces and four-anna pieces; the amount expressed by each kind is the same; if the total number of coins in the bag be 119, find the number of each.

8. Plot the points $(-3, 0)$, $(0-4)$, $(7, -8)$ and find the area of the triangle of which they are the vertices.

PAPER II.

1. Resolve into factors :

(i) $a^4+2a^3x-2ax^3-x^4$, (ii) $(x+1)(x+3)(x+5)(x+7)+15$

2. Solve (i) $\dfrac{2}{5(3x+4)}+\dfrac{4}{2x+3}=\dfrac{6}{3x+4}$.

(ii) $\dfrac{98x-73}{21}=\dfrac{14x-9}{9}-\dfrac{13x-16}{15x-9}$

3. Show that $\dfrac{1}{1+a}+\dfrac{2a}{1+a^2}+\dfrac{4a^3}{1+a^4}+\dfrac{8a^7}{1+a^8}=\dfrac{1}{1-a}-\dfrac{16a^{15}}{1-a^{16}}$.

4. If $2s=a+b+c$, show that

$s^2+(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)=bc+ca+ab$.

5. Simplify $\dfrac{x^3y-y^4}{xy^2+x^2y}\div\left(1+\dfrac{y}{x}\right)^2\div\dfrac{x^4+x^3y+x^2y^2}{(x^2-y^2)^3}$.

6. Extract the square-root of

$9x^8+24x^7+16x^6-30x^5-28x^4+16x^3+25x^2-20x+4$.

7. A number consists of 3 digits, the sum of which is 20; the first and the second together exceed the third by 4 and the first and the third together exceed the second by 14. Find the number.

8. Draw the graphs of (1) $2x-3y=3$, (2) $3x+2y=15$. Find the intercepts made by them on the axes.

PAPER III.

1. Resolve into factors

(i) $x^2+x-y^2-y-z^2-z-2yz$.

(ii) $bcx(y^2+z^2)+cay(z^2+x^2)+abz(x^2+y^2)$
$+xyz(a^2+b^2+c^2)+(ax+by+cz)(ayz+bzx+cxy)$.

2. Solve

(i) $\dfrac{21}{3x+4}+\dfrac{15}{3x-2}=\dfrac{20}{4x-5}+\dfrac{28}{4x+7}$.

(ii) $\dfrac{425}{5x+1}+\dfrac{14}{2x+1}+\dfrac{18}{3x+2}=\dfrac{392}{4x+1}$.

(iii) $\dfrac{x}{a+b}+\dfrac{y}{a-b}=2a$, $\quad\dfrac{x-y}{2ab}=\dfrac{x+y}{a^2+b^2}$.

3. If $xyz=1$, show that

$$\frac{1-x}{1+x}+\frac{1-y}{1+y}+\frac{1-z}{1+z}+\frac{1-x}{1+x}\cdot\frac{1-y}{1+y}\cdot\frac{1-z}{1+z}=0.$$

4. Find the H. C. F. of $18x^3+45x^2+63x+54$ and $24x^3+24x^2+6x+36$; and the L.C.M. of $6x^2-x-1$, $3x^2+7x+2$ and $2x^2+3x-2$.

5. Simplify :—(i) $\dfrac{1}{a-\dfrac{a^2+1}{a-\dfrac{1}{a-1}}}$. (ii) $\dfrac{x^2-\dfrac{y^2}{3}}{9\left(x^3+\dfrac{y^3}{27}\right)}-\dfrac{x-y}{(3x-y)^2+3xy}$.

6. Find the square root of $16a^3(a-2)-8a(1-3a)+1$.

7. A dealer marks an article with a price 20 per cent above the cost price and in selling he deducts 10 per cent from the price marked, and then obtains a profit of 16 shillings Find the cost price

8. Find the equation to a straight line passing through the point $(-3, 4)$ (1) perpendicular, (2) parallel, to $3x+4y=5$.

PAPER IV.

1. Slove

(i) $cx+ay=b(c+a)$, $ay+bz=c(a+b)$, $bz+cx=a(b+c)$.

(ii) $x+y+z=a+b+c$,

$bx+cy+az=cx+ay+bz=a^2+b^2+c^2$.

2. If $x+a$ be a common factor of x^2+px+q and x^2+lx+m, show that $a=\dfrac{m-q}{l-p}$.

3. Find the square root of $(x^2+y^2)(a^2+b^2)+2(ay+bx)(ax-by)$.

4. Resolve into factors :—

(i) $x^2(x+a^3)-a^3(1+x^2)$.

(ii) $(p+aq)(mb-nc)+(p+cq)(na-mb)$.

5. Find the condition that $8lx^3+4mx^2+64x+15$ may be divisible by $4x-3$ and $6x+1$ for all values of x.

6. Reduce to its simplest form

$$\frac{(p^2+q^2)(x+y)^2+2(px-qy)(qx-py)}{(p^2-q^2)(x^2+y^2)}$$

7. A man performed a journey of 913 miles. He travelled part of the way by rails at the rate of 60 miles per hour, part by coach at the rate of 20 miles, and the remainder on foot at the rate of 9 miles per hour. The times occupied by rail, by coach, and on foot were equal. How many hours did the journey take ?

8. Solve graphically $x+y=5$ and $2x-5y=3$.

Paper V.

1. Solve (i) $\dfrac{x-y}{3}-\dfrac{2y-3x}{6}=8,\ \dfrac{x}{6}-\dfrac{y}{3}=1.$

(ii) $2(y-z)=3x-2,\ y-3x=3z-1,\ 2y+3x=4(1-z).$

2. Extract the square root of

$$2y(y^2-x)(y-x)-(y^2-x^2)(y^2+1)+x(y+1)+\tfrac{1}{4}.$$

3. Simplify :—

(i) $$\frac{a(b-c)}{(a-x)(1-bx)(1-cx)}+\frac{b(c-a)}{(b-x)(1-ax)(1-cx)}+\frac{c(a-b)}{(c-x)(1-ax)(1-bx)}.$$

(ii) $$\left(y-\frac{a^2-yx}{y-x}\right)\left(x+\frac{a^2-yx}{x}\right)+\left(\frac{a^2-xy}{y-x}\right)^2.$$

4. Find the equation to the line joining the points (6, 8) and (−5, −12), and draw the line in a diagram.

From the diagram note the co-ordinates of two other points on the line, and verify that they satisfy the equation.

5. Find the values of a and b for which the expression $ax^3+bx^2-25x-6$ is divisible by x^2+x-6.

6. If $a+b+c=0$, prove that $(bc+ca+ab)^3+(a^2-bc)(b^2-ca)(c^2-ab)=0$. [M. F. A. 1888].

7. Multiply together the expressions: $1+ax+\frac{1}{2}a(a-1)x^2+\frac{1}{6}a(a-1)(a-2)x^3$ and $1+bx+\frac{1}{2}b(b-1)x^2+\frac{1}{6}b(b-1)(b-2)x^3$ as far as the term involving x^3 and resolve into factors the co-efficient of x^3 in the product. (B. M. 1897.)

8. An officer can form his men into a hollow square 9 deep. If he had 424 men more, he could form them into a hollow square 10 deep having 10 men more in each side. Find the number of men.

Paper VI.

1. If $x+y+z=0$, prove that

(i) $x(z-x)(x-y)+y(x-y)(y-z)+z(y-z)(z-x)+9xyz=0.$

(ii) $$\frac{x}{(x-y)(x-z)(y+z-x)}+\frac{y}{(y-z)(y-x)(z+x-y)}+\frac{z}{(z-x)(z-y)(x+y-z)}=0.$$

2. Find the H. C. F. and L. C. M. of

$$2x^4-7x^3+8x^2-10x+3 \text{ and } 3x^4-10x^3+8x^2-7x+2.$$

3. Resolve into factors

(i) $x^3+7x^2y-36y^3$. (ii) $2a^2-21ab-11b^2-a+34b-3$.

(iii) $(x+y)^2(z+a)^2+(x-y)^2(z-a)^2-(x^2-y^2)^2-(z^2-a^2)^2$.

4. Extract the square root of :—

$x^2(x^2+y^2+z^2)+2x(y+z)(yz-x^2)+y^2z^2$. (M. M. 1890.)

5. Solve (i) $\frac{1}{x+y}+\frac{1}{y+z}=\frac{8}{15}$,

$$\frac{1}{y+z}+\frac{1}{z+x}=\frac{9}{20},$$

$$\frac{1}{z+x}+\frac{1}{x+y}=\frac{7}{12}.$$

(ii) $\frac{1}{x+y+1}+\frac{1}{x+2y+3}=1$,

$$\frac{5}{x+2y+3}-\frac{3}{x+y+1}=1.$$

6. Plot the points $(8, 5)$, $(-9, 8)$ and $(3, -5)$. Find the area of the triangle formed by them (i) by calculation, (ii) counting squares in the diagram.

Solve graphically the equations $2x-3y=1$, $3x-4y=2$. Show graphically that the equation $5x-7y=3$ is consistent with them.

7. A boy bought a number of oranges for Rs. 2. Had he brought 8 more for the same money, he would have paid 4*p*. less for each. How many did he buy? (M.M. 1893.)

8. Solve

(i) $\frac{a}{ax+1}+\frac{b}{bx+1}=\frac{2c}{cx+1}$.

(ii) $\frac{8x+4y}{4y-3z}=\frac{x-3z}{y-2z}=\frac{x+2y-z}{8-z}=4$.

PAPER VII.

1. Compare the meaning of 23 in arithmetic with that of xy in algebra; also the meaning of $3\frac{5}{7}$ with that of $x\frac{y}{z}$.

2. Solve

(i) $\frac{2x-y}{7}-\frac{5x-2y}{4}=3x-5y+1$,

$x+y=11(x-y)$.

Solve

(ii) $3x+2y+5z=5x+3y+4z=1+x-y+z,$

$5x+3y+5z=2+2x-y.$

(iii) $\dfrac{x+a}{b+c}-\dfrac{a-b}{c}=\dfrac{x+b}{c+a}.$

3. The H.C.F. of two expressions is $x+2y$, and their L.C.M. is $x^4+5xy^3-6y^4$. One of the expressions is $x^3+x^2y+xy^2+6y^3$. What is the other ?

4. Prove that $(x+1)(x+2)(x+3)(x+4)+1$ is a perfect square. Of what number is it the square when $x=200$?

5. Solve graphically the following equations and verify the result :

$$\cdot 8x-4y=2\cdot 4$$

$$\cdot 04x+6\cdot 8y=-\cdot 02$$

6. If a number is equal to the sum of two perfect squares, shew by an algebraical relation that the square of the number is itself the sum of two perfect squares. (B. M. 1896.)

7. The price y in annas of a dish x inches in diameter is given in the following table :—

x	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8	$8\frac{1}{2}$
y	$7\frac{1}{4}$	$7\frac{3}{4}$	$8\frac{1}{4}$	$9\frac{1}{4}$	$11\frac{3}{4}$	$15\frac{1}{4}$	...	$25\frac{1}{2}$

Draw a graph and obtain an approximate price for the 8 in. dish.

8. I bought some pictures at 30s each and some books at 12s each. The total cost was £30. But the number of pictures is 6 more than the number of books. How many were there of each ?

PAPER VIII.

1. Find the square root of

(i) $4x^4+8xy^3-4x^3y-15x^2y^2+16y^4.$

(ii) $(a-b)^2(b-c)^2+(b-c)^2(c-a)^2+(c-a)^2(a-b)^2.$

2. Find the values of p and q for which $2x^4+px^3+qx^2+px+2$ is exactly divisible by x^2-2x-3.

3. Solve : (i) $a(x+y)+b(x-y)=2a,\ y(a+b)-x(a-b)=2b.$

(ii) $x=cy+bz,\ y=az+cx,\ z=bx+ay.$

4. Solve

$$ax+by+cz=0,\ cx+ay+bz=0,$$

$$bx+cy+az=a^3+b^3+c^3-3abc.$$

5. Prove the identity

$$(x^2+2x+7)^2+(2x^2+2x-4)^2+(2x^2-6x+4)^2=(3x^2-2x+9)^2$$

6. Draw graphs of the functions

$$\frac{x}{2\cdot 5} - 1\cdot 5 \text{ and } \frac{1\cdot 8 - 2x}{1\cdot 5}.$$

Read off (to one place of decimals) the value of x for which the functions have equal values.

7. Find graphically when the two hands of a watch are 15 minutes apart between 6 and 7 o'clock.

8. A train 60 yds. long passed another train 72 yds. long, which was travelling in the same direction on a parallel line of rails, in 12 seconds. Had the slower train been travelling half as fast again it would have been passed in 24 seconds. Find the rates at which the trains were travelling.

PAPER IX.

1. Find an expression containing no higher power of x than the first, which added to $x^4+6x^3+13x^2+6x+1$ will make it a complete square. (B. M. 1896.)

2. For what value of x is the expression $x^5-8x^3+11x^2+7x-1789$ exactly divisible by x^2+7x-1?

3. Simplify

$$\frac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2}+\frac{16z^2-(2x-3y)^2}{(3y+4z)^2-4x^2}+\frac{4x^2-(3y-4z)^2}{(4z+2x)^2-9y^2}.$$

(B. M. 1890.)

4. Solve the equations

(i) $$\frac{a-b}{x+1}+\frac{b-c}{x+2}=\frac{a-c}{x+3}.$$

(ii) $$\frac{a-b}{x}+\frac{a+b}{y}=\frac{2(a^2+b^2)}{a^2-b^2}, \quad \frac{a+b}{x}+\frac{a-b}{y}=2.$$

(B. M. 1888.)

5. Find the G. C. M. of $x^4+3x+20$ and $3x^5-2x^4+81x-85$.

6. Extract the square root of

$$9x^4-2x^3y+\frac{163}{9}x^2y^2-2xy^3+9y^4.$$

7. Draw the graphs of $3x=4y$, $2x-y=15$, $x+2y=30$, and prove that they represent three straight lines passing through the same point.

8. Show that if a number of two digits is four times the sum of its digits, the number formed by interchanging the digits is seven times their sum. (B. M. 1889.)

Paper X.

1. Find the cube root of

$$8x^9-12x^8+6x^7-37x^6+36x^5-9x^4+54x^3-27x^2-27.$$

(B. M. 1896).

2. Find the G. C. M. and L. C. M. of

$$x^5+x^4-4x^3+2x^2+6x-9,\ x^4-x^2+6x-9,$$
and $x^4+2x^3-5x^2-6x+9$. (B. M. 1886).

3. Simplify

$$\frac{(2k-3l)^2-k^2}{4k^2-(3l+k)^2}+\frac{4k^2-(3l-k)^2}{9(k^2-l^2)}+\frac{9l^2-k^2}{(2k+3l)^2-k^2}.$$

4. Resolve into factors $x^4-(p^2+2)x^2y^2+y^4$.

5. Extract the square root of

$$(a^2+b^2+c^2)(x^2+y^2+z^2)-(bz-cy)^2-(cx-az)^2-(ay-bx)^2.$$

6. Solve :—

(i) $(x+b+c-a)^3+(x+c+a-b)^3=2(x+c)^3$

(ii) $\dfrac{x-a}{b+c}+\dfrac{x-b}{c+a}+\dfrac{x-c}{a+b}=\dfrac{x+a}{b+c+2a}+\dfrac{x+b}{c+a+2b}+\dfrac{x+c}{a+b+2c}.$

7. Draw the triangle formed by the straight lines $x+3y+7=0$, $y-3x=1$, and $y+7x=11$; and find the co-ordinates of its angular points.

8. A man walks one-third of the distance from A to B at the rate of a miles per hour and the remainder at the rate of $2b$ miles per hour, and travelling back from B to A at the rate of $3c$ miles per hour takes the same time.

Prove that $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{c}$. (B. M. 1885).

CHAPTER XXII.

RATIO AND PROPORTION.

1. When two quantities **of the same kind** are compared as to their magnitude by considering what multiple (part or parts) the one is of the other, the relation between the quantities is called their **ratio.**

The ratio of a miles to b miles or of a maunds to b maunds may be written thus :— $a:b$ where a and b are called the **terms** of the ratio of which a is the **antecedent** and b the **consequent.**

It is evident that the ratio of two quantities is expressed by the *fraction*, of which the numerator is the measure of the first quantity and the denominator, that of the second quantity in terms

of some *common unit*. Thus the ratio of rupees 5 to rupees 7 is measured by the fraction $\frac{5}{7}$, the numerator and the denominator of the fraction being respectively the antecedent and the consequent of the ratio; and we may write $5:7=\frac{5}{7}$ Hence every question on ratios may be treated as a question on fractions.

2. A ratio is called a ratio of **greater inequality**, of **equality** or of **less inequality** according as the antecedent is greater than, equal to or less than the consequent. A ratio of greater inequality is sometimes called a **ratio of majority** and one of less inequality, a **ratio of minority**.

Thus $5:3$ and $9:7$ are ratios of greater inequality which correspond to improper fractions; $7:7$, $11:11$ are ratios of equality which are equal to unity; and $2:3$, $4:7$ are ratios of less inequality which correspond to proper fractions.

3. *A ratio is not altered if both its terms are multiplied or divided by the same quantity.*

For, $\frac{a}{b}=\frac{ma}{mb}$; $\therefore$ $a:b=ma:mb$.

4. A ratio of greater inequality is diminished and a ratio of less inequality is increased by adding the same positive quantity to both its terms,

Let $\frac{a}{b}$ be any ratio and let $\frac{a+x}{b+x}$ be the ratio formed by adding x to both its terms.

Now, $\frac{a+x}{b+x}-\frac{a}{b}=\frac{b(a+x)-a(b+x)}{b(b+x)}=\frac{x(b-a)}{b(b+x)}$.........(1)

If $\frac{a}{b}$ be a ratio of greater inequality, then $b-a$ is negative and since x, b, $b+x$ are all positive, hence from (1).

$\frac{a+x}{b+x}-\frac{a}{b}$ is negative or $\frac{a+x}{b+x}<\frac{a}{b}$.

If $\frac{a}{b}$ be a ratio of less inequality, then $b-a$ is positive; hence

$\frac{a+x}{b+x}-\frac{a}{b}$ is positive or $\frac{a+x}{b+x}>\frac{a}{b}$.

Note. We have the ratio $\frac{a+x}{b+x}=\dfrac{\frac{a}{x}+1}{\frac{b}{x}+1}$ and $\therefore$ the value of the ratio, when x increases more and more, tends to the limit 1. Hence *a ratio is made more nearly equal to unity by adding the same positive quantity to both its terms.*

5. A ratio of greater inequality is increased and a ratio of less inequality is diminished by taking from both its terms the same quantity which is less than either of the terms.

Let $\frac{a}{b}$ be any ratio and let $\frac{a-x}{b-x}$ be the ratio formed by subtracting x from both its terms, x being less than both a and b.

$$\text{Now } \frac{a-x}{b-x}-\frac{a}{b}=\frac{b(a-x)-a(b-x)}{b(b-x)}=\frac{x(a-b)}{b(b-x)} \quad \ldots\ldots\ldots(1)$$

If $\frac{a}{b}$ be a ratio of greater inequality, then $a-b$ is positive; and since x, b, $b-x$ are all positive, we have from (1),

$$\frac{a-x}{b-x}-\frac{a}{b} \text{ positive or } \frac{a-x}{b-x}>\frac{a}{b}.$$

If $\frac{a}{b}$ be a ratio of less inequality, then $a-b$ is negative, hence

$$\frac{a-x}{b-x}-\frac{a}{b} \text{ is negative or } \frac{a-x}{b-x}<\frac{a}{b}.$$

NOTE. If $\frac{a}{b}>1$, the resulting ratio $\frac{a-x}{b-x}$ is infinite when $x=b$, and if $\frac{a}{b}<1$, the ratio $\frac{a-x}{b-x}$ is zero when $x=a$.

6. If the antecedents of two or more ratios are multiplied together, so also the consequents, the ratio of the products is called the ratio **Compounded** of the given ratios.

Thus $ac : bd$ is compounded of the ratios $a : b$ and $c : d$.

The compound of two equal ratios is called the **duplicate ratio**, and that of three equal ratios, the **triplicate** ratio of the given ratios.

Thus $a^2 : b^2$ is the duplicate ratio of $a : b$ and $a^3 : b^3$ is the triplicate ratio of $a : b$.

The ratio $\sqrt{a} : \sqrt{b}$ is called the *sub-duplicate ratio* of $a : b$ and the ratio $\sqrt[3]{a} : \sqrt[3]{b}$, the *sub-triplicate ratio* of $a : b$.

Ex. 1. Find $x : y$ from $\frac{8}{3}x+\frac{7}{4}y=\frac{11}{9}x+\frac{9}{2}y$.

Dividing both sides by y,

$$\frac{8}{3}\frac{x}{y}+\frac{7}{4}=\frac{11}{9}\frac{x}{y}+\frac{9}{2}.$$

$$\therefore \frac{x}{y}\left(\frac{8}{3}-\frac{11}{9}\right)=\frac{9}{2}-\frac{7}{4}.$$

$$\therefore \frac{x}{y}\cdot\frac{13}{9}=\frac{11}{4}, \text{ or, } \frac{x}{y}=\frac{11}{4}\times\frac{9}{13}=\frac{99}{52}.$$

Hence $x : y = 99 : 52$.

Ex. 2. If $x : y = 3 : 2$, find the value of $\dfrac{5x+7y}{3x-2y}$.

We have $\dfrac{5x+7y}{3x-2y} = \dfrac{5\dfrac{x}{y}+7}{3\dfrac{x}{y}-2}$ [dividing top and bottom by y].

$$= \frac{5\times\frac{3}{2}+7}{3\times\frac{3}{2}-2} = \frac{29}{5}.$$

Ex. 3. What quantity must be added to each term of the ratio $a : b$ so that the resulting ratio may be equal to $c : d$?

Let x be the required quantity.

Then $a+x : b+x = c : d$ or $\dfrac{a+x}{b+x} = \dfrac{c}{d}$.

$\therefore d(a+x) = c(b+x)$, on multiplying cross-wise. Solving this equation in x we find,

$$x = \frac{ad-bc}{c-d}.$$

Ex. 4. The ratio of two numbers is as $3 : 4$ and if 8 be added to them they are in the ratio of $13 : 16$. Find the numbers.

Since the numbers are in the ratio of $3 : 4$, they can be represented by $3x$ and $4x$ respectively. Hence by the question $\dfrac{3x+8}{4x+8} = \dfrac{13}{16}$. Solving $x = 6$. Thus the numbers are 18 and 24.

EXERCISE XCVI.

1. Find $x : y$ from

   (1) $3x+9y = 5x+6y$. (2) $4x-5y : 3x+2y = 2 : 3$.

   (3) $ax+by : a'x+b'y = c : c'$.

2. If $x : y = 2 : 3$, find the value of

   (1) $2x+3y : 3x+2y$. (2) $4x+3y : 12x-7y$.

   (3) $\dfrac{3x-2y}{4x+3y}$. (4) $\dfrac{6x+5y}{3y-7x}$.

3. If $\dfrac{5x+2y}{3x-y} = \dfrac{?}{?}$, find the value of $4x+7y : 3x-5y$.

4. Find $x : y : z$ from

   (1) $3x-4y+5z=0$, $2x+3y-4z=0$.

   (2) $ax+by+cz=0$, $a^2x+b^2y+c^2z=0$.

5. Compare the ratios (1) $2 : 7$, $4 : 15$. (2) $5 : 6$, $13 : 16$.

6. Two armies number 11,000 and 7,000 men respectively; before they fight each is re-inforced by 1,000 men; in favour of which army is the increase? (C. E. 1879).

7. Find the ratio compound of

(1) $3 : 4, 8 : 9, 2 : 3$. (2) $x^2 - y^2 : a^2 - b^2, a + b : x + y$.

8. Find the ratio compounded of the duplicate ratio of $4 : 5$, the triplicate ratio of $5 : 8$ and the sub-duplicate ratio of $64 : 9$.

9. If $x + 2 : x - 2$ be in the duplicate ratio of $5 : 3$, find x.

10. What quantity must be added to the terms of the ratio $3 : 5$ to make the resulting ratio equal to $5 : 6$?

11. What quantity must be subtracted from the terms of the ratio $12 : 11$ to make the resulting ratio equal to $3 : 2$?

12. What must be subtracted from each term of $2 : 3$ so that the result may be in the duplicate ratio of $4 : 5$?

13. Find two numbers in the ratio of $5 : 3$ so that their difference may be 102.

14. Two numbers are in the ratio of $4 : 5$ and if 4 be added to them, they are in the ratio of $6 : 7$. Find the numbers.

15. Two numbers are in the ratio of $2 : 3$ and if 15 and 9 be added to them respectively they are in a ratio of equality; find the numbers.

16. Two numbers consisting of the same two digits are in the ratio of $4 : 7$, find the numbers. (P. M. 1896.)

17. A's present age to B's present age as $8 : 7$; 27 years ago their ages were as $5 : 4$. Find their present ages. (A. E. 1909).

18. Two vessels contain wine and water in the ration of 7 to 5 and 4 to 5 respectively. In what ratio must the liquids be taken from each vessel so as to give a mixture in the ratio of 6 to 7?

PROPORTION.

7. Four quantities are said to be **proportional** or **in proportion** when the ratio of the first to the second is equal to the ratio of the third to the fourth.

Thus if $a : b = c : d$, then a, b, c, d are proportional and this is read as "a is to b equals (or as) c is to d."

Instead of the sign of equality between the ratios the symbol $::$ (supposed to be the extremities of $=$) is also used, and in this notation the above relation will be $a : b :: c : d$.

If $a : b = c : d$, then a and d are called the **extremes** and b and c are called the **means**, while d is called the **fourth proportional to** a, b, c.

Four quantities are said to be **inversely proportional** when the first is to the second as the fourth is to the third.

Thus, a, b, c, d are inversely proportional if $a : b = d : c$.

8. Since the equality of two ratios means the equality of two fractions, we can deduce various properties of proportion from propositions regarding fractions; and we consider below some important ones.

(*1*). *If four quantities are proportional, the product of the extremes is equal to the product of the means; and conversely.*

$$\text{Let } a : b = c : d, \text{ then } \frac{a}{b} = \frac{c}{d}.$$

Multiplying both sides by bd,

$$\frac{a}{b} \times bd = \frac{c}{d} \times bd, \text{ whence } ad = bc.$$

Again let $ad = bc$.

Then dividing both sides by bd.

$$\frac{ad}{bd} = \frac{bc}{bd}, \text{ or, } \frac{a}{b} = \frac{c}{d}; \text{ i. e. } a : b = c : d.$$

Note. If three terms of a proportion are given, the remaining one can be immediately determined.

For, the four quantities a, b, c, d forming a proportion are connected by the relation $ad = bc$. Hence three of these being known, the fourth can be found by solving a simple equation.

(2) If $a : b = c : d$, then $b : a = d : c$.

From the given relation, $\frac{a}{b} = \frac{c}{d}$.

Hence $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$, or $\frac{b}{a} = \frac{d}{c}$; *i. e.* $b : a = d : c$.

This process is called **invertendo**.

(3) If $a : b = c : d$, then $a : c = b : d$.

From the given relation $\frac{a}{b} = \frac{c}{d}$.

Multiplying both sides by $\frac{b}{c}$,

$$\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}, \text{ or, } \frac{a}{c} = \frac{b}{d}, \text{ i. e. } a : c = b : d.$$

This process is called **alternando**.

(4) If $a : b = c : d$, the $a : a-b = c : c-d$

For, by invertendo, $\frac{b}{a} = \frac{d}{c}$.

$\therefore 1 - \frac{b}{a} = 1 - \frac{d}{c}$, or, $\frac{a-b}{a} = \frac{c-d}{c}$.

Hence by invertendo $\frac{a}{a-b} = \frac{c}{c-d}$, $\therefore a : a-b = c : c-d$.

This process is called **convertendo**.

(5) If $a : b = c : d$, then $a+b : b = c+d : d$.

For we have $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1$.

Hence $\frac{a+b}{b} = \frac{c+d}{d}$ or $a+b : b = c+d : d$

This process is called **componendo**.

(6) If $a : b = c : d$, then $a-b : b = c-d : d$.

For we have $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} - 1 = \frac{c}{d} - 1$.

Hence $\frac{a-b}{b} = \frac{c-d}{d}$ or $a-b : b = c-d : d$.

This process is called **dividendo**.

(7) If $a : b = c : d$, then $a+b : a-b = c+d : c-d$.

Since $a : b = c : d$, we have

$\frac{a+b}{b} = \frac{c+d}{d}$ by componendo;

$\frac{a-b}{b} = \frac{c-d}{d}$ by dividendo.

$\therefore \frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d}$;

or $\frac{a+b}{a-b} = \frac{c+d}{c-d}$, *i.e.*, $a+b : a-b = c+d : c-d$.

This process is called **componendo and dividendo** and has been independently proved before. (see p 192.)

Ex. 1. What quantity must be added to each of a, b, c, d to bring them into proportion?

Let x be the required quantity.

Then $a+x : b+x = c+x : d+x$, whence

$$(a+x)(d+x) = (b+x)(c+x).$$

$$\text{Solving } x=\frac{bc-ad}{a-b-c+d}.$$

Note.—If a, b, c, d are in proportion, then $bc=ad$ and x vanishes; and if $a+d=b+c$ (*i.e.*, a, b, c, d are in Arithmetical progression, as the student will see afterwards) then x is infinite.

Ex. 2. If $a:b=c:d$, and $a':b'=c':d'$, prove that

$$\frac{aa'+bb'}{aa'-bb'}=\frac{cc'+dd'}{cc'-dd'}.$$

$$\text{we have } \frac{a}{b}=\frac{c}{d} \text{ and } \frac{a'}{b'}=\frac{c'}{d'}.$$

$$\therefore \frac{a}{b}\times\frac{a'}{b'}=\frac{c}{d}\times\frac{c'}{d'} \text{ or } \frac{aa'}{bb'}=\frac{cc'}{dd'}.$$

$$\therefore \text{By comp. and divid. } \frac{aa'+bb'}{aa'-bb'}=\frac{cc'+dd'}{cc'-dd'}.$$

Ex. 3. If $a:b=c:d$, prove that

$$\frac{a+2b+c+2d}{a+2b-c-2d}=\frac{a-2b+c-2d}{a-2b-c+2d}.$$

$$\because \frac{a}{b}=\frac{c}{d},\ \therefore \frac{a}{2b}=\frac{c}{2d}.$$

$$\text{by comp. and divid. } \frac{a+2b}{a-2b}=\frac{c+2d}{c-2d}.$$

$$\therefore \text{by alternando, } \frac{a+2b}{c+2d}=\frac{a-2b}{c-2d}.$$

Again, by comp. and divid.

$$\frac{a+2b+c+2d}{a+2b-c-2d}=\frac{a-2b+c-2d}{a-2b-c+2d}.$$

Ex. 4. If $7a+11b:7a-11b=7c+11d:7c-11d$, prove that a, b, c, d are in proportion.

$$\text{We have } \frac{7a+11b}{7a-11b}=\frac{7c+11d}{7c-11d}.$$

$\therefore$ By comp. and divid.

$$\frac{(7a+11b)+(7a-11b)}{(7a+11b)-(7a-11b)}=\frac{(7c+11d)+(7c-11d)}{(7c+11d)-(7c-11d)}.$$

$$\therefore \frac{14a}{22b}=\frac{14c}{22d} \text{ or } \frac{a}{b}=\frac{c}{d}.$$

Hence a, b, c, d are in proportion.

Ex. 5. If $(ab+ca)^2=(a^2+c^2)(b^2+d^2)$, prove that a, b, c, d are in proportion.

Multiplying out, $a^2b^2+c^2d^2+2abcd=a^2b^2+a^2d^2+b^2c^2+c^2d^2$.

$\therefore\ 2abcd=a^2d^2+b^2c^2$,

or $a^2d^2+b^2c^2-2abcd=0$, or $(ad-bc)^2=0$

$\therefore\ ad-bc=0$, or, $ad=bc$.

$\therefore\ a, b, c, d$ are proportional.

EXERCISE XCVII.

1. Find a fourth proportional to

(1) 2, 3, 4. (2) $a+b, a-b, a^2-b^2$.

2. The last three terms of a proportion being 7, 4, 14, find the first term.

3. What number must be added to 2, 12, 6, 26 to obtain four proportionals?

4. What number must be subtracted from 7, 10, 13, 19 to obtain four proportionals?

5. If the ratio $a : b$ is unaltered when x and y are respectively added to its terms, then $x : y=a : b$.

6. If the ratio $a : b$ gives the same result when x and y are respectively added to its terms and when y and x are added, prove that either $x=y$ or $x+y+a+b=0$.

7. If $a+b : b+c=c+d : d+a$, prove that $a=c$ or $a+b+c+d=0$. (C. E. 1891).

8. If $x=\dfrac{ab}{a^2+b^2}$, find the value of $\dfrac{ax+b}{ax-b}-\dfrac{bx+a}{bx-a}$.

If $a : b=c : d$ prove that

9. $9a+10b : 9a-10b=9c+10d : 9c-10d$.

10. $3a+2b+6c+4d : 3a+2b-6c-4d=3a-2b+6c-4d : 3a-2b-6c+4d$.

11. If $3a+4b : 3a-4b=3c+4d : 3c-4d$, prove that a, b, c, d are in proportion.

12. If $2a+5b : 2a-5b=2c+5d : 2c-5d$, then $7a+6b : 7a-6b=7c+6d : 7c-6d$.

---

9. The following theorem concerning equal ratios is very important and has universal application.

If $a : b=c : d=e : f=$etc., then each of these ratios $=\sqrt[n]{\left(\dfrac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right)}$ where $n, p, q, r\ldots$are any quantities whatever.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=k$,

so that $a=bk$, $c=dk$, $e=fk$,...

Then $pa^n+qc^n+re^n+\cdots=p(bk)^n+q(dk)^n+r(fk)^n+\ldots$

$$=pb^nk^n+qd^nk^n+rf^nk^n+\ldots$$

$$=k^n(pb^n+qd^n+rf^n+\ldots)$$

$$\therefore \sqrt[n]{(pa^n+qc^n+re^n+\ldots)}=k\sqrt[n]{(pb^n+qd^n+rf^n+\ldots)};$$

or $\sqrt[n]{\left(\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right)}=k=$ each of the given ratios.

The following important particular cases of the theorem should be noted and may be proved by the same method.

(i) If $\frac{a}{b}=\frac{c}{d}$, then each $=\frac{a+c}{b+d}=\frac{a-c}{b-d}$.

Thus, *if two fractions are equal then each*

$$=\frac{\textit{sum of numerators}}{\textit{sum of denominators}}=\frac{\textit{diff. of numerators}}{\textit{diff. of denominators}}.$$

(ii) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots$, then each

$$=\frac{a+c+e+\ldots}{b+d+f+\ldots}$$

Thus, *if any number of fractions are equal, then each is equal to a fraction of which the numerator is equal to the sum of the numerators and the denominator is equal to the sum of the denominators of the given fractions* (see also art. 18, Chap. XVI).

10. If $a:b=c:d=e:f=\ldots$, then each

$=\sqrt[n]{\frac{ace\ldots}{bdf\ldots}}$ where n is the number of ratios.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=k$.

Then $\frac{ace\ldots}{bdf\ldots}=\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}\times\ldots$

$=k\times k\times k\ldots$to n factors.

$=k^n$.

$\therefore \sqrt[n]{\frac{ace\ldots}{bdf\ldots}}=k=$ each of the given ratios.

The principle of the article will be readily admitted by the student, for it is only a particular case of the following :

If n quantities be equal, then each is equal to the n th root of their product.

Ex. 1. If $a : b = c : d$, then each $= \sqrt{\dfrac{ac}{bd}} = \dfrac{a+c}{b+d}$.

Let $\dfrac{a}{b} = \dfrac{c}{d} = k$, so that $a = bk$, $c = dk$.

Then $\dfrac{a+c}{b+d} = \dfrac{bk+dk}{b+d} = k$;

$\sqrt{\dfrac{ac}{bd}} = \sqrt{\dfrac{bk.dk}{bd}} = \sqrt{k^2} = k$.

Hence the result to be proved follows.

Ex. 2. If $a : b = c : d$, then

$$\frac{la^n + mb^n}{lc^n + md^n} = \frac{pa^n + qb^n}{pc^n + qd^n}.$$

Let $\dfrac{a}{b} = \dfrac{c}{d} = k$, so that $a = bk$, $c = dk$.

Then $\dfrac{la^n + mb^n}{lc^n + md^n} = \dfrac{l.b^nk^n + mb^n}{l.d^nk^n + md^n} = \dfrac{b^n(lk^n+m)}{d^n(lk^n+m)} = \dfrac{b^n}{d^n}$.

Also $\dfrac{pa^n + qb^n}{pc^n + qd^n} = \dfrac{p.b^nk^n + qb^n}{p.d^nk^n + qd^n} = \dfrac{b^n(pk^n+q)}{d^n(pk^n+q)} = \dfrac{b^n}{d^n}$.

Hence the equality follows

Ex. 3. If $a : b = c : d = e : f$, then

$$\frac{2a+3c+4e}{2b+3d+4f} = \frac{3a+4c+5e}{3b+4d+5f}.$$

Let $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = k$; so that $a = bk$, $c = dk$, $e = fk$.

$\therefore \dfrac{2a+3c+4e}{2b+3d+4f} = \dfrac{2bk+3dk+4fk}{2b+3d+4f} = \dfrac{k(2b+3d+4f)}{2b+3d+4f} = k$.

Also $\dfrac{3a+4c+5e}{3b+4d+5f} = \dfrac{3bk+4dk+5fk}{3b+4d+5f} = \dfrac{k(3b+4d+5f)}{3b+4d+5f} = k$.

Hence the result follows.

Otherwise thus, without using the "k" method :—

From the given relation we have each ratio

$$= \frac{2a}{2b} = \frac{3c}{3d} = \frac{4e}{4f}$$

$$= \frac{\text{sum of numerators}}{\text{sum of denominators}} = \frac{2a+3c+4e}{2b+3d+4f}.$$

Again, each ratio $=\frac{3a}{3b}=\frac{4c}{4d}=\frac{5e}{5f}$

$$=\frac{\text{sum of numerators}}{\text{sum of denominators}}=\frac{3a+4c+5e}{3b+4d+5f}.$$

$$\therefore \frac{2a+3c+4e}{2b+3d+4f}=\frac{3a+4c+5e}{3b+4d+5f}.$$

The second method is more elegant than the first and can be used only after a little experience. The advantage of the first method lies in the fact that it will never fail.

Ex. 4. If $a:b=c:d=e:f$, prove that

$$\frac{ac+ce+ea}{bd+df+fb}=\frac{a^2+c^2+e^2}{b^2+d^2+f^2}.$$

Denoting the equal ratios by k, we have

$$k^2=\frac{a}{b}\times\frac{c}{d}=\frac{c}{d}\times\frac{e}{f}=\frac{e}{f}\times\frac{a}{b}$$

$$i.e., =\frac{ac}{bd}=\frac{ce}{df}=\frac{ea}{fb} \text{ and } \therefore =\frac{ac+ce+ea}{bd+df+fb}.$$

$$\text{Again } k^2=\frac{a^2}{b^2}=\frac{c^2}{d^2}=\frac{e^2}{f^2} \text{ and } \therefore =\frac{a^2+c^2+e^2}{b^2+d^2+f^2}.$$

Hence the result follows.

Ex. 5. If $x:a=y:b=z:c$, prove that

$$\frac{x^2+y^2+z^2}{a^2+b^2+c^2}=\left(\frac{lx+my+nz}{la+mb+nc}\right)^2.$$

Denoting the equal ratios by k, we have

$$k^2=\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2} \text{ and } \therefore =\frac{x^2+y^2+z^2}{a^2+b^2+c^2}.$$

Again we have k

$$=\frac{lx}{la}=\frac{my}{mb}=\frac{nz}{nc} \text{ and } \therefore =\frac{lx+my+nz}{la+mb+nc}.$$

$$\therefore \frac{x^2+y^2+z^2}{a^2+b^2+c^2}=\left(\frac{lx+my+nz}{la+mb+nc}\right)^2,$$ each being equal to k^2.

EXERCISE XCVIII.

1. If $a:b=c:d$, prove that

(1) $3a+4b:3c+4d=7a+11b:7c+11d.$

(2) $la+mb:lc+md=pa+qb:pc+qd.$

(3) $ma+nb:mc+nd=b^2c:d^2a.$ (C.E. 1876).

(4) $ma+nc:mb+nd=\sqrt{(a^2+c^2)}:\sqrt{(b^2+d^2)}.$

(5) $a^2+b^2 : a^2-b^2=ac+bd : ac-bd.$ (C.E. 1879).

(6) $a^2+c^2 : b^2+d^2=ac : bd.$ (C.E. 1877).

(7) $3a+5c : 3b+5d=\sqrt{(5a^2-7c^2)} : \sqrt{(5b^2-7d^2)}.$

(8) $4a^3+5c^3 : 4b^3+5d^3=3a^2c+7ac^2 : 3b^2d+7bd^2.$

2. If $a : b=c : d$, prove the equalities :—

(1) $a^2d-bc^2=ac(b-a).$ (C.E. 1890).

(2) $(a^2+c^2)(b^2+d^2)=(ab+cd)^2.$ (A. E. 1890).

(3) $\dfrac{a^2}{b^2}+\dfrac{c^2}{d^2}=\dfrac{2ac}{bd}.$ (4) $\dfrac{a^3}{b^3}+\dfrac{c^3}{d^3}=\dfrac{a^2c}{b^2d}+\dfrac{ac^2}{bd^2}.$

(5) $\dfrac{(a-b)(a-c)}{a}=(a+d)-(b+c).$ (B. P. 1890).

3. If $a : b=c : d=e : f$, prove that

(1) $\dfrac{5a+6c+7e}{5b+6d+7f}=\dfrac{3a+5c+6e}{3b+5d+6f}.$

(2) $\dfrac{a^2+2c^2+3e^2}{b^2+2d^2+3f^2}=\dfrac{2ac+4ce+5ea}{2bd+4df+5fb}.$

(3) $\dfrac{pa^3+qc^3+re^3}{pb^3+qd^3+rf^3}=\dfrac{ace}{bdf}.$

(4) $(a^2+c^2+e^2)(b^2+d^2+f^2)=(ab+cd+ef)^2.$

4. If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$ prove that

(1) $\dfrac{x^3}{a^2}+\dfrac{y^3}{b^2}+\dfrac{z^3}{c^2}=\dfrac{(x+y+z)^3}{(a+b+c)^2}.$

(2) $\dfrac{xyz}{abc}=\left(\dfrac{x+y+z}{a+b+c}\right)^3=\dfrac{x^3+y^3+z^3}{a^3+b^3+c^3}.$

5. If $a : b=c : d=e : f=g : h$, prove that

(1) $\dfrac{ac+ce+eg+cg}{bd+df+fh+dh}=\dfrac{a^2+c^2+e^2+g^2}{b^2+d^2+f^2+h^2}.$

(2) $\dfrac{ace+ceg+gea}{bdf+dfh+hfb}=\dfrac{a^3+c^3+e^3+g^3}{b^3+d^3+f^3+h^3}.$

(3) $(a^2+c^2+e^2+g^2)(b^2+d^2+f^2+h^2)=(ab+cd+ef+gh)^2.$

---

11. Quantities are said to be in **Continued proportion** when the ratio of the first to the second, the ratio of the second to the third, the ratio of the third to the fourth and so on, are equal.

Thus a, b, c, d...are in continued proportion if

$$a : b=b : c=c : d=\cdots$$

When three quantities a, b, c are in continued proportion then b is called the **mean proportional** between a and c, and c the **third proportional** to a and b.

When four quantities a, b, c, d are in continued proportion, then b and c are called **two mean proportionals** between a and d;

Note. It should be noted that a number of quantities in continued proportion may be expressed by ka, ka^2, $ka^3\cdots$

12. *If three quantities are in continued proportion the product of the extremes is equal to the square of the mean; and conversely.*

For if a, b, c are in continued proportion, then $\frac{a}{b}=\frac{b}{c}$.

$\therefore\ \frac{a}{b}\times bc=\frac{b}{c}\times bc$, or, $ac=b^2$.

Again, if $ac=b^2$, then $\frac{ac}{bc}=\frac{b^2}{bc}$ or $\frac{a}{b}=\frac{b}{c}$.

$\therefore\ a$, b, c are in continued proportion.

Ex. 1. *If three quantities are in continued proportion the first is to the third in the duplicate ratio of the first to the second.*

Let a, b, c be in continued proportion.

Then $\frac{a}{b}=\frac{b}{c}=k$ (suppose).

$\therefore\ k^2=\frac{a}{b}\times\frac{b}{c}=\frac{a}{c}$, and also $k^2=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}$.

$\therefore\ \frac{a}{c}=\frac{a^2}{b^2}$ and $\frac{a^2}{b^2}$ is the duplicate ratio of $a:b$ Hence the proposition.

Ex. 2. Find a mean proportional betweeu a and b.

Let x be the mean proportional between a and b.

Then a, x, b are in continued proportion;

or $x^2=ab$ (Art. 12). $\therefore\ x=\sqrt{ab}$.

Ex. 3. If a, b, c are in continued proportion, prove that

$$\frac{2a+5b+3c}{2a-b-3c}=\frac{2a+b-3c}{2a-5b+3c}.$$

Let $\frac{a}{b}=\frac{b}{c}=k$; then $b=ck$, $a=bk=ck^2$.

Substituting, left side $=\dfrac{2ck^2+5ck+3c}{2ck^2-ck-3c}$

$$=\frac{2k^2+5k+3}{2k^2-k-3}=\frac{(2k+3)(k+1)}{(2k-3)(k+1)}$$

$$=\frac{2k+3}{2k-3} \qquad \ldots \quad (1)$$

Also right side $=\dfrac{2ck^2+ck-3c}{2ck^2-5ck+3c}=\dfrac{2k^2+k-3}{2k^2-5k+3}$

$$=\frac{(2k+3)(k-1)}{(2k-3)(k-1)}=\frac{2k+3}{2k-3}. \qquad \ldots \quad (2)$$

The result follows from (1) and (2).

Ex. 4. If a, b, c, d are in continued proportion, prove that $(ab+bc+cd)^2=(a^2+b^2+c^2)(b^2+c^2+d^2)$ (C. E. 1887).

Let $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}=k$;

then $a=bk$, $b=ck$, $c=dk$.

$\therefore$ the left hand side $=(bk.\ b+ck.\ c+dk.\ d)^2$.

$$=k^2(b^2+c^2+d^2)^2.$$

Right hand side $=(b^2k^2+c^2k^2+d^2k^2)(b^2+c^2+d^2)$

$$=k^2(b^2+c^2+d^2)^2.$$

Hence the result to be proved follows.

Ex. 5. If a, b, c, d are in continued proportion, then $a : d=pa^3+qb^3+rc^3 : pb^3+qc^3+rd^3$.

We have $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}=k$ (suppose).

Then $k^3=\dfrac{a}{b}\times\dfrac{b}{c}\times\dfrac{c}{d}=\dfrac{a}{d}$.

Also $k^3=\dfrac{pa^3}{pb^3}=\dfrac{qb^3}{qc^3}=\dfrac{rc^3}{rd^3}=\dfrac{\text{sum of numerators}}{\text{sum of denominators}}$

$$=\frac{pa^3+qb^3+rc^3}{pb^3+qc^3+rd^3}$$

$$\therefore \frac{a}{d}=\frac{pa^3+qb^3+rc^3}{pb^3+qc^3+rd^3}.$$

EXERCISE XCIX.

1. Find a third proportional to
(1) 4 and 8. (2) 9 and 3. (3) $a+b$ and $a-b$.

2. Find a mean proportional between

(1) 4 and 9. (2) $a^7 b^4$ and $a^4 b^2$. (3) $a+b$ and a^3+b^3.

3. Find to two decimal places a mean proportional between 2 and 3.

4. If $x : y = y : z$, find the simplest value of

$$\frac{xyz\,(x+y+z)^3}{(xy+yz+zx)^3}.$$ (C. E. 1892).

5. If $(x^2-y^2)\,z=(y^2-z^2)\,x$, show that x is to z in the duplicate ratio of x and y. (C. F. 1867).

6. If y is a mean propotional between x and z, show that $xy+yz$ is a mean propotional between x^2+y^2 and y^2+z^2.

7. What must be added to a, b, c to bring them into continued proportion ?

8. If a, b, c are in continued proportion, prove that

(1) $3a^2+5ab+7b^2 : 3b^2+5bc+7c^2=a : c$.

(2) $a^4+b^4+c^4=(a+b+c)\,(a-b+c)\,(a^2-b^2+c^2)$.

(3) $a^4+a^2b^2+b^4=a^2\,(a+b+c)\,(a-b+c)$.

(4) $(ab+bc)^2=(a^2+b^2)\,(b^2+c^2)$.

9. If a, b, c are in continued proportion, prove that

$(3a+8b+5c)(4a-11b+7c)=(3a+2b-5c)(4a-3b-7c)$.

10. If a, b, c, d are in continued proportion, prove that

(1) $a : d=a^3 : b^3$. (C. E. 1887 and 1902).

(2) $(ab+cd)\,(b^2-d^2)=(ab-cd)\,(b^2+d^2)$.

(3) $(a-d)^2=(b-c)^2+(a-c)^2+(b-d)^2$.

(4) $(b+c)^2=(a+b)\,(c+d)$.

(5) $(b-c)^2=(a-c)\,(b-d)-(a-d)\,(b-c)$.

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13. We add some miscellaneous examples on proportion.

Ex. 1. If a, b, c, d are in proportion, prove that

$$\frac{a^2+b^2+c^2+d^2}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}}=abcd.$$

We have $ad=bc=k$ (suppose).

$$\therefore\ a=\frac{k}{d};\ b=\frac{k}{c};\ c=\frac{k}{b};\ d=\frac{k}{a}.$$

Hence squaring and adding,

$$a^2+b^2+c^2+d^2=k^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right),$$

or $\dfrac{a^2+b^2+c^2+d^2}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}}=k^2=ad.\ bc=abcd.$

Ex. 2. If $2a+3b : 2c+3d=5a+7b : 5c+7d$, prove that a, b, c, d are proportional.

Let $\dfrac{2a+3b}{2c+3d}=\dfrac{5a+7b}{5c+7d}=k.$

Then $k=\dfrac{10a+15b}{10c+15d}=\dfrac{10a+14b}{10c+14d}=\dfrac{\text{difference of numerators}}{\text{difference of denominators}}=\dfrac{b}{d}.$

Again, $k=\dfrac{14a+21b}{14c+21d}=\dfrac{15a+21b}{15c+21d}=\dfrac{\text{diff. of numerators}}{\text{diff. of denominators}}=\dfrac{a}{c}.$

$\therefore\ \dfrac{a}{c}=\dfrac{b}{d}$, or, $\dfrac{a}{b}=\dfrac{c}{d}.$

The multipliers are so chosen that once on subtraction of the numerators a is destroyed and again on subtraction b is destroyed. The ordinary proof may be given thus :—

From the given relation, $(2a+3b)(5c+7d)=(2c+3d)(5a+7b)$.

$\therefore\ 10ac+14ad+15bc+21bd=10ac+14bc+15ad+21bd.$

Simplifying $bc=ad$, or, a, b, c, d are in proportion.

Ex. 3. If $(x^2+y^2+z^2)(a^2+b^2+c^2)=(ax+by+cz)^2$,

prove that $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}.$

From the given relation, we have

$b^2x^2+c^2x^2+a^2y^2+c^2y^2+a^2z^2+b^2z^2=2abxy+2bcyz+2cazx.$

Transposing,

$(b^2x^2+a^2y^2-2abxy)+(c^2y^2+b^2z^2-2bcyz)+(a^2z^2+c^2x^2-2cazx)=0.$

$\therefore\ (bx-ay)^2+(cy-bz)^2+(az-cx)^2=0.$

But each term on the left being a square is positive; hence the sum cannot be zero unless *each* term is zero.

$\therefore\ bx-ay=0$ or $\dfrac{x}{a}=\dfrac{y}{b}$; $\quad cy-bz=0$ or $\dfrac{y}{b}=\dfrac{z}{c}$;

$az-cx=0$ or $\dfrac{z}{c}=\dfrac{x}{a}$. $\quad\therefore\ \dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}.$

Ex. 4. *If $a:b=c:d$, show that*

$$4(a+b)\ (c+d)=bd\left\{\frac{a+b}{b}+\frac{c+d}{d}\right\}^2. \quad (C.\ E.\ 1874.)$$

From the given relation by componendo,

$$\frac{a+b}{b}=\frac{c+d}{d}=k\ (suppose).$$

$$\therefore \left(\frac{a+b}{b}+\frac{c+d}{d}\right)^2=(2k)^2$$
$$=2k.\ 2k$$
$$=\frac{2(a+b)}{b}.\ \frac{2(c+d)}{d}$$
$$=\frac{4(a+b)(c+d)}{bd}.$$

$$\therefore bd\left(\frac{a+b}{b}+\frac{c+d}{d}\right)^2=4(a+b)(c+d).$$

Ex. 5. If a, b, c, are in continued proportion, prove that $a^{2n}+b^{2n}+c^{2n}=(a^n+b^n+c^n)(a^n-b^n+c^n)$.

From the given relation $b^2=ac$; hence $(b^2)^n=(ac)^n$ or $b^{2n}=a^nc^n$.

$$\therefore a^{2n}+b^{2n}+c^{2n}$$
$$=a^{2n}+a^nc^n+c^{2n}$$
$$=a^{2n}+2a^nc^n+c^{2n}-b^{2n} \quad \left\{\begin{array}{l}\text{adding } a^nc^n \text{ and subtracting its}\\ \text{equivalent } b^{2n}.\end{array}\right.$$
$$=(a^n+c^n)^2-(b^n)^2$$
$$=(a^n+b^n+c^n)\ (a^n-b^n+c^n).$$

Ex. 6. If $\dfrac{x}{b+c}=\dfrac{y}{c+a}=\dfrac{z}{a+b}$, prove that

$$y+z-x \quad z+x-y \quad x+y-z \qquad \text{(C. E. 1903).}$$

Put each of the equal ratios $=k$.

Then $x=k(b+c)$, $y=k(c+a)$, $z=k(a+b)$.

$$\therefore y+z-x=k.\ 2a, \text{ or } \frac{a}{y+z-x}=\frac{1}{2k};$$

$$z+x-y=k.\ 2b, \text{ or } \frac{b}{z+x-y}=\frac{1}{2k};$$

$$x+y-z=k.\ 2c, \text{ or } \frac{c}{x+y-z}=\frac{1}{2k}.$$

Hence $\dfrac{a}{y+z-x}=\dfrac{b}{z+x-y}=\dfrac{c}{x+y-z}$.

Ex. 7. If $\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}$, find the value of

$$(b-c)x+(c-a)y+(a-b)z.$$

Put each of the ratios $=k$;

then $x=k(b+c)$, $y=k(c+a)$, $z=k(a+b)$.

$\therefore\quad (b-c)x=k(b^2-c^2),\ (c-a)y=k(c^2-a^2),\ (a-b)z=k(a^2-b^2)$

$\therefore\quad (b-c)x+(c-a)y+(a-b)z=k\{(b^2-c^2)+(c^2-a^2)+(a^2-b^2)\}$
$$=k\times o=o.$$

Ex. 8. If $\frac{a(b-c)}{b+c-a}=\frac{b(c-a)}{c+a-b}=\frac{c(a-b)}{a+b-c}$, then each of the ratios is zero, unless $a+b+c=o$.

We have each ratio $=\frac{\text{Sum of numerators}}{\text{Sum of denominators}}$

$$=\frac{a(b-c)+b(c-a)+c(a-b)}{(b+c-a)+(c+a-b)+(a+b-c)}=\frac{o}{a+b+c}.$$

Hence each ratio is o if $a+b+c$ is not zero. If however $a+b+c=o$, then each ratio is $\frac{o}{o}$ or indeterminate. (See Chap. XXV).

Ex. 9. If $x=\frac{\sqrt{(a+b)}+\sqrt{(a-b)}}{\sqrt{(a+b)}-\sqrt{(a-b)}}$, then $bx^2-2ax+b=o$.

By comp. and divid. $\frac{x+1}{x-1}=\frac{2\sqrt{(a+b)}}{2\sqrt{(a-b)}}=\frac{\sqrt{(a+b)}}{\sqrt{(a-b)}}$

Squaring $\frac{x^2+2x+1}{x^2-2x+1}=\frac{a+b}{a-b}$.

Again by comp. and divid. $\frac{2(x^2+1)}{2.2x}=\frac{2a}{2b}$, or $\frac{x^2+1}{2x}=\frac{a}{b}$;

whence $bx^2-2ax+b=o$

EXERCISE C.

1. If a, b, c, d are in proportion, prove that

(1) $a^4+b^4+c^4+d^4 : 1/a^4+1/b^4+1/c^4+1/d^4=a^2b^2c^2d^2$.

(2) $a^n+b^n+c^n+d^n : 1/a^n+1/b^n+1/c^n+1/d^n=(abcd)^n/^2$.

(3) $\left(\frac{a+2c}{b+2d}+\frac{2a+c}{2b+d}\right)^2=\frac{4ac}{bd}$.

2. If $\frac{x}{a}=\frac{y}{b}$, prove that

(1) $\frac{x^2+a^2}{x+a}+\frac{y^2+b}{y+b}=\frac{(x+y)^2+(a+b)^2}{(x+y)+(a+b)}$ (A. E. 1899).

(2) $\dfrac{x^3+a^3}{x^2+a^2}+\dfrac{y^3+b^3}{y^2+b^2}=\dfrac{(x+y)^3+(a+b)^3}{(x+y)^2+(a+b)^2}$.

3. If $a : b=c : d$ and $p : q=r : s$, prove that

$ap+cr : bq+ds=\sqrt{acpr} : \sqrt{bdqs}$ (A. E. 1896).

$=ar+cp : bs+dq$.

4. If $3a+7b : 3c+7d=2a+9b : 2c+9d$, prove that a, b, c, d are in proportion.

5. If $la+mb : pa+qb=lc+md : pc+qd$, then a, b, c, d are proportionals.

6. If $4a+5b : 4c+5d=2a+11b : 2c+11d$, prove

(1) $a+b : c+d=\sqrt{a^2+b^2} : \sqrt{c^2+d^2}$.

(2) $a^2+c^2 : b^2+d^2=\sqrt{a^4+c^4} : \sqrt{b^4+d^4}$.

7. If $(2a+5b+3c)\ (2a-5b+3c)=(2a+b-3c)(2a-b-3c)$, prove that a, b, c, are in continued proportion.

8. If $a : b=c : d=e : f$, prove that

(1) $\sqrt{(a+c+e)(b+d+f)}=\sqrt{ab}+\sqrt{cd}+\sqrt{ef}$.

(2) $27(a+b)(c+d)(e+f)=bdf\left(\dfrac{a+b}{b}+\dfrac{c+d}{d}+\dfrac{e+f}{f}\right)^3$.

(3) $\left(\dfrac{a^2+c^2}{ab+cd}+\dfrac{c^2+e^2}{cd+ef}\right)^2=4\left(\dfrac{a+c}{b+d}\right)\left(\dfrac{c+e}{d+f}\right)$.

9. If a, b, c are in continued proportion, prove that

(1) $a^2-b^2+c^2=b^4(1/a^2-1/b^2+1/c^2)$

(2) $a^3+b^3+c^3=a^2b^2c^2(1/a^3+1/b^3+1/c^3)$.

10. If a, b, c, d are in continued proportion, prove that

(1) $ac^2 : bd^2=abc+a^2b+b^2c : bcd+b^2c+c^2a$.

(2) $\sqrt{(ab)}+\sqrt{(bc)}+\sqrt{(cd)}=\sqrt{(a+b+c)(b+c+d)}$.

(3) $3a^3+5ab^2+7bc^2 : 3b^3+5bc^2+7cd^2=bc^2 : cd^2$.

11. If $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}$, find the value of

$(b-c)x+(c-a)y+(a-b)z$. (C. E. 1878)

12. If $\dfrac{x}{b-c}=\dfrac{y}{c-a}=\dfrac{z}{a-b}$, find the value of

(1) $x+y+z$, (2) $ax+by+cz$

(3) $(b+c)x+(c+a)y+(a+b)z$.

13. If $\dfrac{a}{b+c-a}=\dfrac{b}{c+a-b}=\dfrac{c}{a+b-c}$, then each ratio $=1$, and hence prove that $a=b=c$.

14. If $\dfrac{a+b}{b+c-a}=\dfrac{b+c}{c+a-b}=\dfrac{c+a}{a+b-c}$, then each ratio $=2$.

15. If $\dfrac{x}{ax+by+cz}=\dfrac{y}{bx+cy+az}=\dfrac{z}{cx+ay+bz}$, show that each of these $=\dfrac{1}{a+b+c}$, supposing $x+y+z$ is not equal to zero. (C. E. 1902).

16. If $\dfrac{a+bx}{b+cy}=\dfrac{b+cx}{c+ay}=\dfrac{c+ax}{a+by}$, then each $=\dfrac{1+x}{1+y}$, unless $a+b+c$ is equal to zero.

17. If $\dfrac{b+c-2a}{x}=\dfrac{c+a-2b}{y}=\dfrac{a+b-2c}{z}$, then each $=0$ unless $x+y+z=0$.

18. If $\dfrac{bz-cy}{b-c}=\dfrac{cx-az}{c-a}$, then each $=\dfrac{ay-bx}{a-b}$.

19. If $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}$, prove that

$$\frac{a}{y+z}=\frac{b}{z+x}=\frac{c}{x+y}.$$

20. If $\dfrac{x}{b+c}=\dfrac{y}{c+a}=\dfrac{z}{a+b}$, prove that $\dfrac{x+y+z}{a+b+c}=\dfrac{ax+by+cz}{bc+ca+ab}$.

21. If $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}$, prove that

(i) $\dfrac{y-z}{b-c}=\dfrac{z-x}{c-a}=\dfrac{x-y}{a-b}$, (ii) $bc(y+z)=ca(z+x)=ab(x+y)$,

(iii) $\dfrac{y^2-z^2}{a(b-c)}=\dfrac{z^2-x^2}{b(c-a)}=\dfrac{x^2-y^2}{c(a-b)}$.

22. If $\dfrac{2x}{3y}=\dfrac{\sqrt{(4a+3b)}+\sqrt{(4a-3b)}}{\sqrt{(4a+3b)}-\sqrt{(4a-3b)}}$, prove that $b(4x^2+9y^2)=16\,axy$.

HARDER PROPORTION

14. The following examples illustrate harder work on ratio and proportion.

Ex. 1. A is 24 years old, B is 15 years old. What is the least number of years after which the ratio of their ages will be less than 7 : 5? (B. P. 1893)

Let x be the least number of years.

Then the ratio $24+x : 15+x$ diminishes and becomes less and less than $24 : 15$ as x goes on increasing, until for some value of x it becomes *equal* to $7 : 5$, and if we still increase x more and more the ratio becomes less and less.

Hence putting $\frac{24+x}{15+x}=\frac{7}{5}$ *we get* $x=7\frac{1}{2}$; *thus the required number of years*$=8$.

Ex. 2. *If* $x : y$ *be the ratio of* $a : b$ *in its lowest terms, prove, that* $\frac{x+1}{y+1}>\frac{a+1}{b+1}$, if $b>a$. (C. F. 1882).

Let $a=mx$, $b=my$, where $m>1$.

$$\text{Then } \frac{x+1}{y+1}=\frac{mx+m}{my+m}=\frac{a+m}{b+m}=\frac{(a+1)+(m-1)}{(b+1)+(m-1)}.$$

But $\frac{(a+1)+(m-1)}{(b+1)+(m-1)}>\frac{a+1}{b+1}$, since $m-1$ is positive, and $a<b$; hence the result to be proved follows.

Ex. 3. If $(3a+5b+3c+5d)(3a-5b-3c+5d)=(3a+5b-3c-5d)(3a-5b+3c-5d)$, prove that a,b,c,d are proportional.

From the given relation,

$$\frac{3a+5b+3c+5d}{3a+5b-3c-5d}=\frac{3a-5b+3c-5d}{3a-5b-3c+5d}.$$

$\therefore$ by componendo and dividendo,

$$\frac{2(3a+5b)}{2(3c+5d)}=\frac{2(3a-5b)}{2(3c-5d)}, \text{ or } \frac{3a+5b}{3c+5d}=\frac{3a-5b}{3c-5d}$$

$\therefore$ taking alternately, $\frac{3a+5b}{3a-5b}=\frac{3c+5d}{3c-5d}$.

$\therefore$ Again by componendo and dividendo,

$$\frac{(3a+5b)+(3a-5b)}{(3a+5b)-(3a-5b)}=\frac{(3c+5d)+(3c-5d)}{(3c+5d)-(3c-5d)}.$$

$$\therefore \frac{6a}{10b}=\frac{6c}{10d} \text{ or } \frac{a}{b}=\frac{c}{d}.$$

$\therefore$ a, b, c, d are proportional.

Otherwise :—From the given relation by multiplication,
$(3a+5d)^2-(5b+3c)^2=(3a-5d)^2-(5b-3c)^2$.

Transposing, $(3a+5d)^2-(3a-5d)^2=(5b+3c)^2-(5b-3c)^2$.

$\therefore$ $6a.10d=10b.6c$, or $ad=bc$.

$\therefore$ a, b, c, d are proportional.

EXERCISE CI.

1. Two children are respectively 5 and 6 years old. What is the least number of years before which the ratio of their ages was less than $11 : 15$?

2. What is the least integer which when added to the terms of $19 : 7$ will give a ratio less than $11 : 6$?

3. What is the greatest integer which when subtracted from the terms of $7 : 11$ will give a ratio greater than $2 : 15$?

4. If $a : b$ is a ratio of greater inequality, show that $a : b$ is greater than $a^2+b^2 : 2ab$. (B. P. 1880).

5. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, prove that a, b, c, d are proportional. (C. F. 1893).

6. If $(pa+qb+rc+sd)(pa-qb-rc+sd)=(pa-qb+rc-sd)$ $\times(pa+qb-rc-sd)$, then bc, ad, ps, qr are in proportion.

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15. The following are some more illustrative examples on proportion.

Ex. 1. If $\dfrac{x+2y}{a+b}=\dfrac{y+2z}{b+c}=\dfrac{z+2x}{c+a}$, prove that

$$\frac{10x+7y+10z}{6a+5b+7c}=\frac{13x+2y-3z}{8a-b+c}.$$

We have each ratio $=\dfrac{p(x+2y)+q(y+2z)+r(z+2x)}{p(a+b)+q(b+c)+r(c+a)}$

$$=\frac{(p+2r)x+(2p+q)y+(2q+r)z}{(p+r)a+(p+q)b+(q+r)c}\quad\text{......(1)}$$

Let p, q, r be so chosen that we may have identically

$$(p+2r)x+(2p+q)y+(2q+r)z=10x+7y+10z.$$

Then $p+2r=10$, $2p+q=7$, $2q+r=10$;

whence $p=2$, $q=3$, $r=4$.

$\therefore$ from (1) each ratio $=\dfrac{10x+7y+10z}{6a+5b+7c}$.

Again, choose p, q, r so that we may have identically

$$(p+2r)x+(2p+q)y+(2q+r)z=13x+2y-3z.$$

Then $p+2r=13$, $2p+q=2$, $2q+r=-3$;

whence $p=3$, $q=-4$, $r=5$.

$\therefore$ from (1) each ratio $=\dfrac{13x+2y-3z}{8a-b+c}$. Hence the result.

The student should mark the above solution and use this method when the multipliers p, q, r are not evident on inspection.

EX. 2. If $a(y+z)=b(z+x)=c(x+y)$, prove that

(i) $\dfrac{y}{ab+bc-ca}=\dfrac{x}{ca+ab-bc}=\dfrac{z}{bc+ca-ab}$.

(ii) $\dfrac{y-z}{a(b-c)}=\dfrac{z-x}{b(c-a)}=\dfrac{x-y}{c(a-b)}$.

(iii) $\dfrac{y^2-z^2}{b-c}=\dfrac{z^2-x^2}{c-a}=\dfrac{x^2-y^2}{a-b}$.

(i) From the given relation

$$\frac{y+z}{\frac{1}{a}}=\frac{z+x}{\frac{1}{b}}=\frac{x+y}{\frac{1}{c}}\text{.........(1).}$$

$$\therefore \text{ each ratio}=\frac{(x+y)+(y+z)-(z+x)}{\frac{1}{c}+\frac{1}{a}-\frac{1}{b}}$$

$$=\frac{(z+x)+(x+y)-(y+z)}{\frac{1}{b}+\frac{1}{c}-\frac{1}{a}}=\frac{(y+z)+(z+x)-(x+y)}{\frac{1}{a}+\frac{1}{b}-\frac{1}{c}}.$$

$$\therefore \frac{y}{\frac{1}{c}+\frac{1}{a}-\frac{1}{b}}=\frac{x}{\frac{1}{b}+\frac{1}{c}-\frac{1}{a}}=\frac{z}{\frac{1}{a}+\frac{1}{b}-\frac{1}{c}}.$$

Multiplying the denominators by abc,

$$\frac{y}{ab+bc-ca}=\frac{x}{ca+ab-bc}=\frac{z}{bc+ca-ab}.$$

(ii) Again from (1) each ratio

$$=\frac{(x+y)-(z+x)}{\frac{1}{c}-\frac{1}{b}}=\frac{(y+z)-(x+y)}{\frac{1}{a}-\frac{1}{c}}=\frac{(z+x)-(y+z)}{\frac{1}{b}-\frac{1}{a}}.$$

$$\therefore \frac{y-z}{\frac{b-c}{bc}}=\frac{z-x}{\frac{c-a}{ca}}=\frac{x-y}{\frac{a-b}{ab}}.$$

Multiplying the denominators by abc,

$$\frac{y-z}{a(b-c)}=\frac{z-x}{b(c-a)}=\frac{x-y}{c(a-b)}.$$

(iii) This follows by multiplying the corresponding members of (ii) and the given relations.

Ex. 3. If $\dfrac{cy-bz}{a}=\dfrac{az-cx}{b}=\dfrac{bx-ay}{c}$, prove that $x/a=y/b=z/c$.

Put each of the given ratios $=k$.

Then $ak=cy-bz$, or $a^2k=cay-abz$,

$bk=az-cx$, or $b^2k=abz-bcx$,

$ck=bx-ay$, or $c^2k=bcx-cay$.

$\therefore$ adding, $k\,(a^2+b^2+c^2)=0$.

$\therefore$ $k=0$, since $a^2+b^2+c^2$ is positive and cannot vanish.

Hence $cy-bz=0$ or $y/b=z/c$, $az-cx=0$, or $x/a=z/c$.

$\therefore$ $x/a=y/b=z/c$.

Ex. 4. If $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}$, prove that $\dfrac{x+y+z}{a+b+c}$

$$=\frac{yz+xz+xy}{ax+by+cz}.$$

Each of the given ratios $=\dfrac{\text{sum of numerators}}{\text{sum of denominators}}=\dfrac{x+y+z}{a+b+c}$.

Also each ratio $=\dfrac{x(y+z)}{(b+c-a)(y+z)}=\dfrac{y(z+x)}{(c+a-b)(z+x)}$

$$=\frac{z(x+y)}{(a+b-c)(x+y)}$$

and $\therefore$ $=\dfrac{x(y+z)+y(z+x)+z(x+y)}{(b+c-a)(y+z)+(c+a-b)(z+x)+(a+b-c)(x+y)}$

$=\dfrac{yz+xz+xy}{ax+by+cz}$. $\qquad\therefore\ \dfrac{x+y+z}{a+b+c}=\dfrac{yz+zx+xy}{ax+by+cz}$.

EXERCISE CII.

1. If $a:b=c:d$, prove that
$a^2+b^2+c^2+d^2:(a+b)^2+(c+d)^2=(a+c)^2+(b+d)^2:(a+b+c+d)^2$

2. If $a:b=c:d=e:f$, then $\dfrac{a^3b+3ce^2+4ae^2f}{b^4+3df^2+4bf^3}=\dfrac{e^3}{f^3}$.

3. If a, b, c, d are in continued proportion, prove that
$$(c-a)^2+(d-b)^2+(c-a)(d-b)=(a-d)(a+b-c-d).$$

4. If $x/a=y/b=z/c$, prove that

(i) $\dfrac{x^2+a^2}{x+a}+\dfrac{y^2+b^2}{y+b}+\dfrac{z^2+c^2}{z+c}=\dfrac{(x+y+z)^2+(a+b+c)^2}{x+y+z+a+b+c}$.

(ii) $\dfrac{x^3+a^3}{x^2+a^2}+\dfrac{y^3+b^3}{y^2+b^2}+\dfrac{z^3+c^3}{z^2+c^2}=\dfrac{(x+y+z)^3+(a+b+c)^3}{(x+y+z)^2+(a+b+c)^2}$.

(iii) $(ax^2+by^2+cz^2)(a^2x+b^2y+c^2z)=(a^3+b^3+c^3)\times(x^3+y^3+z^3)$.

5. If $\dfrac{2y+3z}{b-c}=\dfrac{2z+3x}{2(c-a)}=\dfrac{2x+3y}{5(a-b)}$, then $19x+26y+40z=0$.

6. If $\dfrac{y+z}{b+c-2a}=\dfrac{z+x}{2(c+a-2b)}=\dfrac{x+y}{3(a+b-2c)}$,

prove that $5x+8y+9z=0$.

7. If $\dfrac{x+y}{3a-b}=\dfrac{y+z}{3b-c}=\dfrac{z+x}{3c-a}$, prove that

$$\frac{x+y+z}{a+b+c}=\frac{ax+by+cz}{a^2+b^2+c^2}.$$

8. If $\dfrac{y-3z}{b-2c}=\dfrac{z-3x}{c-2a}=\dfrac{x-3y}{a-2b}$, prove that

$$\frac{3x+8y+z}{a+5b}=\frac{x+3y+8z}{b+5c}=\frac{8x+y+3z}{c+5a}.$$

9. If $\dfrac{x+2y-5z}{b+c-3a}=\dfrac{y+2z-5x}{c+a-3b}=\dfrac{z+2x-5y}{a+b-3c}$, prove that

$$\frac{x+y+z}{a+b+c}=\frac{3x+11y-2z}{-4a+4b+12c}.$$

10. If $\dfrac{x+2y+3z}{5a+4b+3c}=\dfrac{2x+3y+4z}{7a+6b+5c}=\dfrac{4x+5y+7z}{12a+11b+9c}$,

prove that $x/(b+c)=y/(c+a)=z/(a+b)$.

SOLUTIONS OF EQUATIONS BY PROPORTION.

In solutions of some equations the methods of ratio and proportion may sometimes be advantageously employed to shorten work. The following are illustrative examples.

Ex. 1. Solve $\dfrac{12x+11}{4x-9}=\dfrac{6x-5}{2x-1}$.

Multiplying top and bottom of the right side by 2,

$$\frac{12x+11}{4x-9}=\frac{12x-10}{4x-2}.$$

$$\therefore \text{ each member } = \frac{(12x+11)-(12x-10)}{(4x-9)-(4x-2)} = \frac{21}{-7} = -3.$$

$$\therefore \frac{12x+11}{4x-9} = -3 \text{ or } -12x+27 = 12x+11.$$

$$\therefore 24x = 16, \qquad \therefore x = \tfrac{24}{16} = \tfrac{3}{2} = 1\tfrac{1}{2}.$$

Ex. 2. Solve $\dfrac{ax^2+bx+c}{px^2+qx+r} = \dfrac{ax+b}{px+q}$.

Put each member $=k$, then $k = \dfrac{ax^2+bx+c}{px^2+qx+r} = \dfrac{ax^2+bx}{px^2+qx}$

$$= \frac{\text{difference of numerators}}{\text{difference of denominators}} = \frac{c}{r}.$$

$\therefore (ax+b)/(px+q) = c/r$, or $r(ax+b) = c(px+q)$.

$\therefore x(ar-cp) = cq-br$ or $x = (cq-br)/(ar-cp)$.

Ex. 3. Solve $\dfrac{x^2+px+q}{x^2+mx+n} = \dfrac{x^2+px+q+r}{x^2+mx+n+r}$.

Here each member $= \dfrac{\text{diff. of numerators}}{\text{diff. of denominators}} = \dfrac{r}{r} = 1.$

$$\therefore \frac{x^2+px+q}{x^2+mx+n} = 1 \text{ or } x^2+px+q = x^2+mx+n.$$

$$\therefore x = \frac{n-q}{p-m}.$$

Ex. 4. Solve $(x-5)(x+4)(x+6)(x-1) = (x-4)(x+3)(x+7)(x-2)$

Here $\{(x-1)(x-5)\} \times \{(x+4)(x+6)\} = \{(x-2)(x-4)\} \times \{(x+3)(x+7)\}$;

$\therefore (x^2-6x+5)(x^2+10x+24) = (x^2-6x+8)(x^2+10x+21)$,

$\therefore$ dividing by $(x^2+10x+24)(x^2+10x+21)$,

we have $\dfrac{x^2-6x+5}{x^2+10x+21} = \dfrac{x^2-6x+8}{x^2+10x+24}$

and $\therefore = \dfrac{(x^2-6x+5)-(x^2-6x+8)}{(x^2+10x+21)-(x^2+10x+24)} = \dfrac{-3}{-3} = 1.$

$\therefore \dfrac{x^2-6x+5}{x^2+10x+21} = 1$, whence $x^2-6x+5 = x^2+10x+21$.

$\therefore -16x = 16$ or $x = -1$.

Note. Observe the arrangement of factors two together on the two sides of the given equation.

Ex. 5. Solve $\left(\dfrac{x+a}{x+b}\right)^3 = \dfrac{x+2a-b}{x-a+2b}$.

Multiplying both sides by $\dfrac{x+b}{x+a}$ we have

$$\left(\frac{x+a}{x+b}\right)^2=\frac{(x+2a-b)(x+b)}{(x-a+2b)(x+a)};$$

or, $\dfrac{x^2+2ax+a^2}{x^2+2bx+b^2}=\dfrac{x^2+2ax+2ab-b^2}{x^2+2bx-a^2+2ab}$

and $\therefore\ =\dfrac{\text{diff. of numerators}}{\text{diff. of denominators}}=\dfrac{a^2+b^2-2ab}{a^2+b^2-2ab}=1.$

$\therefore\ \dfrac{x^2+2bx+a^2}{x^2+2ax+b^2}=1$ or $x^2+2ax+a^2=x^2+2bx+b^2.$

Cancelling x^2 and transposing, $2x(a-b)=-(a^2-b^2).$

$\therefore\ x=-\frac{1}{2}(a+b).$

Otherwise thus :—By alternando,

$$\frac{(x+a)^3}{x+2a-b}=\frac{(x+b)^3}{x-a+2b}.$$

Actually dividing,

$$x^2+(a+b)x+(a^2-ab+b^2)-\frac{(a-b)^3}{x+2a-b}$$

$$=x^2+(a+b)x+(a^2-ab+b^2)+\frac{(a-b)^3}{x-a+2b}.$$

$$\therefore\ \frac{-(a-b)^3}{x+2a-b}=\frac{(a-b)^3}{x-a+2b},$$

$$\text{or } \frac{-1}{x+2a-b}=\frac{1}{x-a+2b}.$$

$$\therefore\ x+2a-b=-x+a-2b,$$

$$\therefore\ 2x=-(a+b) \text{ whence } x=-\tfrac{1}{2}(a+b).$$

For a third method see ex. 1, *p.* 245.

Ex. 6. Solve $\left(\dfrac{2x+a+c}{2x+b+c}\right)^2=\dfrac{x+a}{x+b}.$

Put each member $=k$; then

$$k=\frac{4x^2+4x(a+c)+(a+c)^2}{4x^2+4x(b+c)+(b+c)^2}=\frac{4x(x+a)}{4x(x+b)}$$

and $\therefore\ =\dfrac{\text{diff. of numerators}}{\text{diff. of denominators}}=\dfrac{4cx+(a+c)^2}{4cx+(b+c)^2}.$

Thus $k=\dfrac{4cx+(a+c)^2}{4cx+(b+c)^2}=\dfrac{4c(x+a)}{4c(x+b)}$ and $\therefore\ =\dfrac{(a+c)^2-4ac}{(b+c)^2-4bc}$

Hence $k=\dfrac{(a+c)^2-4ac}{(b+c)^2-4bc}=\dfrac{a^2-2ac+c^2}{b^2-2bc+c^2}$,

i.e. $\dfrac{x+a}{x+b}=\dfrac{a^2-2ac+c^2}{b^2-2cb+c^2}$.

$\therefore$ by convertendo $\dfrac{x+a}{(x+a)-(x+b)}=\dfrac{a^2-2ac+c^2}{a^2-b^2-2c(a-b)}$,

i.e. $\dfrac{x+a}{a-b}=\dfrac{a^2-2ac+c^2}{(a-b)(a+b-2c)}$.

$\therefore x+a=\dfrac{a^2-2ac+c^2}{a+b-2c}$, $\therefore x=\dfrac{a^2-2ac+c^2}{a+b-2c}-a=\dfrac{c^2-ab}{a+b-2c}$,

Otherwise thus :—From the given equation by *alternando*,

$$\frac{(2x+a+c)^2}{x+a}=\frac{(2x+b+c)^2}{x+b}.$$

or, $\dfrac{4x^2+4x(a+c)+(a+c)^2}{x+a}=\dfrac{4x^2+4x(b+c)+(b+c)^2}{x+b}$.

Dividing out, $4x+4c+\dfrac{(a-c)^2}{x+a}=4x+4c+\dfrac{(b-c)^2}{x+b}$;

$\therefore \dfrac{(a-c)^2}{x+a}=\dfrac{(b-c)^2}{x+b}$; hence etc.

For a third method see ex 6, page 243.

Ex. 7. Solve $\dfrac{(5x^4+10x^2+1)(5a^4+10a^2+1)}{(x^4+10x^2+5)(a^4+10a^2+5)}=ax$.

The equation may be written

$$(5x^4+10x^2+1)(5a^4+10a^2+1)=(x^5+10x^3+5x)(a^5+10a^3+5a)$$

or $\dfrac{x^5+10x^3+5x}{5x^4+10x^2+1}=\dfrac{5a^4+10a^2+1}{a^5+10a^3+5a}$.

By comp. and divid., $\left(\dfrac{x+1}{x-1}\right)^5=\left(\dfrac{1+a}{1-a}\right)^5$

$\therefore \dfrac{x+1}{x-1}=\dfrac{1+a}{1-a}$.

Again by comp. and divid. $\dfrac{2x}{2}=\dfrac{2}{2a}$, or $x=\dfrac{1}{a}$.

Ex. 8. Solve $\dfrac{\sqrt{(x+9)}+\sqrt{(x+2)}}{\sqrt{(x+9)}-\sqrt{(x+2)}}=7$.

By comp. and divid. $\frac{2\sqrt{(x+9)}}{2\sqrt{(x+2)}}=\frac{8}{6}$, or $\frac{\sqrt{(x+9)}}{\sqrt{(x+2)}}=\frac{4}{3}$.

By squaring $\frac{x+9}{x+2}=\frac{16}{9}$.

$\therefore$ by comp. and divid. $\frac{2x+11}{7}=\frac{25}{7}$.

$$\therefore 2x+11=25, \text{ or } x=7.$$

Ex. 9. Solve $\frac{y+z}{b+c}=\frac{z+x}{c+a}=\frac{x+y}{a+b}$.........(1).

$$x+y+z=a+b+c\text{.........(2)}.$$

Each member of (1) $=\frac{(z+x)+(x+y)-(y+z)}{(c+a)+(a+b)-(b+c)}$

$=$ two similar expressions and $\therefore =\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

Also each member of (1)

$$=\frac{(y+z)+(z+x)+(x+y)}{(b+c)+(c+a)+(a+b)}=\frac{x+y+z}{a+b+c}=1 \text{ from (2)}.$$

$$\therefore \frac{x}{a}=\frac{y}{b}=\frac{z}{c}=1. \quad \therefore x=a,\ y=b,\ z=c.$$

Ex. 10. Solve $\frac{bz+cy}{l}=\frac{cx+az}{m}=\frac{ay+bx}{n}=d$.

Here $\frac{abz+cay}{al}=\frac{bcx+abz}{bm}=\frac{cay+bcx}{cn}=d$.

Hence $d=\frac{(bcx+abz)+(cay+bcx)-(abz+cay)}{bm+cn-al}$

$=$ two similar expressions ;

and $\therefore$ $d=\frac{2bcx}{bm+cn-al}=\frac{2cay}{cn+al-bm}=\frac{2abz}{al+bm-cn}$.

$$\therefore x=\frac{d(bm+cn-al)}{2bc},\ y=\frac{d(cn+al-bm)}{2ca},\ z=\frac{d(al+bm-cn)}{2ab}$$

EXERCISE CIII.

Solve the following equations :—

1. $\frac{2x-7}{9x+2}=\frac{6x+5}{27x-7}$.

2. $\frac{3x^2+5x+7}{2x^2+3x+5}=\frac{3x+5}{2x+3}$.

3. $\frac{5x-1}{4x+7}=\frac{3-2x+10x^2}{2+14x+8x^2}$.

4. $\frac{(x+a)(x+b)}{x+a+b}=\frac{(x+c)(x+d)}{x+c+d}$.

Solve the following equations :—

5. $\dfrac{(x+a)(x+b)}{(x+c)(x+d)}=\dfrac{x-c-d}{x-a-b}$.

6. $\dfrac{x^2+7x+20}{x^2+3x+11}=\dfrac{x^2+7x+25}{x^2+3x+16}$.

7. $\dfrac{(x+2)(x+3)}{(x+1)(x+7)}=\dfrac{x+5}{x+8}$.

8. $(x+1)(x+2)(x+9)=(x+3)(x+4)(x+5)$.

9. $(x-1)(x-2)(x-6)=(x-3)^3$.

10. $\dfrac{(x+1)(x+9)}{(x+2)(x+4)}=\dfrac{(x+6)(x+10)}{(x+5)(x+7)}$. (M. M. 1889).

11. $(x+4)(x+5)(x+8)(x+11)=(x+3)(x+6)(x+9)(x+10)$.

12. $(x^2+15x+36)(x^2+19x+78)=(x^2+9x+20)(x^2+25x+154)$.

13. $\left(\dfrac{x+2}{x+3}\right)^3=\dfrac{x+1}{x+4}$.

14. $\left(\dfrac{x-1}{x+5}\right)^3=\dfrac{x-7}{x+11}$.

15. $(x+3)^3=(x+5)(x+2)^2$.

16. $\dfrac{x+2}{x+4}=\left(\dfrac{2x+3}{2x+5}\right)^2$.

17. $\dfrac{x+6}{x+4}=\left(\dfrac{x+3}{x+2}\right)^2$.

18. $\dfrac{x+5}{x+7}=\left(\dfrac{x+1}{x+2}\right)^2$

19. $\dfrac{x^4+6x^2+1}{4x^3+4x}=\dfrac{a^4+1}{a^4-1}$.

20. $\dfrac{5x^4+10x^2+1}{x^5+10x^3+5x}=\dfrac{a^5+1}{a^5-1}$.

21. $\dfrac{\sqrt{(x+a)}+\sqrt{(x+b)}}{\sqrt{(x+a)}-\sqrt{(x+b)}}=c$.

22. $\dfrac{\sqrt{(x+1)}+\sqrt{(x-1)}}{\sqrt{(x+1)}-\sqrt{(x-1)}}=\dfrac{4x-1}{2}$

23. $\dfrac{\sqrt{(x+a)}+\sqrt{(x-a)}}{\sqrt{(x+a)}-\sqrt{(x-a)}}=\dfrac{2x+b}{a}$.

24. $\dfrac{\sqrt{(x+a)}+\sqrt{(x-b)}}{\sqrt{(x+a)}-\sqrt{(x-b)}}=\dfrac{a+b}{a-b}$.

25. $\dfrac{y+z}{a}=\dfrac{z+x}{b}=\dfrac{x+y}{c}=d$.

26. $a(y+z)=b(z+x)=c(x+y)=d$.

27. $\dfrac{x+a}{b+c}=\dfrac{y+b}{c+a}=\dfrac{z+c}{a+b}$, $x+y+z=a+b+c$.

28. $\dfrac{y+z}{b+c-a}=\dfrac{z+x}{c+a-b}=\dfrac{x+y}{a+b-c}$, $x+y+z=a+b+c$.

CHAPTER XXIII.

FACTORS AND IDENTITIES.

1. We have already defined an *integral* expression in x, *viz.*, an expression of which the terms contain positive integral powers of x. Thus $3x^4-4x^3+5x^2-7x+10$ is an integral expression in x of the fourth degree.

2. Remainder Theorem. If an integral expression in x is divided by $x-a$, the remainder is obtained by putting $x=a$ in the expression.

Let Q be the quotient and R the remainder when the expression is divided by $x-a$, so that R does not contain x. Then since

dividend = quotient × divisor + remainder, we have

the expression $= Q\,(x-a)+R$, identically.

If we put $x=a$ in this identity, the right hand side $=R$ and the left hand side = the value of the expression obtained by putting $x=a$.

Hence, since R is the remainder, the theorem is proved.

Obs. When an expression is divided by $x+a$, we find by writing $x+a$ in the form $x-(-a)$, that the remainder is obtained by putting $x=-a$ in the expression.

Ex. 1. Find the remainder when $3x^3-4x+7$ is divided by $x-2$.

The remainder $=3.\ 2^3-4.\ 2+7=23$.

Ex. 2. Find the remainder when $3x^3+4x^2+5x-1$ is divided by $x+2$.

Here the divisor $=x+2=x-(-2)$.

$\therefore$ the remainder $=3(-2)^3+4(-2)^2+5(-2)-1=-24+16-10-1=-19$.

Ex. 3. Find the remainder when ax^3+bx^2+cx+d is divided by $x-h$.

The remainder $=ah^3+bh^2+ch+d$, which the student should verify by actual division.

3. Factor Theorem. If an integral expression in x vanishes when $x=a$, the expression is exactly divisible by $x-a$, or $x-a$ is a factor of the expression.

If the expression is divided by $x-a$, the remainder is obtained by putting $x=a$ in the expression, but this by hypothesis is zero. Hence the expression is exactly divisible by $x-a$.

Obs. If an integral expression in x vanishes when $x=-a$, the expression is divisible by $x+a$.

Ex. 1. Prove that $3x^5-8x^2+x+4$ is divisible by $x-1$.

If the expression is divided by $x-1$, the remainder

$=3\times 1-8\times 1+1+4=0$; hence the expression is divisible by $x-1$.

Ex. 2. Prove that $4x^3+7x^2+2x+51$ is divisible by $x+3$.

If the expression is divided by $x+3$, the remainder

$$=4(-3)^3+7(-3)^2+2(-3)+51$$

$=-108+63-6+51=0$; hence the expression is divisible by $x+3$.

Ex. 3. Resolve x^3+4x^2+x-6 into factors.

Putting $x=1$, the expression $1+4+1-6=0$: hence $x-1$ is a factor. This suggests the following method.

$$\begin{aligned} x^3+4x^2+x-6 &= x^2(x-1)+5x(x-1)+6(x-1) \\ &= (x-1)(x^2+5x+6) \\ &= (x-1)(x+2)(x+3). \end{aligned}$$

Ex. 4. Find the value of c for which $2x^3+(4+c)x^2-9x-30$ is divisible by $x-2$.

If the expression is divided by $x-2$ the remainder $=2.2^3+(4+c)2^2-9\times 2-30=16+16+4c-18-30=4c-16$.

If the expression is divisible by $x-2$ then $4c-16=0$ or $c=4$.

4. Divisibility of $x^n \pm y^n$.

(i). *x^n-y^n is divisible by $x-y$ where n is any integer.*

Let x^n-y^n be divided by $x-y$ and let Q be the quotient and R the remainder, so that R does not contain x. Then we have $x^n-y^n=Q(x-y)+R$ identically.

Putting $x=y$, we get $y^n-y^n=R$ or $R=0$;

whence x^n-y^n is divisible by $x-y$.

Continuing the division we get

$$\frac{x^n-y^n}{x-y}=x^{n-1}+x^{n-2}y+x^{n-3}y^2+\ldots\ldots+xy^{n-2}+y^{n-1}$$

(ii). *x^n+y^n is divisible by $x+y$ only when n is an odd integer but not when n is an even integer.*

Let x^n+y^n be divided by $x+y$ and let Q be the quotient and R the remainder, so that R does not contain x. Then we have $x^n+y^n=Q(x+y)+R$ identically.

Putting $x=-y$, we get $(-y)^n+y^n=R$.

Now when n is odd $(-y)^n+y^n=-y^n+y^n=0$,

and when n is even $(-y)^n+y^n=y^n+y^n=2y^n$.

Hence R is 0 when n is odd but not when n is even. Hence x^n+y^n is divisible by $x+y$ only when n is odd, and not so when n is even.

Continuing the division we get, when n is odd,

$$\frac{x^n+y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-\ldots\ldots-xy^{n-2}+y^{n-1}.$$

(iii.) *x^n-y^n is divisible by $x+y$ when n is an even integer, but not when n is an odd integer.*

Here using the previous notation, we have

$x^n-y^n=Q(x+y)+R$ identically.

Putting $x=-y$, we get $(-y)^n-y^n=R$.

Now when n is even $(-y)^n-y^n=y^n-y^n=0$, and when n is odd $(-y)^n-y^n=-y^n-y^n=-2y^n$.

Hence R is 0 when n is even but not when n is odd. Hence x^n-y^n is divisible by $x+y$ only when n is even but not so when n is odd.

Continuing the division we get, when n is even,

$$\frac{x^n-y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-\ \cdots\ +xy^{n-2}-y^{n-1}.$$

(iv.) *x^n+y^n is never divisible by $x-y$.*

As before, we put $x^n+y^n=Q(x-y)+R$.

Making $x=y$, $y^n+y^n=R$, or, $R=2y^n$ and cannot be zero for any value of n.

Hence x^n+y^n is never divisible by $x-y$.

Ex. 1. Find the complete quotient when x^5-y^5 is divided by $x+y$.

$$\frac{x^5-y^5}{x+y}=\frac{(x^5+y^5)-2y^5}{x+y}=\frac{x^5+y^5}{x+y}-\frac{2y^5}{x+y}$$

$$=x^4-x^3y+x^2y^2-xy^3+y^4-\frac{2y^5}{x+y}.$$

Ex. 2. Find the complete quotient when x^6+y^6 is divided by $x-y$.

$$\frac{x^6+y^6}{x-y}=\frac{(x^6-y^6)+2y^6}{x-y}=\frac{x^6-y^6}{x-y}+\frac{2y^6}{x-y}$$

$$=x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5+\frac{2y^6}{x-y}.$$

EXERCISE CIV.

Find the remainder when

1. $5x^3-7x+2$ is divided by $x-3$.
2. $7x^4-5x^3+2x^2-9x+1$ is divided by $x-1$.
3. $3x^3+5x^2-3x+9$ is divided by $x+2$.
4. $4x^3-7x^2+x-5$ is divided by $2x-1$.
5. $6x^3+5x^2-2x+4$ is divided by $3x+2$

Find the value of c when

6. $3x^2-4cx+5$ is divisible by $x-1$.
7. x^2-7x+c is divisible by $x-4$.
8. $6x^3-17x^2+c$ is divisible by $3x-4$.
9. $x^3-6x^2+5(x+c)-14$ is divisible by $x-2$.
10. $x^3+cx^2-14x-24$ is divisible by $x+3$.

Prove that a, b, and $a+b$ are factors of

11. $(a+b)^5-a^5-b^5$. 12. $(a+b)^7-a^7-b^7$.

Find the result of dividing

13. x^5-y^5 by $x-y$, x^6-y^6 by $x-y$, x^8-y^8 by $x-y$.
14. x^5+y^5 by $x+y$, x^7+y^7 by $x+y$, x^9+y^9 by $x+y$.
15. x^6-y^6 by $x+y$, x^8+y^8 by $x-y$, $x^{10}+y^{10}$ by $x-y$.
16. x^7-y^7 by $x+y$, x^8+y^8 by $x+y$, x^7+y^7 by $x-y$.

Resolve into factors.

17. $x^3+5x^2-2x-24$. 18. x^3+4x^2+x-6.
19. $x^3-9x^2+26x-24$. 20. $x^3+9x^2+23x+15$.
21. $x^4+5x^3-7x^2-41x-30$. 22. $x^4+5x^3+5x^2-5x-6$.

---

5. Factorization of expressions of second degree in x and y.

We shall here use a method of inspection as in the following examples.

Ex. 1. Factorise $6x^2+19xy+15y^2-x-y-2$.

Terms of the second degree in x and y in the expression

$$=6x^2+19xy+15y^2=(2x+3y)(3x+5y) \quad ...(1).$$

Terms without $y=6x^2-x-2=(2x+1)(3x-2)$ ...(2).

Terms without $x=15y^2-y-2=(3y+1)(5y-2)$ ...(3).

Hence in agreement with (1), (2), (3), the factors of the given expression are $2x+3y+1$ and $3x+5y-2$.

Ex. 2. Factorize $12x^2-10xy+2y^2+11x-5y+2$.

In the expression,

terms without $y=12x^2+11x+2=(3x+2)(4x+1)\ldots(1)$;
terms without $x=2y^2-5y+2=(-y+2)(-2y+1)\ldots(2)$;
terms of the second degree $=12x^2-10xy+2y^2$
$$=2(3x-y)(2x-y)$$
$$=(3x-y)(4x-2y)\ldots(3).$$

Hence in agreement with (1), (2), and (3) the factors are $3x-y+2$ and $4x-2y+1$.

NOTE.—In (2) we do not write $2y^2-5y-2=(y-2)(2y-1)$ but $=(-y+2)(-2y+1)$, by examining the terms of the factors of (1). Again, in (3) we put the factors in the forms in which they are, by examining the terms of the factors of (1) and (2).

6. Formula XVII. The student is advised to revise formula XVII at this stage (see p. 144). We shall here consider it more fully.

Ex. 1. Prove that $(b-c)(b+c)^2+(c-a)(c+a)^2+(a-b)(a+b)^2$ $=-(b-c)(c-a)(a-b)$.

Put $b+c=A$, $c+a=B$, $a+b=C$; then $A-B=-(a-b)$, $B-C=-(b-c)$, $C-A=-(c-a)$.

$$\therefore \text{ left side}=-A^2(B-C)-B^2(C-A)-C^2(A-B)$$
$$=(B-C)(C-A)(A-B)$$
$$=-(b-c)(c-a)(a-b).$$

Ex. 2. Prove that $(a-b)(x-a)(x-b)+(b-c)(x-b)(x-c)+(c-a)$ $\times(x-c)(x-a)=-(b-c)(c-a)(a-b)$.

Put $x-a=A$, $x-b=B$, $x-c=C$; then $A-B=-(a-b)$, $B-C=-(b-c)$, $C-A=-(c-a)$.

$$\therefore \text{ left side}=-AB(A-B)-BC(B-C)-CA(C-A),$$
$$=(B-C)(C-A)(A-B)$$
$$=-(b-c)(c-a)(a-b).$$

Ex. 3. Prove that $a(b-c)^3+b(c-a)^3+c(a-b)^3$
$$=(b-c)(c-a)(a-b)(a+b+c).$$

Left side $=a\{(b^3-c^3)-3bc(b-c)\}+b\{(c^3-a^3)-3ca(c-a)\}$
$+c\{(a^3-b^3)-3ab(a-b)\}$
$$=a(b^3-c^3)-3abc(b-c)+b(c^3-a^3)-3abc(c-a)$$
$$+c(a^3-b^3)-3abc(a-b)$$
$$=a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)$$
$$-3abc\{(b-c)+(c-a)+(a-b)\}.$$
$$=a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3).$$
$$=-a^3(b-c)-b^3(c-a)-c^3(a-b), \text{ re-arranging.}$$
$$=(b-c)(c-a)(a-b)(a+b+c). \quad \text{See p. 147.}$$

EXERCISE CV.

Resolve into factors.

1. $(2a+b)^2(2b-c-a)+(2b+c)^2(2c-a-b)+(2c+a)^2(2a-b-c)$.
2. $(a+2b)(b+2c)(a+b-2c)+(b+2c)(c+2a)(b+c-2a)$
$+(c+2a)(a+2b)(c+a-2b)$.
3. $a^2(b+c)-b^2(c+a)+c^2(a-b)$.
4. $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.
5. $(2a+b+c)^2(b-c)+(2b+c+a)^2(c-a)+(2c+a+b)^2(a-b)$.
6. $(a^2+1)(b-c)+(b^2+1)(c-a)+(c^2+1)(a-b)$.
7. $(a+1)^2(b-c)+(b+1)^2(c-a)+(c+1)^2(a-b)$.
8. $(pa^2+qa+r)(b-c)+(pb^2+qb+r)(c-a)+(pc^2+qc+r)(a-b)$.
9. $(a+1)^3(b-c)+(b+1)^3(c-a)+(c+1)^3(a-b)$.
10. $b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$.
11. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$.
12. $b^3c^3(b-c)+c^3a^3(c-a)+a^3b^3(a-b)$.
13. $a^4(b-c)+b^4(c-a)+c^4(a-b)$.
14. $a^5(b-c)+b^5(c-a)+c^5(a-b)$.
15. $a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3$.
16. $(b-c)(b+c)^3+(c-a)(c+a)^3+(a-b)(a+b)^3$.
17. $(b+c)(b-c)^3+(c+a)(c-a)^3+(a+b)(a-b)^3$.
18. $a(b-c)(x-b)(x-c)+b(c-a)(x-c)(x-a)+c(a-b)(x-a)$
$\times(x-b)$.
19. $(b-c)(1+ab)(1+ac)+(c-a)(1+bc)(1+ba)+(a-b)(1+ca)$
$\times(1+cb)$.
20. $a(b-c)(1+ab)(1+ac)+b(c-a)(1+bc)(1+ba)+c(a-b)(1+ca)$
$\times(1+cb)$.

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7. Formula XVIII.—The student is recommended to revise formula XVIII (see p. 145). If we put P for any one of the equivalent forms

$$a^2b+ab^2+b^2c+bc^2+c^2a+ca^2,$$
$$a^2(b+c)+b^2(c+a)+c^2(a+b),$$
$$a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2),$$
$$bc(b+c)+ca(c+a)+ab(a+b),$$

the formula XVIII may be written as

$$P+2abc=(b+c)(c+a)(a+b) \quad \dots \quad (1)$$

Again, from ex. 5, *p.* 116, we have

$$P+3abc=(a+b+c)(bc+ca+ab) \quad \dots \quad (2)$$

Hence from (1) and (2) by subtraction,

$$(b+c)(c+a)(a+b)=(bc+ca+ab)(a+b+c)-abc, \quad \ldots \quad (3)$$

a result already proved independently. [See Ex. 6, p. 116.]

Also, we have (see *p.* 140)

$$(a+b+c)^3-a^3-b^3-c^3=3(b+c)(c+a)(a+b);$$
$$=3(P+2abc) \text{ from (1)}$$
$$=3P+6abc \quad \ldots \ (4)$$

Ex. 1. Prove that $a(b-c)^2+b(c-a)^2+c(a-b)^2+9abc$
$$=(a+b+c)(bc+ca+ab).$$

Left side $=a(b^2+c^2-2bc)+b(c^2+a^2-2ca)+c(a^2+b^2-2ab)+9abc$
$$=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc$$
$$=P+3abc$$
$$=(a+b+c)(bc+ca+ab).$$

Ex. 2. Factorize $(x+a)^2(2x+b+c)+(x+b)^2(2x+c+a)$
$$+(x+c)^2(2x+a+b)+2(x+a)(x+b)(x+c).$$

Putting $x+a=A$, $x+b=B$, $x+c=C$,

the expr. $=A^2(B+C)+B^2(C+A)+C^2(A+B)+2ABC$
$$=(B+C)(C+A)(A+B)$$
$$=(2x+b+c)(2x+c+a)(2x+a+b), \text{ on substitution.}$$

Ex. 3. Prove that $a(1+b^2)(1+c^2)+b(1+c^2)(1+a^2)$
$$+c(1+a^2)(1+b^2)+4abc=(1+bc+ca+ab)(a+b+c+abc).$$

Left side $=a\{1+(b^2+c^2)+b^2c^2\}$
$$+b\{1+(c^2+a^2)+c^2a^2\}$$
$$+c\{1+(a^2+b^2)+a^2b^2\}+4abc$$
$$=(a+b+c)+a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$$
$$+ab^2c^2+bc^2a^2+ca^2b^2+4abc$$
$$=(a+b+c)+(P+3abc)+abc(bc+ca+ab)+abc$$
$$=(a+b+c)+(a+b+c)(bc+ca+ab)+abc(bc+ca+ab+1)$$
$$=(a+b+c)(1+bc+ca+ab)+abc(1+bc+ca+ab)$$
$$=(1+bc+ca+ab)(a+b+c+abc).$$

EXERCISE CVI.

Factorize

1. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc.$
2. $bc(b+c)+ca(c+a)+ab(a+b)+2abc.$
3. $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc.$
4. $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc.$

Factorize

5. $bc(b+c)+ca(c+a)+ab(a+b)+3abc.$

6. $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc.$

7. $a^2(b-c)+b^2(a-c)+c^2(a+b)-2abc.$

8. $a(b^2+c^2)+b(c^2+a^2)-c(a^2+b^2)-3abc.$

9. $a^2(2b+3c)+4b^2(3c+a)+9c^2(a+2b)+12abc.$

10. $6bc(2b+3c)+3ca(3c+a)+2ab(a+2b)+1$[illegible]$abc.$

11. $a(b+c)^2+b(c+a)^2+c(a+b)^2-3abc.$

12. $a(b-c)^2+b(c-a)^2+c(a-b)^2+8abc.$

If $a+b+c=p$, $bc+ca+ab=q$, $abc=r$, find the value of

13. $(b+c)(c+a)(a+b).$ **14.** $a^3+b^3+c^3.$

15. $bc(b+c)+ca(c+a)+ab(a+b).$

8. Prove that

$$(a^n+b^n)-(a+b)(a^{n-1}+b^{n-1})+ab(a^{n-2}+b^{n-2})=0.$$

This can be proved by multiplication or thus :—

We have $a^2-a(a+b)+ab=0 \ldots\ldots(1).$

$$b^2-b(a+b)+ab=0 \ldots\ldots(2).$$

Multiply (1) by a^{n-2} and (2) by b^{n-2} and add the products ; then the result to be proved follows.

Ex. 1. Prove that $(a+b)^5-a^5-b^5=5ab(a+b)(a^2+ab+b^2).$

Left side $=(a+b)^5-(a^5+b^5).$

$=5a^4b+10a^3b^2+10a^2b^3+5ab^4$, expanding $(a+b)^5$

$=5ab(a^3+2a^2b+2ab^2+b^3)$

$=5ab\{(a^3+b^3)+2ab(a+b)\}$

$=5ab(a+b)(a^2+ab+b^2).$

9. Prove that

$$a^n+b^n+c^n-(a+b+c)(a^{n-1}+b^{n-1}+c^{n-1})+(bc+ca+ab)$$
$$\times(a^{n-2}+b^{n-2}+c^{n-2})-abc(a^{n-3}+b^{n-3}+c^{n-3})=0.$$

Evidently $(a-a)(a-b)(a-c)=0.$

$\therefore\ a^3-a^2(a+b+c)+a(bc+ca+ab)-abc=0 \ldots(1).$

Similarly, $b^3-b^2(a+b+c)+b(bc+ca+ab)-abc=0 \ldots(2).$

$c^3-c^2(a+b+c)+c(bc+ca+ab)-abc=0 \ldots(3).$

Multiply (1) by a^{n-3}, (2) by b^{n-3}, (3) by c^{n-3} and add the products column by column ; then the result to be proved follows.

10. Conditional Identities. If $a+b+c=0$, we have proved before that

(i) $a^2+b^2+c^2=-2(bc+ca+ab)$.

(ii) $a^3+b^3+c^3=3abc$.

Ex. 1. Express $2(y-x)(y-z)+2(z-y)(z-x)+2(x-y)(x-z)$ as the sum of three squares.

Put $x-y=a$, $y-z=b$, $z-x=c$, so that $a+b+c=0$.

$$\begin{aligned}\therefore \text{ the expr. } &= -2(ab+ac+bc)\\ &= a^2+b^2+c^2 \text{ by (i) above, for } a+b+c=0\\ &= (x-y)^2+(y-z)^2+(z-x)^2.\end{aligned}$$

Ex. 2. Express $3(x-y)(y-z)(z-x)$ as the sum of three cubes.

Put $x-y=a$, $y-z=b$, $z-x=c$; then $a+b+c=0$.

$\therefore 3abc=a^3+b^3+c^3$ by (ii) above,

Or $3(x-y)(y-z)(z-x)=(x-y)^3+(y-z)^3+(z-x)^3$.

Ex. 3. If $a+b+c=0$, prove that

$$a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)=\tfrac{1}{2}(a^2+b^2+c^2)^2.$$

From the given condition $a+b=-c$.

$\therefore$ squaring, $a^2+2ab+b^2=c^2$.

$\therefore$ transposing, $a^2+b^2-c^2=-2ab$.

Squaring, $a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2=4a^2b^2$.

Transposing, $a^4+b^4+c^4=2a^2b^2+2b^2c^2+2c^2a^2$.

Hence also $2(a^4+b^4+c^4)=2a^2b^2+2b^2c^2+2c^2a^2+a^4+b^4+c^4$.

$$=(a^2+b^2+c^2)^2.$$

or $a^4+b^4+c^4=\frac{1}{2}(a^2+b^2+c^2)^2$.

Ex. 4. If $a+b+c=0$, prove that

(i) $a^5+b^5+c^5=-5abc(bc+ca+ab)$

(ii) $a^7+b^7+c^7=7abc(bc+ca+ab)^2$.

These results might be deduced from art. 8 or 9, or we may proceed thus :

Let $bc+ca+ab=q$, $abc=r$; then $\because a+b+c=0$,

$a^2+b^2+c^2=-2(bc+ca+ab)=-2q$.

Also $a^3+b^3+c^3=3abc=3r$.

Now $(a-a)(a-b)(a-c)=0$,

or $a^3-a^2(a+b+c)+a(bc+ca+ab)-abc=0$.

$\therefore$ putting $a+b+c=0$, $bc+ca+ab=q$, $abc=r$, we get

$$a^3+aq-r=0, \text{ or, } a^3=r-aq\ldots(1)$$

Similarly $b^3=r-bq\ldots(2)$, $c^3=r-cq\ldots(3)$.

(*i*) Multiplying (1), (2), (3) by a^2, b^2, c^2 respectively,

$$a^5=a^2r-a^3q,\ b^5=b^2r-b^3q,\ c^5=c^2r-c^3q.$$

$\therefore$ adding, $a^5+b^5+c^5=r\ (a^2+b^2+c^2)-q\ (a^3+b^3+c^3)$

$$=r\times(-2q)-q\times 3r$$
$$=-5qr$$
$$=-5abc(bc+ca+ab).$$
$$\text{or}=\tfrac{5}{2}abc(a^2+b^2+c^2).$$

(*ii*) Again, $a^7=a.a^6=a(r^2-2aqr+a^2q^2)$, from (1) by squaring.

$\therefore\ a^7=ar^2-2a^2qr+a^3q^2$

Similarly $b^7=br^2-2b^2\ qr+b^3q^2$

$c^7=cr^2-2c^2qr+c^3q^2$;

$\therefore$ adding, $a^7+b^7+c^7$

$$=r^2(a+b+c)-2qr(a^2+b^2+c^2)+q^2(a^3+b^3+c^3)$$
$$=-2qr(-2q)+q^2.\ 3r, \text{ substituting,}$$
$$=7q^2r$$
$$=7abc(bc+ca+ab)^2.$$

Note. By substituting for a, b, c any three quantities whose sum is zero, we can deduce various identities from the preceding examples.

Ex. 5. If $xy+yz+zx=1$, prove that

$$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz:$$

Left side$=x\{1-(y^2+z^2)+y^2z^2\}$

$$+y\{1-(z^2+x^2)+z^2x^2\}$$
$$+z\{1-(x^2+y^2)+x^2y^2\}$$
$$=(x+y+z)-\{x(y^2+z^2)+y(z^2+x^2)+z(x^2+y^2)\}$$
$$+xyz(yz+zx+xy)$$
$$=(x+y+z)-\{(x+y+z)(yz+zx+xy)-3xyz\}$$
$$+xyz(yz+zx+xy), \text{ by (2) art. 7}$$
$$=(x+y+z)-\{(x+y+z)\times 1-3xyz\}+xyz\times 1$$
$$=4xyz.$$

11. Illustrative Examples.

Ex. 1. Prove that $(bc-a^2)(ca-b^2)(ab-c^2)$

$$=(bc+ca+ab)^3-abc(a+b+c)^3$$

We have $bc-a^2=bc+ca+ab-a(a+b+c)$

$$=q-ap, \text{ where } bc+ac+ab=q,\ a+b+c=p.$$

Similarly, $ca-b^2=q-bp$; $ab-c^2=q-cp$.

$\therefore\quad (bc-a^2)(ca-b^2)(ab-c^2)$

$=(q-cp)(q-bp)(q-cp)$

$=q^3-q^2(ap+bp+cp)+q(bcp^2+cap^2+abp^2)-abcp^3.$

$=q^3-q^2.p^2+q.p^2q-abcp^3$

$=q^3-abcp^3=(bc+ca+ab)^3-abc(a+b+c)^3.$

Ex. 2. Prove that $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$

$=16(s-a)(s-b)(s-c)(s-d)$ where $2s=a+b+c+d$.

Left side $=(2ab+2cd)^2-(a^2+b^2-c^2-d^2)^2.$

$=(2ab+2cd+a^2+b^2-c^2-d^2)$

$\times(2ab+2cd-a^2-b^2+c^2+d^2).$

$=\{(a+b)^2-(c-d)^2\}\{(c+d)^2-(a-b)^2\}$

$=(a+b+c-d)(a+b-c+d)(c+d+a-b)$

$\times(c+d-a+b).$

Now $2s=a+b+c+d$,

$\therefore\ 2s-2a=-a+b+c+d,\ 2s-2b=a-b+c+d,$

$2s-2c=a+b-c+d,\ 2s-2d=a+b+c-d.$

$\therefore$ left side $=(2s-2a)(2s-2b)(2s-2c)(2s-2d)$

$=16(s-a)(s-b)(s-c)(s-d).$

Ex. 3. Express

(*i*) $2(4x-1)^2+2(2x-3)^2$ as the sum of two squares.

(*ii*) $(5x^2-4x+1)(3x^2+2x-7)$, as the difference of two squares.

(*i*) We know $2a^2+2b^2=(a+b)^2+(a-b)^2\ldots(1)$

Put $a=4x-1$, $b=2x-3$;

then $a+b=(4x-1)+(2x-3)=6x-4$;

$a-b=(4x-1)-(2x-3)=2x+2.$

Hence substituting in (1),

$2(4x-1)^2+2(2x-3)^2=(6x-4)^2+(2x+2)^2.$

(*ii*) We know $ab=\left(\dfrac{a+b}{2}\right)^2-\left(\dfrac{a-b}{2}\right)^2\ldots(2).$

Put $a=5x^2-4x+1$, $b=3x^2+2x-7$;

then $\dfrac{a+b}{2}=\dfrac{(5x^2-4x+1)+(3x^2+2x-7)}{2}=4x^2-x-3,$

$\dfrac{a-b}{2}=\dfrac{(5x^2-4x+1)-(3x^2+2x-7)}{2}=x^2-3x+4.$

Hence substituting in (2)

$$(5x^2-4x+1)(3x^2+2x-7)=(4x^2-x-3)^2-(x^2-3x+4)^2.$$

Ex. 4. Prove that $(b+c)(b-c)^3+(c+a)(c-a)^3+(a+b)(a-b)^3$

$$=2(b-c)(c-a)(a-b)(a+b+c)$$

We have $(b+c)(b-c)^3=(b+c)(b-c)\times(b-c)^2$

$$=(b^2-c^2)\times(b^2+c^2-2bc)$$

$$=(b^4-c^4)-2bc(b^2-c^2)$$

Similarly, $(c+a)(c-a)^3=(c^4-a^4)-2ca(c^2-a^2)$,

$$(a+b)(a-b)^3=(a^4-b^4)-2ab(a^2-b^2).$$

$\therefore$ by adding $(b+c)(b-c)^3+(c+a)(c-a)^3+(a+b)(a-b)^3$

$$-2bc(b^2-c^2)-2ca(c^2-a^2)-2ab(a^2-b^2)$$

$=-2\{a^3(b-c)+b^3(c-a)+c^3(a-b)\}$, re-arranging

$=2(b-c)(c-a)(a-b)(a+b+c)$. See p. 147.

Ex. 5. Prove that $(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3$

$$-3(x^2-yz)(y^2-zx)(z^2-xy)=(x^3+y^3+z^3-3xyz)^2.$$

Putting $x^2-yz=a$, $y^2-zx=b$, $z^2-xy=c$,

the left-side $=a^3+b^3+c^3-3abc$

$$=\tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}\ldots(1).$$

Now $a+b+c=x^2+y^2+z^2-yz-zx-xy$.

Again, $a-b=(x^2-yz)-(y^2-zx)$

$$=(x^2-y^2)+z(x-y)=(x-y)(x+y+z).$$

Similarly $b-c=(y-z)(x+y+z)$, $c-a=(z-x)(x+y+z)$.

$\therefore (a-b)^2+(b-c)^2+(c-a)^2$

$=(x-y)^2(x+y+z)^2+(y-z)^2(x+y+z)^2+(z-x)^2(x+y+z)^2$

$=(x+y+z)^2\{(x-y)^2+(y-z)^2+(z-x)^2\}$

$\therefore$ from (1) left-side

$=\tfrac{1}{2}(x+y+z)\{(x-y)^2+(y-z)^2+(z-x)^2\}$

$\times\{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)\}.$

$=(x^3+y^3+z^3-3xyz)^2.$

Ex. 6. Prove that $(x-y)^4+(y-z)^4+(z-x)^4$

$$=\tfrac{1}{2}\{(x-y)^2+(y-z)^2+(z-x)^2\}^2.$$

Put $x-y=a$, $y-z=b$, $z-x=c$; then $a+b+c=0$.

$\therefore a^4+b^4+c^4=\tfrac{1}{2}(a^2+b^2+c^2)^2$, ex. 3, art. 10.

Hence substituting for a, b, c, we get the required result.

Ex. 7. If $a+b+c+d=0$, prove that

$$a^3+b^3+c^3+d^3=3(bcd+cda+dab+abc).$$

We have $a+b=-(c+d)$.

$\therefore$ Cubing, $a^3+b^3+3ab(a+b)=-\{c^3+d^3+3cd(c+d)\}$;

transposing, $a^3+b^3+c^3+d^3=-3ab(a+b)-3cd(c+d)$

$=-3ab\{-(c+d)\}-3cd\{-(a+b)\}$

$=3(abc+abd+acd+bcd)$.

Ex. 8. If $x^2=3x+2$, prove that

$x^5-3x^4+2x^3-10x^2=14x+4$.

$\because$ $x^2=3x+2$, we have

$x^3=3x^2+2x=3(3x+2)+2x=11x+6$,

$\therefore$ $x^4=11x^2+6x=11(3x+2)+6x=39x+22$,

$\therefore$ $x^5=39x^2+22x=39(3x+2)+22x=139x+78$.

Hence $x^5-3x^4+2x^3-10x^2$

$=(139x+78)-3(39x+22)+2(11x+6)-10(3x+2)$

$=139x+78-117x-66+22x+12-30x-20$

$=14x+4$.

EXERCISE CVII.

1. If $a+b=3$, $ab=4$, find the value of

(i) a^4+b^4, (ii) a^5+b^5, (iii) a^6+b^6.

2. If $a+b+c=0$, $bc+ca+ab=5$, $abc=6$, find the value of

(i) $a^4+b^4+c^4$, (ii) $a^5+b^5+c^5$, (iii) $a^6+b^6+c^6$.

3. If $x^2=2x+1$, prove that

(i) $x^5-4x^4+3x^3-2x^2+8x+4=0$

(ii) $x^6-2x^5+x^4-3x^3+x^2=11x+5$.

4. Express as the difference of two squares :—

(i) $(6x^2-7x+5)(2x^2+x-3)$, (ii) $(x+1)(x+2)(x+3)(x+4)$.

(iii) $(x+3a)(x+5a)(x+7a)(x+9a)$.

Prove the following :—

5. (i) $x(y^2+z^2-x^2)+y(z^2+x^2-y^2)+z(x^2+y^2-z^2)+6xyz$

$=(x+y+z)(2xy+2yz+2zx-x^2-y^2-z^2)$.

(ii) $x^2(y^4+z^4-x^4)+y^2(z^4+x^4-y^4)+z^2(x^4+y^4-z^4)+6x^2y^2z^2$

$=(x^2+y^2+z^2)(x+y+z)(y+z-x)(z+x-y)(x+y-z)$.

6. $a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2$

$+(b+c-a)(c+a-b)(a+b-c)=4abc$

7. $a^3+b^3+c^3+24abc$.

$=(a+b+c)^3-3\{a(b-c)^2+b(c-a)^2+c(a-b)^2\}$

Prove the following :—

8. $a(b+c)(b^2+c^2-a^2)+b(c+a)(c^2+a^2-b^2)+c(a+b)(a^2+b^2-c^2)$
$=2abc(a+b+c)$.

9. $(s-a)^2(s-b)+(s-b)^2(s-c)+(s-c)^2(s-a)+a^2b+b^2c+c^2a$
$=s^3$, if $2s=a+b+c$.

10. $9(a^3+b^3+c^3)-(a+b+c)^3$
$=(4b+4c+a)(b-c)^2+(4c+4a+b)(c-a)^2+(4a+4b+c)(a-b)^2$.

11. $(2x+z)^3+9(2x+z)^2(y+z)+27(2x+z)(y+z)^2+27(y+z)^3$
$=8(x+z)^3+12(x+z)^2(3y+2z)+6(x+z)(3y+2z)^2+(3y+2z)^3$.

12. $x^3+6(y+z)x^2+12(y+z)^2x+8(y+z)^3$
$=4(3x+2y+6z)y^2+(x+6y+2z)(x+2z)^2$. (M. M. 1881).

13. $(a+b)^3-(b+c)^3+(c+d)^3-(d+a)^3$
$=3(a-c)(b-d)(a+b+c+d)$.

14. $(ma-nb)(mb-nc)(mc-na)+(na-mb)(nb-mc)(nc-ma)$
$=mn(m+n)(b-c)(c-a)(a-b)$.

15. $(x+a)^3(b-c)+(x+b)^3(c-a)+(x+c)^3(a-b)$
$=-(b-c)(c-a)(a-b)(a+b+c+3x)$.

16. $(x+a)(b-c)^3+(x+b)(c-a)^3+(x+c)(a-b)^3$
$=(b-c)(c-a)(a-b)(a+b+c+3x)$.

17. $(b-c)(b+c-2a)^3+(c-a)(c+a-2b)^3+(a-b)(a+b-2c)^3=0$.

18. $(b-c)^3(b+c-2a)+(c-a)^3(c+a-2b)+(a-b)^3(a+b-2c)=0$.

19. $x^3+y^3+z^3-3xyz=4(a^3+b^3+c^3-3abc)$, if $x=b+c-a$,
$y=c+a-b$, $z=a+b-c$.

20. $(x^2+2yz)^3+(y^2+2zx)^3+(z^2+2xy)^3-3(x^2+2yz)$
$\times(y^2+2zx)(z^2+2xy)=(x^3+y^3+z^3-3xyz)^2$.

21. If $a+b+c=0$, prove that

(i) $\dfrac{a^5+b^5+c^5}{5}=\dfrac{a^3+b^3+c^3}{3}\cdot\dfrac{a^2+b^2+c^2}{2}$.

(ii) $\dfrac{a^7+b^7+c^7}{7}=\dfrac{a^5+b^5+c^5}{5}\cdot\dfrac{a^2+b^2+c^2}{2}$.

(iii) $\dfrac{a^7+b^7+c^7}{7}\cdot\dfrac{a^3+b^3+c^3}{3}=\left(\dfrac{a^5+b^5+c^5}{5}\right)^2$.

22. If $bc+ca+ab=0$, prove that
$$(a+b+c)^3=a^3+b^3+c^3-3abc.$$

23. If $x+y+z=xyz$, then

(i) $x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz$.

(ii) $(x+y)(1-yz)(1-zx)+(y+z)(1-zx)(1-xy)$
$+(z+x)(1-xy)(1-yz)=(x+y)(y+z)(z+x)$.

24. If $xy+yz+zx=1$, then

(i) $(1+x^2)(1+y^2)(1+z^2)=(x+y)^2(y+z)^2(z+x)^2$.

(ii) $(1+x^2)(y+z)+(1+y^2)(z+x)+(1+z^2)(x+y)$
$=3(x+y)(y+z)(z+x)$.

Prove the following :—

25. $(a+1)^3(b-c)^3+(b+1)^3(c-a)^3+(c+1)^3(a-b)^3$
$=3(a+1)(b+1)(c+1)(b-c)(c-a)(a-b)$.

26. $a^4(b-c)^4+b^4(c-a)^4+c^4(a-b)^4$
$=2a^2b^2(b-c)(c-a)+2b^2c^2(c-a)(a-b)+2c^2a^2(a-b)(b-c)$.

27. $(b-c)^4+(c-a)^4+(a-b)^4$
$=2\{(b-c)^2(c-a)^2+(c-a)^2(a-b)^2+(a-b)^2(b-c)^2\}$
$=2(a^2+b^2+c^2-bc-ca-ab)^2$.

28. $(b-c)^5+(c-a)^5+(a-b)^5=5(b-c)(c-a)(a-b)$
$\times(a^2+b^2+c^2-bc-ca-ab)$.

29. $(b-c)^7+(c-a)^7+(a-b)^7=7(b-c)(c-a)(a-b)$
$\times(a^2+b^2+c^2-bc-ca-ab)^2$.

30. $2\{(b+c-2a)^4+(c+a-2b)^4+(a+b-2c)^4\}$
$=\{(b+c-2a)^2+(c+a-2b)^2+(a+b-2c)^2\}^2$ (C. E. 1896).

12. Fractional Identities. The following are some illustrative examples on fractions.

Ex. 1. Simplify

$$\frac{(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3}{(y+z)(y-z)^3+(z+x)(z-x)^3+(x+y)(x-y)^3}$$ (M. M. 1892.)

Let N be the numerator and D the denominator of the fraction. Then we have

$$\begin{aligned}N+D&=\{(y-z)(y+z)^3+(y+z)(y-z)^3\}\\&\quad+\{(z-x)(z+x)^3+(z+x)(z-x)^3\}\\&\quad+\{(x-y)(x+y)^3+(x+y)(x-y)^3\}\\&=(y^2-z^2)\{(y+z)^2+(y-z)^2\}+\text{ two similar terms}\\&=2(y^2-z^2)(y^2+z^2)+2(z^2-x^2)(z^2+x^2)+2(x^2-y^2)(x^2+y^2)\\&=2(y^4-z^4+z^4-x^4+x^4-y^4)=0.\end{aligned}$$

$\therefore\ N=-D$: hence the fraction $=-1$.

Ex. 2. If $x+y+z=xyz$, prove that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$ (C. E. 1898).

$$\text{Left side} = \frac{x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)}{(1-x^2)(1-y^2)(1-z^2)}$$

$$\begin{aligned}
\text{Numerator} &= x[1-(y^2+z^2)+y^2z^2] \\
&\quad +y[1-(z^2+x^2)+z^2x^2] \\
&\quad +z[1-(x^2+y^2)+x^2y^2] \\
&=(x+y+z)-[x(y^2+z^2)+y(z^2+x^2)+z(x^2+y^2)] \\
&\qquad\qquad +xyz(yz+zx+xy) \\
&=(x+y+z)-[(x+y+z)(yz+zx+xy)-3xyz] \\
&\qquad\qquad +xyz(yz+zx+xy), \text{ by (2), art. 7.} \\
&=xyz-[xyz(yz+zx+xy-3xyz]+xyz(yz+zx+xy) \\
&\qquad\qquad [\text{substituting for } x+y+z] \\
&=4xyz.
\end{aligned}$$

$$\therefore \text{ Left side} = \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

Ex. 3. If $x+y+z=xyz$, prove that

$$\frac{x+y}{1-xy}+\frac{y+z}{1-yz}+\frac{z+x}{1-zx}=\frac{x+y}{1-xy}\cdot\frac{y+z}{1-yz}\cdot\frac{z+x}{1-zx}.$$

From the hypothesis, by transposition,

$$x+y=xyz-z=-z(1-xy).$$

$$\therefore \frac{x+y}{1-xy}=-z\ldots(1).$$

Similarly, $\dfrac{y+z}{1-yz}=-x\ldots(2).$

$$\frac{z+x}{1-zx}=-y\ldots(3).$$

$\therefore$ Adding (1), (2), (3) we have

$$\begin{aligned}
&\frac{x+y}{1-xy}+\frac{y+z}{1-yz}+\frac{z+x}{1-zx}. \\
&=-(x+y+z). \\
&=-xyz, \text{ from hypothesis.} \\
&=(-z)\times(-x)\times(-y). \\
&=\frac{x+y}{1-xy}\cdot\frac{y+z}{1-yz}\cdot\frac{z+x}{1-zx}, \text{ from (1), (2), (3).}
\end{aligned}$$

Ex. 4. If $x=cy+bz$, $y=az+cx$, $z=bx+ay$,

show that $\dfrac{x^2}{1-a^2}=\dfrac{y^2}{1-b^2}=\dfrac{z^2}{1-c^2}.$

The given relations are

$$x-cy-bz=0\ldots(1),\quad cx-y+az=0\ldots(2),\quad bx+ay-z=0\ldots(3)$$

From (1) and (2) by cross-multiplication.

$$\frac{x}{ac+b}=\frac{y}{bc+a}=\frac{z}{1-c^2}\ldots(4).$$

Similarly from (2) and (3),

$$\frac{x}{1-a^2}=\frac{y}{ab+c}=\frac{z}{ac+b}\ldots(5).$$

Also from (1) and (3),

$$\frac{x}{ab+c}=\frac{y}{1-b^2}=\frac{z}{bc+a}\ldots(6).$$

Hence from (4) and (5),

$$\frac{x^2}{(ac+b)(1-a^2)}=\frac{z^2}{(1-c^2)(ac+b)} \text{ or } \frac{x^2}{1-a^2}=\frac{z^2}{1-c^2}$$

Also from (5), and (6),

$$\frac{x^2}{(1-a^2)(ab+c)}=\frac{y^2}{(ab+c)(1-b^2)} \text{ or } \frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}$$

Hence $\dfrac{x^2}{1-a^2}=\dfrac{y^2}{1-b^2}=\dfrac{z^2}{1-c^2}$.

Ex. 5. If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a+b+c}$, shew that

$$\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n+b^n+c^n}, \text{ where } n \text{ is odd.}$$

From the given relation $\dfrac{bc+ca+ab}{abc}-\dfrac{1}{a+b+c}=0.$

$\therefore\ (a+b+c)(bc+ca+ab)-abc=0$

or $(b+c)(c+a)(a+b)=0$

$\therefore\ (b+c)=0$, or, $(c+a)=0$, or, $a+b=0$.

Taking $b+c=0$, we have $b=-c$

$\therefore\ b^n=(-c)^n=-c^n$, since n is odd

$\therefore\ b^n+c^n=0\ldots(1)$

Again, since $b=-c$, we have

$$\frac{1}{b}=-\frac{1}{c}, \therefore \frac{1}{b^n}=\left(-\frac{1}{c}\right)^n=-\frac{1}{c^n}, \text{ since } n \text{ is odd.}$$

$$\therefore \frac{1}{b^n}+\frac{1}{c^n}=0. \quad (2)$$

The result to be proved follows from (2) and (1). We would get the same thing by taking $c+a=o$ or $a+b=o$.

Ex. 6. Simplify

$$\frac{b-a}{(ax+y)(bx+y)}+\frac{c-b}{(bx+y)(cx+y)}+\frac{d-c}{(cx+y)(dx+y)}$$

$$\text{1st term}=\frac{1}{x}\left\{\frac{bx-ax}{(ax+y)(bx+y)}\right\}=\frac{1}{x}\left(\frac{1}{ax+y}-\frac{1}{bx+y}\right)$$

$$\text{2nd term}=\frac{1}{x}\left\{\frac{cx-bx}{(bx+y)(cx+y)}\right\}=\frac{1}{x}\left(\frac{1}{bx+y}-\frac{1}{cx+y}\right)$$

$$\text{3rd term}=\frac{1}{x}\left\{\frac{dx-cx}{(cx+y)(dx+y)}\right\}=\frac{1}{x}\left(\frac{1}{cx+y}-\frac{1}{dx+y}\right)$$

$\therefore$ adding and cancelling terms on the right,

$$\text{the given expr}=\frac{1}{x}\left(\frac{1}{ax+y}-\frac{1}{dx+y}\right)=\frac{1}{x}\cdot\frac{x(d-a)}{(ax+y)(dx+y)}$$

$$=\frac{d-a}{(ax+y)(dx+y)}.$$

This example illustrates the importance of judicious breaking up of terms. The following example is another instance.

Ex. 7. Prove that

$$\frac{a}{a^2-1}+\frac{a^2}{a^4-1}+\frac{a^4}{a^8-1}=\frac{1}{2}\left(\frac{a+1}{a-1}-\frac{a^8+1}{a^8-1}\right).$$

We have $\dfrac{a}{a^2-1}=\dfrac{1}{2}\cdot\dfrac{2a}{a^2-1}=\dfrac{1}{2}\cdot\dfrac{(a+1)^2-(a^2+1)}{a^2-1}$

$$\therefore \frac{a}{a^2-1}=\frac{1}{2}\left(\frac{a+1}{a-1}-\frac{a^2+1}{a^2-1}\right).$$

Similarly, $\dfrac{a^2}{a^4-1}=\dfrac{1}{2}\left(\dfrac{a^2+1}{a^2-1}-\dfrac{a^4+1}{a^4-1}\right),$

$$\frac{a^4}{a^8-1}=\frac{1}{2}\left(\frac{a^4+1}{a^4-1}-\frac{a^8+1}{a^8-1}\right).$$

$\therefore$ adding and cancelling terms on the right, the required result follows.

Ex. 8. If $\frac{a}{b}+\frac{c}{d}=\frac{b}{a}+\frac{d}{c}$, prove that $\frac{a^3}{b^3}+\frac{c^3}{d^3}=\frac{b^3}{a^3}+\frac{d^3}{c^3}$ (M. M. 1866).

From the given relation, transposing, $\frac{a}{b}-\frac{b}{a}=\frac{d}{c}-\frac{c}{d}$...(1).

$$\therefore \text{ cubing, } \frac{a^3}{b^3}-\frac{b^3}{a^3}-3\left(\frac{a}{b}-\frac{b}{a}\right)=\frac{d^3}{c^3}-\frac{c^3}{d^3}-3\left(\frac{d}{c}-\frac{c}{d}\right)$$

$$\therefore \frac{a^3}{b^3}-\frac{b^3}{a^3}=\frac{d^3}{c^3}-\frac{c^3}{d^3}, \because 3\left(\frac{a}{b}-\frac{b}{a}\right)=3\left(\frac{d}{c}-\frac{c}{d}\right) \text{ from (1)}$$

Hence transposing, $\frac{a^3}{b^3}+\frac{c^3}{d^3}=\frac{b^3}{a^3}+\frac{d^3}{c^3}$.

Ex. 9. If $a+b+c=3$, $ab+bc+ca=2$, and $abc=1$, find the value of

$$\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}.$$

$$\text{The expr.}=\frac{a}{a^2+abc}+\frac{b}{b^2+abc}+\frac{c}{c^2+abc}$$

$$=\frac{a}{1+a^2}+\frac{b}{1+b^2}+\frac{c}{1+c^2}, \text{ for } abc=1$$

$$=\frac{a(1+b^2)(1+c^2)+b(1+c^2)(1+a^2)+c(1+a^2)(1+b^2)}{(1+a^2)(1+b^2)(1+c^2)}.$$

Now numerator

$$=a+a(b^2+c^2)+ab^2c^2+b+b(c^2+a^2)+bc^2a^2+c+c(a^2+b^2)+ca^2b^2$$
$$=(a+b+c)+\{a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)\}+abc(bc+ca+ab)$$
$$=(a+b+c)+\{(a+b+c)(bc+ca+ab)-3abc\}+abc(bc+ca+ab)$$
$$=3+3\times 2-3\times 1+1\times 2, \text{ substituting}$$
$$=3+6-3+2=8.$$

Also denominator

$$=1+(a^2+b^2+c^2)+(b^2c^2+c^2a^2+a^2b^2)+a^2b^2c^2$$
$$=1+\{(a+b+c)^2-2(bc+ca+ab)\}+\{(bc+ca+ab)^2$$
$$-2abc(a+b+c)\}+a^2b^2c^2$$
$$=1+3^2-2\times 2+2^2-2\times 1\times 3+1^2, \text{ substituting}$$
$$=1+9-4+4-6+1$$
$$=5. \quad \therefore \text{ the expr.}=\tfrac{8}{5}.$$

Ex. 10. Simplify $\dfrac{(x-y)^2}{(y+z-2x)(z+x-2y)}+\dfrac{(y-z)^2}{(z+x-2y)(x+y-2z)}$

$$+\frac{(z-x)^2}{(x+y-2z)(y+z-2x)}.$$

Put $x-y=a$, $y-z=b$, $z-x=c$,

then $y+z-2x=(z-x)-(x-y)=c-a$,

$z+x-2y=(x-y)-(y-z)=a-b$.

$x+y-2z=(y-z)-(z-x)=b-c$.

Hence the expression becomes

$$\frac{a^2}{(c-a)(a-b)}+\frac{b^2}{(a-b)(b-c)}+\frac{c^2}{(b-c)(c-a)}$$

$$=\frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{(a-b)(b-c)(c-a)}$$

$$=\frac{-(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)}$$

Ex. 11. Simplify

$$\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-c)(b-a)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}.$$

The expression

$$=-\frac{a}{(a-b)(c-a)(x-a)}-\frac{b}{(b-c)(a-b)(x-b)}-\frac{c}{(c-a)(b-c)(x-c)}$$

$$=\frac{-a(b-c)(x-b)(x-c)-b(c-a)(x-c)(x-a)-c(a-b)(x-a)(x-b)}{(b-c)(c-a)(a-b)(x-a)(x-b)(x-c)}.$$

In the numerator,

1st term $=-a(b-c)x^2+a(b^2-c^2)x-abc(b-c)$,

2nd term $=-b(c-a)x^2+b(c^2-a^2)x-abc(c-a)$,

3rd term $=-c(a-b)x^2+c(a^2-b^2)x-abc(a-b)$.

$\therefore$ adding column by column and observing that the first and third columns vanish identically, we have numerator

$$=a(b^2-c^2)x+b(c^2-a^2)x+c(a^2-b^2)x$$
$$=(b-c)(c-a)(a-b)x.$$

$\therefore$ the expression $=\dfrac{x}{(x-a)(x-b)(x-c)}$.

Obs. The following results of which the second is proved above may be remembered by the student :—

(i) $\dfrac{1}{(a-b)(a-c)(x-a)}+\dfrac{1}{(b-a)(b-c)(x-b)}+\dfrac{1}{(c-a)(c-b)(x-c)}$

$$=\frac{1}{(x-a)(x-b)(x-c)}.$$

(ii) $$\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-a)(b-c)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}$$
$$=\frac{x}{(x-a)(x-b)(x-c)}.$$

(iii) $$\frac{a^2}{(a-b)(a-c)(x-a)}+\frac{b^2}{(b-a)(b-c)(x-b)}+\frac{c^2}{(c-a)(c-b)(x-c)}$$
$$=\frac{x^2}{(x-a)(x-b)(x-c)}.$$

Ex. 12. Simplify

$$\frac{a^2+ha+k}{(a-b)(a-c)(x-a)}+\frac{b^2+hb+k}{(b-a)(b-c)(x-b)}+\frac{c^2+hc+k}{(c-a)(c-b)(x-c)}$$

1st term=

$$\frac{a^2}{(a-b)(a-c)(x-a)}+h\frac{a}{(a-b)(a-c)(x-a)}+k\frac{1}{(a-b)(a-c)(x-a)}.$$

Similarly breaking up each term and adding column by column, the expression

$$=\left\{\frac{a^2}{(a-b)(a-c)(x-a)}+\frac{b^2}{(b-a)(b-c)(x-b)}+\frac{c^2}{(c-a)(c-b)(x-c)}\right\}$$
$$+h\left\{\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-a)(b-c)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}\right\}$$
$$+k\left\{\frac{1}{(a-b)(a-c)(x-a)}+\frac{1}{(b-a)(b-c)(x-b)}+\frac{1}{(c-a)(c-b)(x-c)}\right\}$$
$$=\frac{x^2}{(x-a)(x-b)(x-c)}+h\frac{x}{(x-a)(x-b)(x-c)}+k\frac{1}{(x-a)(x-b)(x-c)},$$

simplifying each part within { }

$$=\frac{x^2+hx+k}{(x-a)(x-b)(x-c)}.$$

EXERCISE CVIII.

Simplify the following :—

1. $\dfrac{6x^2+7xy-3y^2+x+7y-2}{2x^2+5xy+3y^2-5x-7y+2}$. 2. $\dfrac{3x^2+2xy-y^2-8x+4y-3}{9x^2-6xy+y^2+18x-6y+5}$.

3. $\dfrac{xy+2x^2-3y^2-4yz-xz-z^2}{2x^2+9xz-5xy+4z^2+8yz-12y^2}$.

4. $\dfrac{(y-z)(y+z)^3+(z-x)(z+x)^3+(x-y)(x+y)^3}{(y-z)(y+z)^2+(z-x)(z+x)^2+(x-y)(x+y)^2}$.

5. $\dfrac{(a+b)^3-(b+c)^3+(c+d)^3-(d+a)^3}{(a+b)^2-(b+c)^2+(c+d)^2-(d+a)^2}$

Simplify the following :—

6. $\dfrac{a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3}{a(b-c)^3+b(c-a)^3+c(a-b)^3}$.

7. $\dfrac{\dfrac{1+x}{1-x}+\dfrac{4x}{1+x^2}+\dfrac{8x}{1-x^2}-\dfrac{1-x}{1+x}}{\dfrac{1+x^2}{1-x^2}+\dfrac{4x^2}{1+x^4}-\dfrac{1-x^2}{1+x^2}}$ (C. E. 1870).

8. $\dfrac{\dfrac{a+b+c}{a+b-c}+\dfrac{a+c-b}{b+c-a}}{\dfrac{a+b-c}{a+c-b}+\dfrac{b+c-a}{a+c+b}} \div \dfrac{\dfrac{a+b+c}{a+b-c}}{\dfrac{b+c-a}{a+c-b}}$ (M. M. 1875).

9. If $\dfrac{a}{b}+\dfrac{b}{a}=\dfrac{c}{d}+\dfrac{d}{c}$, prove that $\dfrac{a^3}{b^3}+\dfrac{b^3}{a^3}=\dfrac{c^3}{d^3}+\dfrac{d^3}{c^3}$.

10. If $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{b}{a}+\dfrac{d}{c}$, prove that $\dfrac{a^5}{b^5}+\dfrac{c^5}{d^5}=\dfrac{b^5}{a^5}+\dfrac{d^5}{c^5}$.

If $a+b+c=0$, prove that

11. $\dfrac{1}{b^2+c^2-a^2}+\dfrac{1}{c^2+a^2-b^2}+\dfrac{1}{a^2+b^2-c^2}=0$.

12. $\dfrac{2a^2}{b^2+c^2-a^2}+\dfrac{2b^2}{c^2+a^2-b^2}+\dfrac{2c^2}{a^2+b^2-c^2}=-3$.

13. $\dfrac{a^2+b^2+c^2}{a^3+b^3+c^3}+\dfrac{2}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)=0$. (M. M. 1877).

14. $\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{1}{c^3}=\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)^3+\dfrac{3}{abc}$. (M. M. 1899).

15. $\left(\dfrac{b-c}{a}+\dfrac{c-a}{b}+\dfrac{a-b}{c}\right)\left(\dfrac{a}{b-c}+\dfrac{b}{c-a}+\dfrac{c}{a-b}\right)=9$.

If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a+b+c}$; prove that

16. $\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{1}{c^3}=\dfrac{1}{a^3+b^3+c^3}=\dfrac{1}{(a+b+c)^3}$.

17. $\dfrac{1}{a^5}+\dfrac{1}{b^5}+\dfrac{1}{c^5}=\dfrac{1}{a^5+b^5+c^5}$.

Prove that

18. $\dfrac{a-b}{(x+a)(x+b)}+\dfrac{b-c}{(x+b)(x+c)}+\dfrac{c-d}{(x+c)(x+d)}=\dfrac{a-d}{(x+a)(x+d)}$

19. $\dfrac{a(a-1)}{(a+1)(a^2+1)}+\dfrac{a^2(a^2-1)}{(a^2+1)(a^4+1)}+\dfrac{a^4(a^4-1)}{(a^4+1)(a^8+1)}$
$=\dfrac{1}{2}\left(\dfrac{a^8-1}{a^8+1}-\dfrac{a-1}{a+1}\right)$.

Simplify the following :—

20. $\dfrac{(a+1)^3}{(a-b)(a-c)}+\dfrac{(b+1)^3}{(b-a)(b-c)}+\dfrac{(c+1)^3}{(c-a)(c-b)}$.

21. $\dfrac{a^2-(b-c)^2}{(a-b)(a-c)}+\dfrac{b^2-(c-a)^2}{(b-c)(b-a)}+\dfrac{c^2-(a-b)^2}{(c-a)(c-b)}$. (M. M. 1890).

22. $\dfrac{b^2+c^2-2a^2}{(a-b)(a-c)}+\dfrac{c^2+a^2-2b^2}{(b-c)(b-a)}+\dfrac{a^2+b^2-2c^2}{(c-a)(c-b)}$. (M. M. 1889).

23. $\dfrac{a^2(b+c)^2}{(a-b)(a-c)}+\dfrac{b^2(c+a)^2}{(b-c)(b-a)}+\dfrac{c^2(a+b)^2}{(c-a)(c-b)}$.

24. $\dfrac{(b+c-a)^2}{(a-b)(a-c)}+\dfrac{(c+a-b)^2}{(b-a)(b-c)}+\dfrac{(a+b-c)^2}{(c-a)(c-b)}$.

25. $\dfrac{(a+b)(a+c)}{(a-b)(a-c)}+\dfrac{(b+a)(b+c)}{(b-a)(b-c)}+\dfrac{(c+a)(c+b)}{(c-a)(c-b)}$.

26. $\dfrac{a(a+b)(a+c)}{(a-b)(a-c)}+\dfrac{b(b+a)(b+c)}{(b-a)(b-c)}+\dfrac{c(c+a)(c+b)}{(c-a)(c-b)}$.

27. $\dfrac{a(b-c)^2}{(a-b)(a-c)}+\dfrac{b(c-a)^2}{(b-a)(b-c)}+\dfrac{c(a-b)^2}{(c-a)(c-b)}$.

28. $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$.

29. $\dfrac{a(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{b(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{c(x-a)(x-b)}{(c-a)(c-b)}$.

30. $\dfrac{a^2(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{b^2(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{c^2(x-a)(x-b)}{(c-a)(c-b)}$.

31. $\dfrac{(a-l)(a-m)}{(a-b)(a-c)}+\dfrac{(b-l)(b-m)}{(b-c)(b-a)}+\dfrac{(c-l)(c-m)}{(c-a)(c-b)}$.

32. $\dfrac{bc(a-l)(a-m)}{(a-b)(a-c)}+\dfrac{ca(b-l)(b-m)}{(b-c)(b-a)}+\dfrac{ab(c-l)(c-m)}{(c-a)(c-b)}$.

33. $\dfrac{(b+c-ma)^2}{(a-b)(a-c)}+\dfrac{(c+a-mb)^2}{(b-c)(b-a)}+\dfrac{(a+b-mc)^2}{(c-a)(c-b)}$.

34. $\dfrac{(1+ab)(1+ac)}{(a-b)(a-c)}+\dfrac{(1+bc)(1+ba)}{(b-c)(b-a)}+\dfrac{(1+ca)(1+cb)}{(c-a)(c-b)}$.

35. $\dfrac{a(1+ab)(1+ac)}{(a-b)(a-c)}+\dfrac{b(1+bc)(1+ba)}{(b-c)(b-a)}+\dfrac{c(1+ca)(1+cb)}{(c-a)(c-b)}$.

36. $\dfrac{a^3(b+c)^2}{(a-b)(a-c)}+\dfrac{b^3(c+a)^2}{(b-c)(b-a)}+\dfrac{c^3(a+b)^2}{(c-a)(c-b)}$.

37. $\dfrac{bc(b+c)^2}{(a-b)(a-c)}+\dfrac{ca(c+a)^2}{(b-c)(b-a)}+\dfrac{ab(a+b)^2}{(c-a)(c-b)}$.

Simplify the following :—

38. $\dfrac{1}{(y+z-2x)(z+x-2y)}+\dfrac{1}{(z+x-2y)(x+y-2z)}$
$+\dfrac{1}{(x+y-2z)(y+z-2x)}.$

39. $\dfrac{x-y}{(y+z-2x)(z+x-2y)}+\dfrac{y-z}{(z+x-2y)(x+y-2z)}$
$+\dfrac{z-x}{(x+y-2z)(y+z-2x)}.$

40. $\dfrac{(y-z)(z-x)}{(y+z-2x)(z+x-2y)}+\dfrac{(z-x)(x-y)}{(z+x-2y)(x+y-2z)}$
$+\dfrac{(x-y)(y-z)}{(x+y-2z)(y+z-2x)}.$

41. $\dfrac{(x-y)^3}{(y+z-2x)(z+x-2y)}+\dfrac{(y-z)^3}{(z+x-2y)(x+y-2z)}$
$+\dfrac{(z-x)^3}{(x+y-2z)(y+z-2x)}.$

42. $\dfrac{b+c-a}{(b+c)(c-a)(a-b)}+\dfrac{c+a-b}{(c+a)(a-b)(b-c)}+\dfrac{a+b-c}{(a+b)(b-c)(c-a)}.$

43. $\dfrac{1}{(a-b)(a-c)(x-a)}+\dfrac{1}{(b-a)(b-c)(x-b)}+\dfrac{1}{(c-a)(c-b)(x-c)}.$

44. $\dfrac{a^2}{(a-b)(a-c)(x-a)}+\dfrac{b^2}{(b-c)(b-a)(x-b)}+\dfrac{c^2}{(c-a)(c-b)(x-c)}.$

45. $\dfrac{3a+4}{(a-b)(a-c)(x-a)}+\dfrac{3b+4}{(b-c)(b-a)(x-b)}+\dfrac{3c+4}{(c-a)(c-b)(x-c)}.$

46. $\dfrac{a^2+a+1}{(a-b)(a-c)(x-a)}+\dfrac{b^2+b+1}{(b-a)(b-c)(x-b)}+\dfrac{c^2+c+1}{(c-a)(c-b)(x-c)}.$

47. $\dfrac{a^3}{(a-b)(a-c)(x-a)}+\dfrac{b^3}{(b-a)(b-c)(x-b)}+\dfrac{c^3}{(c-a)(c-b)(x-c)}.$

48. $\dfrac{(2a+3)^2}{(a-b)(a-c)(x-a)}+\dfrac{(2b+3)^2}{(b-a)(b-c)(x-b)}+\dfrac{(2c+3)^2}{(c-a)(c-b)(x-c)}.$

49. $\dfrac{a^2h+ak+l}{(a-b)(a-c)(x-a)}+\dfrac{b^2h+bk+l}{(b-a)(b-c)(x-b)}+\dfrac{c^2h+ck+l}{(c-a)(c-b)(x-c)}.$

If $x=\dfrac{b^2+c^2-a^2}{2bc}$, $y=\dfrac{c^2+a^2-b^2}{2ca}$, $z=\dfrac{a^2+b^2-c^2}{2ab}$, prove that

50. $(b+c)x+(c+a)y+(a+b)z=a+b+c.$

51. $a(x+yz)=b(y+zx)=c(z+xy).$ (M. M. 1871.)

52. If $\dfrac{x-y}{x+y}=a$, $\dfrac{y-z}{y+z}=b$, $\dfrac{z-x}{z+x}=c$, shew that $(1-a)(1-b)(1-c)$ $=(1+a)(1+b)(1+c)$. (A. E. 1901.)

53. If $x=\dfrac{a+1}{a-1}$, $y=\dfrac{b+1}{b-1}$, $z=\dfrac{c+1}{c-1}$, prove that
$$\frac{(x^2+1)(y^2+1)(z^2+1)}{(yz+1)(zx+1)(xy+1)}=\frac{(a^2+1)(b^2+1)(c^2+1)}{(bc+1)(ca+1)(ab+1)}.$$ (M. M. 1874.)

54. Prove that $\dfrac{bc}{(x-b)(x-c)}+\dfrac{ca}{(x-c)(x-a)}+\dfrac{ab}{(x-a)(x-b)}=0$,

if $\dfrac{1}{x}=\dfrac{1}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$. (M. M. 1873.)

55. If $\dfrac{a-b}{1+ab}+\dfrac{c-d}{1+cd}=0$, then $\dfrac{a+c}{1-ac}=\dfrac{b+d}{1-bd}$, and $\dfrac{a-d}{1+ad}=\dfrac{b-c}{1+bc}$.

If $xy+yz+zx=1$, prove that

56. $\dfrac{1+x^2}{(x+y)(x+z)}+\dfrac{1+y^2}{(y+z)(y+x)}+\dfrac{1+z^2}{(z+x)(z+y)}=3.$

57. $\dfrac{x+yz}{(x+y)(x+z)}+\dfrac{y+zx}{(y+z)(y+x)}+\dfrac{z+xy}{(z+x)(z+y)}=3.$

58. $\dfrac{x+y}{1-xy}+\dfrac{y+z}{1-yz}+\dfrac{z+x}{1-zx}=\dfrac{1}{xyz}.$

59. $\dfrac{x}{1-x^2}+\dfrac{y}{1-y^2}+\dfrac{z}{1-z^2}=\dfrac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$

If $x+y+z=xyz$, prove that

60. $\dfrac{1}{1+x}+\dfrac{1}{1+y}+\dfrac{1}{1+z}-1=\dfrac{2}{(1+x)(1+y)(1+z)}.$

61. $\dfrac{1+x^2}{(x+y)(x+z)}+\dfrac{1+y^2}{(y+z)(y+x)}+\dfrac{1+z^2}{(z+x)(z+y)}=1.$

62. $\dfrac{1}{x-yz}+\dfrac{1}{y-zx}+\dfrac{1}{z-xy}=\dfrac{4xyz}{(x-yz)(y-zx)(z-xy)}.$

63. Shew that if $a+b+c=1$, $bc+ca+ab=\frac{1}{3}$, $abc=\frac{1}{27}$, then $\dfrac{1}{a+bc}+\dfrac{1}{b+ca}+\dfrac{1}{c+ab}=\frac{27}{4}$. (M. M. 1878.)

64. If $a+b+c=2$, $a^2+b^2+c^2=3$, $a^3+b^3+c^3=4$, find the value of $\frac{1}{a-bc}+\frac{1}{b-ca}+\frac{1}{c-ab}$.

65. If $a+b+c=4$, $bc+ca+ab=3$, $abc=2$, find the value of $\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2}$.

CHAPTER XXIV.

ELIMINATION.

1. The subject of elimination is of general use in every branch of Mathematics, and it is proposed to give here an outline of it.

2. Suppose we have two equations involving one unknown quantity x. We can solve for x from one of them and substitute the value of x so obtained in the other, thus getting a relation among the constants which must hold in order that the two equations may be true *for the same value of* x. This relation is called the **eliminant** of the equations, and x is said to be eliminated between them.

Similarly, if we have three equations in x and y we can solve for x and y from any two of them and substitute these values in the third, when x and y are eliminated; and the result is called the **eliminant** of the equations. Reasoning in this way we find that we can eliminate three unknown quantities from four equations, and in general to eliminate n quantities we should have $n+1$ independent equations.

3. We add some illustrative examples on the subject.

Ex. 1. Eliminate x between the equations

$$ax+b=l\ldots\ldots(1),\quad cx+d=m\ldots\ldots\ldots(2)$$

From (1) $ax=l-b$ or $x=(l-b)/a\ldots\ldots\ldots(3)$

From (2) $cx=m-d$ or $x=(m-d)/c\ldots\ldots\ldots(4)$

From (3) and (4) $(l-b)/a=(m-d)/c$, or $a(m-d)=c(l-b)$, which is the required eliminant.

Ex. 2. Eliminate x between the equations

$$ax^2+bx+c=0,\quad a'x^2+b'x+c'=0.$$

By cross-multiplication, $\frac{x^2}{bc'-b'c}=\frac{x}{ca'-c'a}=\frac{1}{ab'-a'b}$.

$\therefore \frac{x^2}{bc'-b'c}\times\frac{1}{ab'-a'b}=\frac{x^2}{(ca'-c'a)^2}$, whence

$(bc'-b'c)(ab'-a'b)=(ca'-c'a)^2$.

Ex. 3. Eliminate x between the equations

$ax^2+bx+c=0.........(1)$, $x^3=1.........(2)$

Multiplying (1) by x, $ax^3+bx^2+cx=0$, hence

from (2), $bx^2+cx+a=0.........(3)$

Eliminating x between (1) and (3) as in the preceding example, we get

$(ab-c^2)(ac-b^2)=(bc-a^2)^2$

$\therefore$ $a^2bc-ab^3-ac^3+b^2c^2=b^2c^2-2a^2bc+a^4$.

Cancelling b^2c^2 from both sides and then dividing by a,

$abc-b^3-c^3=-2abc+a^3$,

or transposing, $a^3+b^3+c^3=3abc$.

Ex. 4. Eliminate x between the equations

$ax^3+bx^2+cx+d=0.....................(1)$

$a'x^2+b'x+c'=0.....................(2)$

Multiply (1) by c' and (2) by d, and subtract;

then $ac'x^3+(bc'-a'd)x^2+(cc'-b'd)x=0$,

or $ac'x^2+(bc'-a'd)x+(cc'-b'd)=0..............(3)$

Eliminating x between (2) and (3) as in the last example,

$\{b'(cc'-b'd)-(bc'-a'd)c'\}\{a'(bc'-a'd)-ac'b'\}$

$=\{ac'^2-a'(cc'-b'd)\}^2$

Ex. 5. Eliminate x and y between the equations

$ax+by+c=0$, $a'x+b'y+c'=0$, $a''x+b''y+c''=0$.

From the first two equations by cross-multiplication,

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{1}{ab'-a'b}$$

$$\therefore\ x=\frac{bc'-b'c}{ab'-a'b},\ y=\frac{ca'-c'a}{ab'-a'b}.$$

Substituting these values of x and y in the third equation,

$$a''.\frac{bc'-b'c}{ab'-a'b}+b''.\frac{ca'-c'a}{ab'-a'b}+c''=0,$$

$\therefore$ $a''(bc'-b'c)+b''(ca'-c'a)+c''(ab'-a'b)=0$, the eliminant required.

Ex. 6. Eliminate x, y, z between the equations

$$x^2-yz=a,\ y^2-zx=b,\ z^2-xy=c,\ x^3+y^3+z^3=3xyz.$$

We have $a+b+c=x^2+y^2+z^2-yz-zx-xy$;

$$\begin{aligned}a^2-bc&=(x^2-yz)^2-(y^2-zx)(z^2-xy)\\&=x^4+xy^3+xz^3-3x^2yz\\&=x(x^3+y^3+z^3-3xyz).\end{aligned}$$

Similarly $b^2-ca=y(x^3+y^3+z^3-3xyz)$,

$$c^2-ab=z(x^3+y^3+z^3-3xyz).$$

$$\begin{aligned}\therefore\ a^3+b^3+c^3-3abc&=(a+b+c)(a^2-bc+b^2-ca+c^2-ab)\\&=(x^2+y^2+z^2-yz-zx-xy)(x+y+z)(x^3+y^3+z^3-3xyz)\\&=(x^3+y^3+z^3-3xyz)^2.\\&=0 \text{ from the fourth equation.}\end{aligned}$$

Hence $a^3+b^3+c^3=3abc$ is the eliminant required.

Ex. 7. Eliminate l, m, n between the equations

$$\frac{m}{n}+\frac{n}{m}=a,\ \frac{n}{l}+\frac{l}{n}=b,\ \frac{l}{m}+\frac{m}{l}=c.$$

We have by multiplying the three equations,

$$\begin{aligned}abc&=\left(\frac{m}{n}+\frac{n}{m}\right)\left(\frac{n}{l}+\frac{l}{n}\right)\left(\frac{l}{m}+\frac{m}{l}\right)\\&=\left(\frac{m}{n}+\frac{n}{m}\right)\left\{\left(\frac{m}{n}+\frac{n}{m}\right)+\left(\frac{mn}{l^2}+\frac{l^2}{mn}\right)\right\}\\&=\left(\frac{m}{n}+\frac{n}{m}\right)^2+\left(\frac{n^2}{l^2}+\frac{l^2}{n^2}\right)+\left(\frac{l^2}{m^2}+\frac{m^2}{l^2}\right)\\&=\left(\frac{m}{n}+\frac{n}{m}\right)^2+\left(\frac{n}{l}+\frac{l}{n}\right)^2+\left(\frac{l}{m}+\frac{m}{l}\right)^2-4.\\&=a^2+b^2+c^2-4.\end{aligned}$$

Hence the required eliminant is $abc=a^2+b^2+c^2-4$.

Ex. 8. Eliminate l, m, n between the equations

$$\frac{m}{n}-\frac{n}{m}=a,\ \frac{n}{l}-\frac{l}{n}=b,\ \frac{l}{m}-\frac{m}{l}=c.$$

Squaring the given equations,

$$\frac{m^2}{n^2}+\frac{n^2}{m^2}=a^2+2,\ \frac{n^2}{l^2}+\frac{l^2}{n^2}=b^2+2,\ \frac{l^2}{m^2}+\frac{m^2}{l^2}=c^2+2.$$

Hence eliminating l^2, m^2, n^2 as in the last example,

$$(a^2+2)(b^2+2)(c^2+2)=(a^2+2)^2+(b^2+2)^2+(c^2+2)^2-4.$$

$\therefore$ simplifying $2b^2c^2+2c^2a^2+2a^2b^2+a^2b^2c^2=a^4+b^4+c^4$.

Ex. 9. Eliminate x, y, z between the equations

$$x=by+cz\ldots(1),\ y=cz+ax\ldots(2),\ z=ax+by\ldots(3)$$

Adding ax to both sides of (1),

$$ax+x=ax+by+cz \text{ or } x(a+1)=ax+by+cz.$$

$$\therefore \frac{x}{ax+by+cz}=\frac{1}{a+1} \text{ or } \frac{ax}{ax+by+cz}=\frac{a}{a+1}\ldots(4).$$

Similarly, adding by to both sides of (2), and cz to both sides of (3),

$$\frac{by}{ax+by+cz}=\frac{b}{b+1}\ldots(5).$$

$$\frac{cz}{ax+by+cz}=\frac{c}{c+1}\ldots(6).$$

Adding (4), (5), (6),

$$\frac{ax+by+cz}{ax+by+cz}=\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}$$

or $1=\dfrac{a}{a+1}+\dfrac{b}{b+1}+\dfrac{c}{c+1}$, the eliminant required.

EXERCISE CIX.

1. Eliminate x between the equations

(1) $ax+b=0$, $cx+d=0$. (2) $ax^2+bx+c=0$, $x^3=d$.

(3) $ax^3+bx+c=0$, $a'x^3+b'x+c'=0$.

2. Eliminate x and y from the equations

(1) $x+y=a$, $x^2+y^2=b^2$, $xy=c^2$.

(2) $x+y=a$, $x^2+y^2=b^2$, $x^3+y^3=c^3$.

(3) $x+y=a$, $x^2+y^2=b^2$, $x^4+y^4=c^4$.

(4) $ax+by+c=0$, $a'x+b'y+c'=0$, $x^2+y^2=d^2$.

Eliminate x, y, z from

3. (1) $x+y+z=a$, $yz+zx+xy=b^2$, $x^3+y^3+z^3-3xyz=c^3$.

(2) $x+y+z=a$, $x^2+y^2+z^2=b^2$, $x^3+y^3+z^3=c^3$, $xyz=d^3$.

(3) $x+y+z=a$, $yz+zx+xy=b^2$, $x^3+y^3+z^3=c^3$, $xyz=d^3$.

(4) $ax+yz=bc$, $by+zx=ca$, $cz+xy=ab$, $xyz=abc$.

4. $ax+hy+gz=0$, $hx+by+fz=0$, $gx+fy+cz=0$.

5. (1) $x^2+2yz=a$, $y^2+2zx=b$, $z^2+2xy=c$, $yz+zx+xy=d$.

(2) $x^2(y-z)=a$, $y^2(z-x)=b$, $z^2(x-y)=c$, $xyz=d$.

(3) $x^2(y+z)=a$, $y^2(z+x)=b$, $z^2(x+y)=c$, $xyz=d$.

(4) $x^2-yz=a$, $y^2-zx=b$, $z^2-xy=c$, $yz+zx+xy=d$.

6. Eliminate m from

$$y=mx-2am-am^3,\quad x=2a+3am^2.$$

7. Eliminate x, y, z from

(1) $yz=a^2,\ zx=b^2,\ xy=c^2,\ xyz=d^3.$

(2) $x=cy+bz,\ y=az+cx,\ z=bx+ay.$

(3) $x-a=y-b=z-c=d,\ x^3+y^3+z^3=3xyz.$

(4) $x+\dfrac{1}{x}=a,\ y+\dfrac{1}{y}=b,\ z+\dfrac{1}{z}=c,\ xyz=1.$

8 Eliminate a, b from

$$\frac{x}{a}+\frac{y}{b}=1,\ a^n+b^n=c^n,\ \frac{x}{a^{n+1}}=\frac{y}{b^{n+1}}.$$

9. Eliminate x from

$x^5+10x+5/x^3=a,\ 5x^3+10/x+1/x^5=b$

10. Eliminate x, y from

$4(x^2+y^2)=ax+by,\ 2(x^2-y^2)=ax-by,\ xy=c^2.$

CHAPTER XXV.

MISCELLANEOUS PROPOSITIONS.

1. We shall here consider some propositions of a miscellaneous kind. We begin with multiplication and division by detached co efficients.

2. **Detached Co-efficients.** The method of detached co-efficients shortens work and can be used after a little practice. The following examples illustrate the process.

Ex. 1. Multiply $3x^3-4x^2+7x-2$ by x^2-2x+3.

Writing down simply the co-efficients, we proceed thus :

$$\begin{array}{l}
3-4+7-2\\
\underline{1-2+3}\\
3-4+7-2\\
-6+8-14+4\\
\underline{+9-12+21-6}\\
3-10+24-28+25-6
\end{array}$$

The product is of the 5th degree in x and since the multiplier and multiplicand are in descending powers of x, the product, will be also in descending powers of x. Hence the required product

$$=3x^5-10x^4+24x^3-28x^2+25x-6.$$

The student may see the truth of the above by writing down the product fully.

Ex. 2. Divide $1-\frac{3}{2}x^2+4x^3-8x^4$ by $1+\frac{5}{3}x-3x^2$.

Supplying $0\,x$, we proceed thus :—

$$
\begin{array}{l}
1+\frac{5}{3}-3\Big)\ 1+0-\frac{3}{2}+4-8\ \Big(1-\frac{5}{3}+\frac{77}{18} \\
\qquad\qquad\quad \underline{1+\frac{5}{3}-3} \\
\qquad\qquad\qquad -\frac{5}{3}+\frac{3}{2}+4 \\
\qquad\qquad\qquad \underline{-\frac{5}{3}-\frac{25}{9}+5} \\
\qquad\qquad\qquad\qquad \frac{77}{18}-1-8 \\
\qquad\qquad\qquad\qquad \underline{\frac{77}{18}+\frac{385}{54}-\frac{77}{6}} \\
\qquad\qquad\qquad\qquad\qquad -\frac{439}{54}+\frac{29}{6}
\end{array}
$$

Hence the quotient $=1-\frac{5}{3}x+\frac{77}{18}x^2$.

remainder $=-\frac{439}{54}x^3+\frac{29}{6}x^4$.

Note. In dividing by the method of detached co-efficients, co-efficients of the same powers of x should be written in the same column. Thus $-\frac{439}{54}$ in the remainder coming under the term $4x^3$ of the dividend means $-\frac{439}{54}x^3$; so $\frac{29}{6}$ means $\frac{29}{6}x^4$.

3. Multiplication by a binomial, by the method of detached co-efficients is useful. We consider a simple case below.

Ex. 1. Multiply $3x^3-4x^2+7x-2$ by $1+x$.

Ordinary method : $3x^3-4x^2+7x-2$

$$
\begin{array}{r}
\underline{x+1\qquad\qquad} \\
3x^4-4x^3+7x^2-2x\qquad \\
\underline{3x^3-4x^2+7x-2} \\
3x^4-x^3+3x^2+5x-2\qquad
\end{array}
$$

Observe that in forming co-efficients in the product we add each co-efficient in the multiplicand to the previous one.

Hence the following method of detached co-efficients .

$$
\begin{array}{l}
\underline{3-4+7-2\quad} \\
3-1+3+5-2
\end{array}
$$

Here $-1=3-4$, $+3=-4+7$, $+5=7-2$.

$\therefore$ product (being of the 4th degree in x)

$$=3x^4-x^3+3x^2+5x-2.$$

Ex. 2. Multiply $5x^4-2x^3+3x^2-6x+10$ by $(1+x)^3$.

Here we multiply three times by $1+x$. The process stands thus :—

$$
\begin{array}{l}
\underline{5-2+3-6+10} \\
\underline{5+3+1-3+4+10} \\
\underline{5+8+4-2+1+14+10} \\
5+13+12+2-1+15+24+10
\end{array}
$$

Thus the product (being of the 7th degree)

$$=5x^7+13x^6+12x^5+2x^4-x^3+15x^2+24x+10.$$

4. We are sometimes required to determine a particular co-efficient in a product. Remembering that terms in the product of two expressions are obtained by multiplying *every* term of one expression by *every* term of the other, it follows that we can pick out any desired terms in the product and hence any desired co-efficient.

Ex. Find the co-efficient of x^4 in the product—

$$(3x^5-2x^4-4x^3+3x^2-x+2)(2x^2-3x+5).$$

The terms giving rise to x^4 in the product are $-2x^4\times 5$,

$-4x^3\times(-3x)$, $3x^2\times 2x^2$. Hence the co-efficient of x^4

$=-10+12+6=8.$

5. Indirect Multiplication and division.—Sometimes a product or a quotient is obtained conveniently in an indirect manner, as in the following examples.

Ex. 1. Multiply $(x+a)(x^2+a^2)(x^4+a^4)(x^8+a^8)$

Denoting the product by P, we have

$$\begin{aligned}
P&=(x+a)(x^2+a^2)(x^4+a^4)(x^8+a^8)\\
\therefore\ P(x-a)&=(x-a)(x+a)(x^2+a^2)(x^4+a^4)(x^8+a^8)\\
&=(x^2-a^2)(x^2+a^2)(x^4+a^4)(x^8+a^8)\\
&=(x^4-a^4)(x^4+a^4)(x^8+a^8)\\
&=(x^8-a^8)(x^8+a^8)=(x^{16}-a^{16}).
\end{aligned}$$

$$\begin{aligned}
\therefore\ P&=\frac{x^{16}-a^{16}}{x-a}\\
&=x^{15}+x^{14}a+x^{13}a^2+\ldots\ldots+xa^{14}+a^{15}.
\end{aligned}$$

Ex. 2. Divide $x^8+x^6y^2+x^4y^4+x^2y^6+y^8$

by $x^4-x^3y+x^2y^2-xy^3+y^4$.

$$\begin{aligned}
\text{Here quotient}&=\frac{x^8+x^6y^2+x^4y^4+x^2y^6+y^8}{x^4-x^3y+x^2y^2-xy^3+y^4}\\
&=\frac{(x^8+x^6y^2+x^4y^4+x^2y^6+y^8)\times(x^2-y^2)}{(x^4-x^3y+x^2y^2-xy^3+y^4)(x+y)\times(x-y)}\\
&=\frac{x^{10}-y^{10}}{(x^5+y^5)(x-y)}=\frac{(x^5+y^5)(x^5-y^5)}{(x^5+y^5)(x-y)}\\
&=\frac{x^5-y^5}{x-y}=x^4+x^3y+x^2y^2+xy^3+y^4.
\end{aligned}$$

EXERCISE CX.

Multiply (using detached co-efficients).

1. $7x^4-5x^3+3x^2-2x+3$ by $3x^2-4x+1$.
2. $x^5+3x^4-2x^3+7x^2-3x+2$ by $2x^2-3x+4$.
3. $\frac{1}{2}x^3-\frac{1}{3}x^2+\frac{2}{3}x-4$ by $\frac{3}{2}x^2-\frac{1}{2}x+\frac{4}{7}$.

Divide (using detached co-efficients)

4. $6x^4-3x^3+4x^2-5x+6$ by $3x^2-2x+1$.

5. $\frac{1}{2}x^4-\frac{2}{3}x^3+2x^2-\frac{3}{2}x+5$ by $\frac{1}{3}x^2-\frac{1}{2}x+2$.

6. Multiply $6x^5-3x^4+7x^3-2x-7$ by $(1+x)^4$.

7. Pick out the co-efficient of x^3 in

$$(3x^5-2x^4+7x^3-5x^2+7x-2)(3x^3-4x^2+7x-7).$$

8. Pick out the co-efficient of x^4 in

$$(\tfrac{1}{2}x^5-2x^4+\tfrac{3}{2}x^3-3x^2+4x-5)(\tfrac{3}{2}x^2-2x+3).$$

---

6. Homogeneous and Symmetrical Expressions. An expression is said to be *homogeneous* in two or more letters when each term of the expression is of the same dimensions in those letters.

Thus $ax^2-2a^2x+x^3$ is homogeneous and of three dimensions in a and x.

An expression is said to be *symmetrical* in two letters when it is not altered by the interchange of these letters.

Thus $3x^3+5xy+3y^3$ is symmetrical with respect to x and y.

7. *The product of two homogeneous expressions is homogeneous and its dimension is the sum of the dimensions of the expressions.*

For, if one expression is of the degree m and another of the degree n, then each term in the product (being obtained by multiplying some term in one expression by some term in the other) is of the degree $m+n$, *i.e.* the product is homogeneous and of $m+n$ dimensions.

Ex. 1. Multiply $3x^4+5x^3y-4x^2y^2+2xy^3-2y^4$ by $x+y$. We multiply by using the method of detached co-efficients :

$$\frac{3+5-4+2-2}{3+8+1-2+0-2}$$

Here $8=3+5$, $1=5-4$, $-2=-4+2$, $0=2-2$.

Now the product is homogeneous and of degree five. Hence the product$=3x^5+8x^4y+x^3y^2-2x^2y^3+0xy^4-2y^5$.

Ex. 2. Find the homogeneous expression of the second degree symmetrical in x, y, z which is equal to 32 when $x=1$, $y=-2$, $z=3$ and equal to 16 when $x=-1$, $y=2$, $z=1$.

A homogeneous expression of the second degree which is symmetrical in x, y, z is of the form

$$k(x^2+y^2+z^2)+l(yz+zx+xy), \text{ where, } k \text{ and } l \text{ are constant.}$$

Putting $x=1, y=-2, z=3$; and again $x=-1, y=2, z=1$, from given conditions, we get

$$14k-5l=32\ldots\ldots(2) \quad 6k-l=16\ldots\ldots(2)$$

Solving we have $k=3$, $l=2$.

Hence the expression is $3(x^2+y^2+z^2)+2(yz+zx+xy)$.

8. Zero and infinity. In Mathematics the student will observe that zero has two meanings. When 5 is subtracted from 5 or a from a, the remainder is nothing, or, *absolute zero*. Again, when a quantity is dimmished gradually so that it can be made smaller than any small quantity which can be named, such a quantity is said to be ultimately equal to *zero*.

The latter definition requires a little explanation Suppose from a finite straight line AB one half is taken away, then one half of the remainder, then one half of what remains and so on, what is the final remainder? It is evident from the process that each remainder goes on diminishing and by carrying the process long enough, we can make the final remainder *as small as we please*. This is shortly stated by saying that the final remainder is zero.

Similarly, the word *infinity* (in symbol ∞) is used to mean a quantity which can be made larger than any large quantity.

A quantity which is neither zero nor infinity is called a *finite quantity*.

9. *Forms* $A\times O$, $O\times A$, $O\div A$, $A\div O$.

We already know that $A\times O=O$, so also $O\times A=O$. Again, to find $O\div A$, we see from the adjacent scheme that $O\div A=O$.

$$A\,)\,O\,(\,O \\ O$$

Lastly, we come to $A\div O$. To understand this we consider the values of $\frac{7}{\frac{1}{10}}, \frac{7}{\frac{1}{10^2}}, \frac{7}{\frac{1}{10^3}}$ .... in which the numerator is constant and the denominator goes on diminshing. It is evident that

$$\frac{7}{\frac{1}{10}}, \frac{7}{\frac{1}{10^2}}, \frac{7}{\frac{1}{10^3}} \quad \ldots\ldots\ldots$$

are fractions which become larger and larger.

Thus $\frac{7}{\frac{1}{10}}=70$, $\frac{7}{\frac{1}{10^2}}=700$, $\frac{7}{\frac{1}{10^3}}=7000$...........

Hence if the denominator of a fraction becomes smaller and smaller (the numerator remaining constant), the fraction tends to become larger and larger or shortly.

$$\frac{A}{0} = \infty \text{ (infinity).}$$

Note. If $AB=0$, then either $A=0$ or $B=0$. In other words, *if the product of two (or more) quantities be zero, any one of them may be zero.*

10. From $\frac{0}{0}$. If we divide 0 by 0, the quotient may be any (finite) quantity as shown in the annexed scheme. Hence a fraction of the form $\frac{0}{0}$ is indeterminate in value.

0) 0 (any quantity
  —

The student will understand more of this form afterwards.

11. Theorem. If the sum of the squares of any number of real quantities be zero, then each of the quantities must be separately zero.

Let $x^2+y^2=0$. Then $\because$ the square of a *real* quantity (positive or negative) is always *positive*, we have x^2, y^2 both positive. Hence the sum of x^2 and y^2 (two positive quantities) cannot be zero, unless $x^2=0$ and $y^2=0$ simultaneously *i.e.* unless x and y be both zero.

Similarly, if $x^2+y^2+z^2=0$ (x, y, z being real), we must have $x=y=z=0$.

Note. The significance of the word *real* in the Theorem will be understood when the student reads about imaginary quantities.

Ex. 1. If $a^3+b^3+c^3=3abc$, prove that either $a+b+c=0$ or $a=b=c$ (a, b, c being all real).

From the hypothesis, $a^3+b^3+c^3-3abc=0$.

$$\therefore \tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}=0.$$

$$\therefore \text{ either } a+b+c=0 \text{ or } (a-b)^2+(b-c)^2+(c-a)^2=0.$$

The latter $(a-b)^2+(b-c)^2+(c-a)^2=0$ gives by the theorem

$$a-b=0,\ b-c=0,\ c-a=0 \text{ or } a=b=c.$$

12. Inequalities. A quantity a is said to be *greater than* another quantity b if $a-b$ is positive. The following is an important theorem on ineqalities.

The sum of the squares of two real unequal quantities is greater than twice their product.

If a and b be real and unequal, the theorem states that

$$a^2+b^2>2ab.$$

We have $(a-b)^2$ positive, whether $a>b$ or $b>a$.

$$\therefore\ a^2+b^2-2ab \text{ is positive}$$

$$\therefore\ a^2+b^2>2ab.$$

Note. If a and b be real and equal, then $a^2+b^2=2ab$, so a^2+b^2 *is never less than* $2ab$.

Ex. 1. If a, b, c be real and unequal, prove that

$$a^2+b^2+c^2>ab+bc+ca.$$

We have
$$a^2+b^2>2ab,$$
$$b^2+c^2>2bc,$$
$$c^2+a^2>2ca.$$

$\therefore$ adding $2(a^2+b^2+c^2)>2(ab+bc+ca)$,

or $a^2+b^2+c^2>ab+bc+ca$.

Ex. 2. A man receives $\frac{x}{y}$ths of Rs. 10 and afterwords $\frac{y}{x}$ths of Rs. 10. He then gives away Rs. 20. Show that he cannot lose by the transaction. (C. E. 1881.)

The man altogether receives $\left(\frac{x}{y}+\frac{y}{x}\right)$ 10 rupees, while he gives away 20 rupees. We assume x and y to be real and positive.

$$\text{Now } \left(\frac{x}{y}+\frac{y}{x}\right)10-20$$

$$=\frac{(x^2+y^2)10}{xy}-20$$

$$=\frac{10(x^2+y^2)-20xy}{xy}$$

$=\frac{10}{xy}(x-y)^2$, which may be *zero* (if x and y be equal) or *positive* (if x and y be unequal) but never negative.

$$\therefore\ \left(\frac{x}{y}+\frac{y}{x}\right)10 \nless 20, \text{ or the man cannot lose.}$$

EXERCISE CXI.

1. A homogeneous and symmetrical expression of the second degree in x and y shall be equal to 3, when x and y are equal to unity, and shall be equal to 11, when $x=2$, $y=1$. Show that $5(x^2+y^2)-7xy$ is the expression.

2. If $a^2+b^2+c^2=bc+ca+ab$, prove that $a=b=c$.

Note. In examples $2-7$ the quantities a, b, c are real.

3. If $(1+a)^2+(1+b)^2+(1+c)^2=4(a+b+c)$, prove that $a=b=c=1$.

4. If $1+a^2+b^2=a+b+ab$, prove that $a=b=1$.

5. If $4a^2+3b^2+5c^2=2(ab+2bc+3ca)$, prove that $a=b=c$.

6. If a, b, c be real, positive and unequal, prove that $(a+b)(b+c)(c+a)>8abc$.

7. A man receives $\dfrac{x+2a}{y}$ths of Rs. 5 and again $\dfrac{y}{x+2a}$ths of Rs. 5. He then gives away Rs. 10. Show that he cannot lose by the transaction. (P. E. 1889).

13. Conditional relations.—We have already considered various conditional identities. The following are additional examples.

Ex. 1. If $x(b-c)+y(c-a)+z(a-b)=0$, prove that

$$\frac{bz-cy}{b-c}=\frac{cx-az}{c-a}=\frac{ay-bx}{a-b}.$$

From hypothesis, $x(b-c)+y(c-a)+z(a-b)=0$.

Also identically, $a(b-c)+b(c-a)+c(a-b)=0$.

$\therefore$ by cross multiplication, $\dfrac{b-c}{cy-bz}=\dfrac{c-a}{az-cx}=\dfrac{a-b}{bx-ay}$.

$$\therefore \frac{bz-cy}{b-c}=\frac{cx-az}{c-a}=\frac{ay-bx}{a-b}.$$

Ex. 2. If $ax+by+cz=0$ and $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=0$, prove that $ax^2+by^2+cz^2+(a+b+c)(xy+yz+zx)=0$.

From the given relations, $ax+by+cz=0$,

$$ayz+bzx+cxy=0.$$

Hence by cross multiplication,

$$\frac{a}{x(y^2-z^2)}=\frac{b}{y(z^2-x^2)}=\frac{c}{z(x^2-y^2)}=k \text{ (suppose)}$$

$\therefore\quad a=kx(y^2-z^2),\ b=ky(z^2-x^2),\ c=kz(x^2-y^2)$...(1)

$\therefore\quad ax^2=kx^3(y^2-z^2),\ by^2=ky^3(z^2-x^2),\ cz^2=kz^3(x^2-y^2)$.

Adding, $ax^2+by^2+cz^2$

$$=k\{x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)\}$$
$$=-k(y-z)(z-x)(x-y)(xy+yz+zx)\text{...(2)}$$

Again from (1) by adding,

$$a+b+c=k\{x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)\}$$
$$=k(y-z)(z-x)(x-y).\qquad\text{...(3)}$$

Hence from (2) and (3) by division,

$$\frac{ax^2+by^2+cz^2}{a+b+c} = -(xy+yz+zx),$$

whence $ax^2+by^2+cz^2+(a+b+c)(xy+yz+zx)=0$.

Ex. 3. Find the condition that the expr. $x^4+ax^3+bx^2+cx+d$ may be a perfect square with respect to x for all values of x.

We proceed to extract the square root thus.

$$\begin{array}{l|l}
 & x^4+ax^3+bx^2+cx+d \;\Big|\; x^2+\frac{ax}{2}+\frac{4b-a^2}{8} \\
 & x^4 \\ \hline
2x^2+\frac{ax}{2} & ax^3+bx^2 \\
 & ax^3+\frac{a^2x^2}{4} \\ \hline
2x^2+ax+\frac{4b-a^2}{8} & \frac{4b-a^2}{4}x^2+cx \qquad +d \\
 & \frac{4b-a^2}{4}x^2+\frac{a(4b-a^2)}{8}x+\left(\frac{4b-a^2}{8}\right) \\ \hline
 & \left\{c-\frac{a(4b-a^2)}{8}\right\}x+d-\left(\frac{4b-a^2}{8}\right)^2
\end{array}$$

If the expression is a perfect square for all values of x, the above remainder must be zero *for all values of* x. Hence $c-\frac{a(4b-a^2)}{8}$ $=0$ and $d-\left(\frac{4b-a^2}{8}\right)^2=0$. The conditions give

$a(4b-a^2)=8c$ and $(4b-a^2)^2=64d$

Ex. 4. If x, y, z are unequal, and if $2a-3y=\frac{(z-x)^2}{y}$, and $2a-3z=\frac{(x-y)^2}{z}$, then will $2a-3x=\frac{(y-z)^2}{x}$, and $x+y+z=a$

From the given relations,

$2ay-3y^2=(z-x)^2$...(1), $2az-3z^2=(x-y)^2$.....................(2)

$\therefore$ By the subtraction, $2a(y-z)-3(y^2-z^2)=(z-y)(z-2x+y)$

$\because$ $y-z$ is not 0 we divide out by it.

Then $2a-3(y+z)=2x-y-z$.

$$\therefore\ x+y+z=a \qquad (3)$$

Now to prove $2a-3x=\frac{(y-z)^2}{x}$, or, $2ax-3x^2=(y-z)^2$......(4)

Evidently (4) is true, for subtracting (1) from (4) we have $2a(x-y)-3(x^2-y^2)=(y-x)(y-2z+x)$ or dividing out by $x-y$ which is not 0, $2a-3(x+y)=2z-x-y$,

or $x+y+z=a$, which is true by (3).

EXERCISE CXII.

1. If $x(b-c)+y(c-a)+z(a-b)=0$, prove that

$$\frac{b-c}{y-z}=\frac{c-a}{z-x}=\frac{a-b}{x-y}.$$

2. If $ax+by+cz=0$ and $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$,

prove that $ax^3+by^3+cz^3+(a+b+c)(y+z)(z+x)(x+y)=0$.

3. If $x+\frac{1}{y}=1$ and $y+\frac{1}{z}=1$, prove that $z+\frac{1}{x}=1$ and $xyz+1=0$. (B. M. 1887).

4. If $x+\frac{1}{y}=y+\frac{1}{z}=z+\frac{1}{x}$, then $x^2y^2z^2=1$, or, $x=y=z$.

5, If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$ prove that $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{2n+1}$

$$=\frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}.$$

6. Show that if $\frac{a-b}{c}+\frac{b-c}{a}+\frac{c+a}{b}=1$ and $a-b+c$ is not $=0$, then $\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$. (C. E. 1875).

7. If $ab+ac+bc=1$, prove that,

$$\left(1-\frac{a^2}{1+a^2}-\frac{b^2}{1+b^2}-\frac{c^2}{1+c^2}\right)^2=\frac{4a^2b^2c^2}{(1+a^2)(1+b^2)(1+c^2)}.$$

8. If $\frac{a}{b+c}+\frac{c}{a+b}=\frac{2b}{c+a}$, prove that $a+b+c=0$

or, $b^2=\frac{1}{2}(c^2+a^2)$.

9. If $x+y=2z$, show that (i) $\frac{x}{x-z}+\frac{y}{y-z}=2$. (B. M. 1882).

(ii) $\frac{x}{x-z}+\frac{z}{y-z}=1$.

If $a+b+c=0$, prove that

10. $\frac{ab}{a^2+ab+b^2}+\frac{bc}{b^2+bc+c^2}+\frac{ca}{c^2+ca+a^2}=-1$.

If $a+b+c=0$, prove that

11. $\dfrac{a^2}{b^2+c^2-a^2}+\dfrac{b^2}{c^2+a^2-b^2}+\dfrac{c^2}{a^2+b^2-c^2}=-\dfrac{3}{2}.$

12. If x, y, z be unequal, and $y^2+z^2+myz=z^2+x^2+mzx$ $=x^2+y^2+mxy$, then each $=\frac{1}{2}(x^2+y^2+z^2)$.

13. If x, y, z be unequal and

$$x-\frac{ayz}{x^2}=y-\frac{azx}{y^2}=z-\frac{axy}{z^2}\text{, then each}=x+y+z-a.$$

14. If $a^2-b^2=b^2-c^2=c^2-a^2$, shew that

$$\frac{ab-c^2}{a-b}+\frac{bc-a^2}{b-c}+\frac{ca-b^2}{c-a}=0. \qquad \text{(B. M. 1891).}$$

15. If $\dfrac{1}{1+l+lm}+\dfrac{1}{1+m+mn}+\dfrac{1}{1+n+nl}=1$, prove that either

$$lm=1 \text{ or } (1+l)(1+m)(1+n)=-1. \qquad \text{(B.P. 1889)}$$

16. If $\dfrac{(2x-y-z)^3}{x}=\dfrac{(2y-z-x)^3}{y}$, then each $=\dfrac{(2z-x-y)^3}{z}$.

17. If $\dfrac{a-1}{x}-\dfrac{a-2}{y}=\dfrac{1}{b}$ and $\dfrac{b-1}{x}-\dfrac{b-2}{y}=\dfrac{1}{a}$,

then $\dfrac{c-1}{x}-\dfrac{c-2}{y}=\dfrac{c}{ab}$. (M. M. 1869)

18. If $\dfrac{1-2bx+b^2}{1-b^2}=\dfrac{1-b^2}{1+2by+b^2}$, then $\dfrac{x-y}{1-xy}=\dfrac{2b}{1+b^2}$ (B. M. 1888).

MISCELLANEOUS EXERCISE PAPERS (IV).

Paper I.

1. Solve the equations :—

   (i) $a(x+y)-b(x-y)=2a^2(a^2-b^2)$, $(x-y)=4a^2b$.

   (ii) $2(x-a)(a+b)-(x-3b)(a-b)=8ab$.

2. Prove that $(b-c)^2(c-a)^2+(c-a)^2(a-b)^2+(a-b)^2(b-c)^2$
$$=(a^2+b^2+c^2-bc-ca-ab)^2.$$

3. Find the values of A, B, C, such that the equation $x^2-2x+1=A(x^2+1)+B(x-2)-C$ may be an identity.

4. Without actual division find the remainder when $x^4-2x^3-2x^2-1$ is divided by $x-a$.

5. If $a:b=c:d=e:f$ prove that $\left(\dfrac{a+5c+7e}{b+5d+7f}\right)^2=\dfrac{ac+ce}{bd+df}$.

6. Prove that $(2a+b+c)^2(b-c)+(2b+c+a)^2(c-a)+(2c+a+b)^2 \times (a-b) = -(b-c)(c-a)(a-b)$.

7. Eliminate x, y, z between

$$bx+ay=cy+bz=az+cx=\frac{x}{a}+\frac{y}{b}+\frac{z}{c}.$$

8. Simplify

$$\frac{a^2(a+b)(a+c)}{(a-b)(a-c)}+\frac{b^2(b+c)(b+a)}{(b-c)(b-a)}+\frac{c^2(c+a)(c+b)}{(c-a)(c-b)}.$$

PAPER II.

1. If $x^2+x+1=0$, then $x^3=1$ and $x^{2n}+x^{2n-1}+1=0$.

2. Simplify :— $\dfrac{x^2-yz}{(x+y)(x+z)}+\dfrac{y^2-zx}{(y+z)(y+x)}+\dfrac{z^2-xy}{(z+x)(z+y)}$.

3. If $2s=a+b+c$, prove that

$$a(b-c)(s-a)^2+b(c-a)(s-b)^2+c(a-b)(s-c)^2=0.$$

4. Eliminate x and y from $x+1/x=a$, $y+1/y=b$, $xy+1/(xy)=c$.

5. The expression $ax-3b$ is equal to 30 when x is 3; and to 42, when x is 7 ; what is its value when x is $4\cdot\dot{3}$; and for what value of x is it zero ? (C. E. 1874).

6. Prove that $a(b+c)(b^2+c^2-a^2)+b(c+a)(c^2+a^2-b^2)+c(a+b) \times (a^2+b^2-c^2)=2abc(a+b+c)$.

7. Resolve into factors $(b-c)(1+ab)(1+ac)+(c-a)(1+bc)(1+ba) +(a-b)(1+ca)(1+cb)$.

8. If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$ prove that $\dfrac{x^3}{a^2}+\dfrac{y^3}{b^2}+\dfrac{z^3}{c^2}=\dfrac{(x+y+z)^3}{(a+b+c)^2}$.

PAPER III.

1. Plot the points P (0, 8), Q (−4, 14), R (−2, −1).

Find (1) the co-ordinates of D, E, F, the middle points of QR, RP, PQ respectively (2) the equations to PD, QE, RF.

2. If $a+b=c+d$, prove that either of them is equal to

$$\frac{abcd}{ab+cd}\left\{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right\} \qquad \text{(B. M. 1887).}$$

3. If $a : b=c : d$, then $a^2 : c^2 :: a^2p+abq+b^2r : c^2p+cdq+d^2r$.

4. Eliminate x, y and z from the equations—

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=a, \frac{x}{z}+\frac{y}{x}+\frac{z}{y}=b, \left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)=c.$$

5. Find a homogeneous and symmetrical expression of the second degree in x and y which shall be equal to 3 when x and y are each equal to unity and shall be equal to 11 when $x=2$, $y=1$.
(P. E. 1900).

6. Prove that $(a+b)^5-a^5-b^5=5ab(a+b)(a^2+ab+b^2)$.

7. Resolve into factors—

(i) $a(a+2b)^3-b(b+2a)^3$

(ii) $4(a-b)^3-(a-4b)(2a+b)^2$

8. Simplify :—

(i) $(b+c-a)^3+(c+a-b)^3+(a+b-c)^3-(a+b+c)^3$.

(ii) $\dfrac{3a+2b+2c}{(a-b)(a-c)}+\dfrac{3b+2c+2a}{(b-c)(b-a)}+\dfrac{3c+2a+2b}{(c-a)(c-b)}$.

PAPER IV.

1. Show that $a+(1-a)b+(1-a)(1-b)c+(1-a)(1-b)(1-c)d$
$=1-(1-a)(1-b)(1-c)(1-d)$.

2. Prove that $x^4+px^3+qx^2+rx+s$ is a perfect square if $p^2s=r^2$ and $p^3-4pq+8r=0$.

3. Find the cube root of
$8a^{12}-36a^{10}+66a^8-63a^6+33a^4-9a^2+1$.

4. Eliminate x, y, z from the equations

$$c\frac{y}{z}+b\frac{z}{y}=2f,\ a\frac{z}{x}+c\frac{x}{z}=2g,\ b\frac{x}{y}+a\frac{y}{x}=2h.$$

5. Show that, if $a:b=c:d$, then

(i) $a+b:c+d::\sqrt{(2a^2-3b^2)}:\sqrt{(2c^2-3d^2)}$.

(ii) $ab+cd$ is a mean proportional to a^2+c^2 and b^2+d^2.

6. If $\dfrac{x}{b+c}=\dfrac{y}{c+a}=\dfrac{z}{a+b}$, prove that

$$\frac{x+y+z}{a+b+c}=\frac{ax+by+cz}{bc+ca+ab}$$

7. Prove that $\{(b-c)^2+(c-a)^2+(a-b)^2\}^2$
$=2\{(b-c)^4+(c-a)^4+(a-b)^4\}$.

8. Prove that $(1-a^2)(1-b^2)(1-c^2)-(a+bc)(b+ca)(c+ab)$
$=(1+abc)(1-a^2-b^2-c^2-2abc)$.

PAPER V.

$a+b$

1. Solve (i) $\frac{bc(ax-1)}{b+c}+\frac{ca(bx-1)}{c+a}+\frac{ab(cx-1)}{a+b}=a+b+c$

(C. E. 1902).

(ii) $\frac{bc(ax-1)}{a(b+c)}+\frac{ca(bx-1)}{b(c+a)}+\frac{ab(cx-1)}{c(a+b)}=3.$

2. If $a : x=b : y=c : z$ and $\frac{x^2}{p^2}+\frac{y^2}{q^2}+\frac{z^2}{r^2}=1$ prove that

$$\frac{a^2}{p^2}+\frac{b^2}{q^2}+\frac{c^2}{r^2}=\frac{a^2+b^2+c^2}{x^2+y^2+z^2}.$$

3. Prove that $x^{mn}-y^{mn}$ is divisible by both x^m-y^m and x^n-y^n.

4. If $a+b+c=0$, show that $a^6+b^6+c^6=3a^2b^2c^2+\frac{1}{4}(a^2+b^2+c^2)^3$.

5. Eliminate x, y, z, from the equations $x^2-yz=a$, $y^2-zx=b$, $z^2-xy=c$, $(x+y+z)(a+b+c)=1$.

6. Simplify $\frac{a^4-a^3b-ab^3+b^4}{a^4+3a^3b+4a^2b^2+3ab^3+b^4}$.

7. If $(a+b)^2+(b+c)^2+(c+d)^2=4(ab+bc+cd)$, then $a=b=c=d$.

8. Simplify $\frac{(a+d)^3}{(a-b)(a-c)}+\frac{(b+d)^3}{(b-a)(b-c)}+\frac{(c+d)^3}{(c-a)(c-b)}$

PAPER VI.

1. Prove that $8a^2b^2c^2+(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)$
$=(a^2+b^2+c^2)(a+b+c)(a+b-c)(a-b+c)(b+c-a)$.

2. Factorise $a^3(b+c)+b^3(c+a)+c^3(a+b)+abc(a+b+c)$

3. Eliminate x, y, z from $\frac{x^2-xy-xz}{p}=\frac{y^2-yz-yx}{q}=\frac{z^2-zx-zy}{r}$,
$px+qy+rz=0$.

4. If $ax+by=a+b$, $a^2x^2+b^2y^2=a^2+b^2$ prove that
$a^nx^n+b^ny^n=a^n+b^n$.

5. Prove that

(i) $(a-5b)(3a+b)^3+(5a-b)(a+3b)^3=32(a+b)(a-b)^3$

(ii) $(a+3x+2b)(a+x)^3-(2a+3x+b)(b+x)^3$
$=(a-b)(a+2x+b)^3$.

6. If $a : b=c : d=e : f$, show that

$$27(a+b)(c+d)(e+f)=bdf\left(\frac{a+b}{b}+\frac{c+d}{d}+\frac{e+f}{f}\right)^3$$

5. F[illegible] ...cond ... [illegible] $3s=a+b+c$, prove that

$$\frac{(s-a)^2}{(s-b)(s-c)}+\frac{(s-b)^2}{(s-c)(s-a)}+\frac{(s-c)^2}{(s-a)(s-b)}=3.$$

8. If $x=by+cz+du$, $y=ax+cz+du$,

$z=ax+by+du$, $u=ax+by+cz$,

then $\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}=1.$

PAPER VII.

1. Prove that $(bcd+cda+dab+abc)^2-abcd(a+b+c+d)^2$
$=(bc-ad)(ca-bd)(ab-cd).$

2. If $a:b=c:d$, prove that
$\sqrt{(3a^2+4c^2)}:\sqrt[3]{(5a^3-6c^3)}=\sqrt{(3b^2+4d^2)}:\sqrt[3]{(5b^3-6d^3)}.$

3. If $a^2=b+c$, $b^2=c+a$, $c^2=a+b$,

prove that $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=1.$

4. Eliminate x, y from the equations
$x+y+z=a$, $yz+zx+xy=b$, $xyz=c$.

5. Divide x^6-6x+5 by x^2-2x+1 by the method of detached co-efficients.

6. If $\frac{a}{b+c}=\frac{b}{c+a}=\frac{c}{a+b}$, shew that each of these fractions $=\frac{1}{2}$ or -1.

7. Prove that $2^{4n}-1$ is divisible by 15, if n be a positive integer. (M. M. 1875.)

8. Find three numbers which are to one another as $2:3:5$ and such that the sum of the greatest and the least exceeds the the third by 24.

PAPER VIII.

1. If $ab+bc+ca=0$, prove that $(a+b+c)^3=a^3+b^3+c^3-3abc$.

2. If $(a+b+c)(a+b+a)=(c+d+a)(c+d+b)$, prove that each of these quantities is equal to

$$\frac{(a-c)(a-d)(b-c)(b-d)}{(a+b-c-d)^2}.$$

3. If $a+b:b+c=c+d:d+a$, prove that $a=c$ or $a+b+c+d=0$.

4. Simplify

$$\frac{(a-1)^2}{(a-b)(a-c))x-a)}+\frac{(b-1)^2}{(b-a)(b-c)(x-b)}+\frac{(c-1)^2}{(c-a)(c-b)(x-c)}.$$

5. Prove that $a(b-c)(1+ca)(1+ab)+b(c-a)(1+ab)(1+a+b)^2$
$+c(a-b)(1+bc)(1+ca)=-abc(b-c)(c-a)(a-b)$.

6. If $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$, prove that

$$\frac{x^2-yz}{a}=\frac{y^2-zx}{b}=\frac{z^2-xy}{c}=(a+b+c)(x+y+z).$$

7. If $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$, prove that

$$\frac{1}{2}\left(\frac{y^2z^2}{x^2}+\frac{z^2x^2}{y^2}+\frac{x^2y^2}{z^2}\right)=(x+y+z)^2.$$

8. Eliminate x, y, z from the equations

$$ax+by+cz=0,\ bx+cy+az=0,\ cx+ay+bz=0.$$

Paper IX.

1. If $(a+b+c)^2=3(bc+ca+ab)$, prove that $a=b=c$.

2. Solve the equations

(i) $\frac{a+c}{x-2b}-\frac{b+c}{x-2a}=\frac{a-c}{x+2b}-\frac{b-c}{x+2a}$. (M. M. 1888.)

(ii) $\frac{a-b}{x+y}+\frac{a+b}{y-x}=\frac{2(a^2+b^2)}{a^2-b^2}$, $\frac{a+b}{x+y}+\frac{a-b}{y-x}=2$.

3. Prove that $a(b-c)(x-b)(x-c)+b(c-a)(x-c)(x-a)$
$+c(a-b)(x-a)(x-b)=-(b-c)(c-a)(a-b)x$.

4. Simplify

$$\frac{bc(x-a)^2}{(a-b)(a-c)}+\frac{ca(x-b)^2}{(b-c)(b-a)}+\frac{ab(x-c)^2}{(c-a)(c-b)}.$$

5. If $\frac{ay-bx}{c}=\frac{cx-az}{b}=\frac{bx-cy}{a}$, prove that $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

6. Show that the last digit in $3^{2n+1}+2^{2n+1}$ is 5, if n be any whole number. (M. M. 1868.)

7. If $a : b=c : d$, prove that

(i) $(\sqrt{a}+\sqrt{b})^2 : (\sqrt{c}+\sqrt{d})^2=a-b : c-d$.

(ii) $a^2b-3ac^2 : b^3-3ad^2=a^2+5c^2 : b^2+5d^2$.

8. Eliminate x, y, z from the equations

$$\frac{y-z}{y+z}=a,\ \frac{z-x}{z+x}=b,\ \frac{x-y}{x+y}=c.$$

PAPER X.

1. If $(ax+by+cz)^2=(a^2+b^2+c^2)(x^2+y^2+z^2)$, prove that

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

2. Find the first four terms of the square root of a^2+x^2 and from the result deduce the square root of 101 correct to six places of decimals. (C. E. 1877).

3. The expression $ax-by$ is equal to 10 when $x=2$ and $y=3$ and it is equal to 25 when $x=3$ and $y=2$; find its value when $x=y=1$.

4. Show that $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3}$

$$=\frac{(a+b)(b+c)(c+a)}{abc}$$ (B. M. 1895).

5. Prove that $\dfrac{a-b}{m+ab}+\dfrac{b-c}{m+bc}+\dfrac{c-a}{m+ca}$

$$=m\frac{(a-b)(b-c)(c-a)}{(m+ab)(m+bc)(m+ca)}$$ (M. M. 1884).

6. What must be the form of m so that a^m-x^m may have both a^n+x^n and a^n-x^n for divisors, n being any positive integer? (M. M. 1875).

7. If $x : a=y : b=z : c$, prove that

$$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=\frac{(x+y+z)^3}{(a+b+c)^2}$$ (C. E. 1901).

8. Eliminate x, y from the equations $x^3+3xy^2=a^3$, $y^3+3x^2y=b^3$, $xy=c^2$.

CHAPTER XXVI.

LAWS OF INDICES.

1. The student is already familiar with the index laws for positive integral exponents. For convenience we state and prove these laws here in one place.

2. First Law. $a^m\times a^n=a^{m+n}$, where m and n are positive integers.

Proof. By the definition of a power,

$a^m=a\times a\times\cdots\cdots$ to m factors ; $a^n=a\times a\times\cdots\cdots$ to n factors.

$\therefore\ a^m\times a^n=(a\times a\times\cdots\cdots$ to m factors$)\times(a\times a\times\cdots$ to n factors$)$

$=a\times a\times\cdots\cdots$ to $(m+n)$ factors

$=a^{m+n}$, by def. of a power.

$(a+b)^2$

Cor. $a^m \times a^n \times a^p \times \cdots = a^{m+n+p+\cdots\cdots}$, m, n, p,...being integers. (see art. 5, Chap. VII).

3. Second Law. $\left.\begin{array}{l} \dfrac{a^m}{a^n} = a^{m-n} \text{ where } m>n \\ \text{and } = \dfrac{1}{a^{n-m}} \text{ where } m<n \end{array}\right\}$ m and n are positive integers.

Proof. $\dfrac{a^m}{a^n} = \dfrac{a \times a \times \cdots \text{to } m \text{ factors}}{a \times a \times \cdots \text{to } n \text{ factors}}$

$\therefore$ when $m>n$, we get

$$\frac{a^m}{a^n} = \frac{(a \times a \times \cdot \text{ to } m \quad n \text{ factors}) \times (a \times a \times \cdots \text{to } n \text{ factors})}{a \times a \times \cdot \text{ to } n \text{ factors}}$$

$= a \times a \times \cdots\cdots \text{to } m-n$ factors

$= a^{m-n}$ (see art. 4, Chap. VIII).

Similarly when $m<n$, $a^m/a^n = 1/a^{n-m}$

4. Third Law. $(a^m)^n = (a^n)^m = a^{mn}$, where m and n are positive integers.

Proof. $(a^m)^n = a^m \times a^m \cdots \text{to } n$ factors $= (a \times a \times \cdots \text{to } m \text{ factors})$
$\times (a \times a \times \cdots \text{to } m \text{ factors})$
$\times \cdots\cdots\cdots\cdots \text{to } n$ rows

$= a \times a \times \cdots \text{to } mn$ factors

$= a^{mn}$ (see art. 3, (i), Chap. XVII).

Similarly, $(a^n)^m = a^{mn}$.

Hence the nth power of the mth power of a quantity = the mth power of the nth power of the quantity = the mnth power of the quantity.

5. Fourth Law. $(ab)^m = a^m b^m$ where m is a positive integer.

Proof. $(ab)^m = ab \times ab \times \cdots \text{to } m$ factors

$= (a \times a \times \cdots \text{to } m \text{ factors}) \times (b \times b \times \cdots \text{ to } m \text{ factors})$

$= a^m b^m$. (see art. 3, (ii), Chap. XVII).

Cor. $(abc\ldots)^m = a^m b^m c^m \ldots$

6. Fifth Law. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$, where m is a positive integer.

Proof. $\left(\dfrac{a}{b}\right)^m = \dfrac{a}{b} \times \dfrac{a}{b} \times \cdots\cdots$ to m factors.

$= \dfrac{a \times a \times \cdots \text{ to } m \text{ factors}}{b \times b \times \cdots \text{to } m \text{ factors}}$

$= \dfrac{a^m}{b^m}$. (see art. 3, (iii), Chap. XVII).

cond The student will observe that only two of these laws are independent, *viz.*, Laws I and IV. The Laws II and III can be deduced from Law I and the Law V can be deduced from Law IV.

Suppose $m>n$, so that $m-n$ is positive; then from Law I $a^{m-n}\times a^n=a^{m-n+n}=a^m$.

Hence dividing both sides by a^n, we get

$$a^{m-n}=\frac{a^m}{a^n}\text{, which is Law II.}$$

Again we have, $(a^m)^n=a^m\, a^m.a^m\ldots$ to n factors.

$$=a^{m+m+\cdots\text{to } n \text{ terms}}\text{ by Law I (cor.)}$$

$\therefore\ (a^m)^n=a^{mn}$, which is Law III.

Lastly, from Law IV, $\left\{\left(\frac{a}{b}\right)\times b\right\}^m=\left(\frac{a}{b}\right)^m\times b^m.$

$$\therefore\ a^m=\left(\frac{a}{b}\right)^m\times b^m.$$

Dividing both sides by b^m, we have

$$\frac{a^m}{b^m}=\left(\frac{a}{b}\right)^m\text{, which is Law V.}$$

8 The five laws of indices have been proved from the fundamental principles of Algebra, on the supposition that the *indices are positive integers.* They become meaningless when the indices are positive fractions, zero or negative quantities.

Thus it is absurd to say that $a^{\frac{3}{2}}=a\times a\times a\cdots\cdots$ to $\frac{3}{2}$ factors or $a^{-4}=a\times a\times\cdots$ to -4 factors. We can, however, define symbols like $a^{\frac{3}{2}}$ or a^{-4}, but as it is necessary that all algebraical symbols should obey the same laws, the meanings which we may give to fractional and negative indices must be consistent with the first index law. Hence taking the first index law *viz.* $a^m\times a^n=a^{m+n}$ to be true for *all values* of m and n let us see what meanings must be given to symbols like $a^{\frac{p}{q}}$, a^0 or a^{-p} which are at present unintelligible.

For example, to find the meaning of $a^{\frac{1}{2}}$

Since the first index law should be always obeyed, we must have

$$a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a^{\frac{1}{2}+\frac{1}{2}},$$

$$\text{or } a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a^1\ (i.\ e.\ a).$$

$\therefore\ a^{\frac{1}{2}}$ is such that its square is equal to a or $a^{\frac{1}{2}}=\sqrt{a}$.

Again, to find the meaning of a^{-2}.

Since the first index law should be always obeyed, we must have

$$a^{-2} \times a^3 = a^{-2+3} = a^1.$$

$$\therefore\ a^{-2} = \frac{a}{a^3} = \frac{1}{a^2}.$$

We proceed to find the meanings of fractional and negative indices in general cases.

9. Positive fractional Index.—*To find the meaning of $a^{\frac{p}{q}}$ where p and q are positive integers.*

By assuming that $a^m \times a^n = a^{m+n}$ for all values of m and n, and by putting $m = n = \frac{p}{q}$ we get $a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}} = a^{\frac{2p}{q}}$,

Similarly, $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{2p}{q}+\frac{p}{q}} = a^{\frac{3p}{q}}$

Proceeding in this way, we get,

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \cdots\cdots \text{to } q \text{ factors} = a^{\frac{qp}{q}} \text{ or } \left(a^{\frac{p}{q}}\right)^q = a^p$$

Hence $a^{\frac{p}{q}}$ is such a quantity that if it is raised to the q^{th} power the result is a^p, or in other words, $a^{\frac{p}{q}}$ is the q^{th} root of a^p.

Hence $a^{\frac{p}{q}} = \sqrt[q]{a^p}$; thus *we may consider $a^{\frac{p}{q}}$ as the q^{th} root of the p^{th} power of a.* As a particular case $a^{\frac{1}{q}}$ is the q^{th} root of a.

Again, by assuming $a^m \times a^n = a^{m+n}$ for all values of m and n, we get

$$a^{\frac{1}{q}} \times a^{\frac{1}{q}} = a^{\frac{2}{q}},$$

$$a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times a^{\frac{1}{q}} = a^{\frac{2}{q}} \times a^{\frac{1}{q}} = a^{\frac{3}{q}}.$$

Similarly, $a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times \cdots\cdots$to p factors$= a^{\frac{p}{q}}$, or $\left(a^{\frac{1}{q}}\right)^p = a^{\frac{p}{q}}$.

Hence $a^{\frac{p}{q}}=(\sqrt[q]{a})^p$; thus *we may also consider* $a^{\frac{p}{q}}$ *as the* p^{th} *power of the* q^{th} *root of* a.

Ex. 1. $a^{\frac{3}{2}}=\sqrt{(a^3)}$ or $=(\sqrt{a})^3$.

Ex. 2. $2^{\frac{3}{4}}=\sqrt[4]{(2^3)}=\sqrt[4]{8}$,
or $=(\sqrt[4]{2})^3$.

Ex. 3. $8^{\frac{2}{3}}=\sqrt[3]{(8^2)}=\sqrt[3]{64}=4$,
or $=(\sqrt[3]{8})^2=2^2=4$.

10. Zero Index. *To find the meaning of* a^0.

Since we have assumed $a^m\times a^n=a^{m+n}$ for *all* values of m and n, we have, putting $m=0$,

$$a^0\times a^n=a^{0+n}=a^n. \quad \therefore\ a^0=a^n\div a^n=1,$$

Thus *any (finite) quantity raised to the zero power is equal to unity.*

Ex. $2^0=1$, $3^0=1$, $x^0=1$.

11. Negative Index. *To find the meaning of* a^{-p} *where* p *is positive.*

Since we have assumed $a^m\times a^n=a^{m+n}$ for all values of m and n, we have putting $m=-p$, $n=p$,

$$a^{-p}\times a^p=a^{-p+p}=a^0.$$

But $a^0=1$; $\therefore$ $a^{-p}\times a^p=1$; *i.e.* $a^{-p}=1/a^p$ or $a^p=1/a^{-p}$.

Thus a^{-p} is the reciprocal of a^p; hence *if we change a quantity into its reciprocal and change the sign of the index, the value* is *unchanged.*

Ex. 1. $a^{-2}=\dfrac{1}{a^2}$, $a^{-\frac{2}{3}}=\dfrac{1}{a^{\frac{2}{3}}}=\dfrac{1}{\sqrt[3]{(a^2)}}$.

Ex. 2. $25^{-3}=\dfrac{1}{5^3}=\dfrac{1}{125}$, $4^{-\frac{3}{2}}=\dfrac{1}{4^{\frac{3}{2}}}=\dfrac{1}{\sqrt{4^3}}=\dfrac{1}{\sqrt{64}}=\dfrac{1}{8}$.

Note. It is useful to remember that $\left(\dfrac{1}{a}\right)^{-m}=a^m$ and $\dfrac{a^{-m}}{b^{-n}}=\dfrac{1}{a^m}\div\dfrac{1}{b^n}$
$=\dfrac{b^n}{a^m}$.

12. We have thus obtained meanings for or definitions of a positive fractional index, a zero index and a negative index on the assumption that the first index law is true *universally*. We are

now to show that these difinitions fully obey the other indi[ces]
or, in other words, the other index laws are true *for all va[lues of]*
the indices.

Since the seeond law is deducible from the first (see art. 7) *without any appeal to the definition of a power*, we infer that the second law *viz.* $a^m \div a^n = a^{m-n}$ is true for *all values of m and n*. In the next two articles we shall prove that the third and the fourth laws are true for all values of the indices ; and since the fifth law is deducible from the fourth *without any appeal to the definition of a power*, (see art. 7) it follows that the fifth law is true universally.

Hence all *the index laws are true, whether the indices are positive or negative, integral or fractional.*

13. *To prove that* $(a^m)^n = a^{mn}$, *for all values of m and n*

Case I. Let n be a positive integer m unrestricted in value.

Then $(a^m)^n = a^m \times a^m \times \cdots\cdots$ to n factors

$$= a^{m+m+\ldots\text{to } n \text{ terms}} \text{ by law I, cor.}$$
$$= a^{mn}.$$

Case II. Let n be a positive fraction $= \dfrac{p}{q}$ (where p and q are positive integers) and m unrestricted as before.

Then $(a^m)^n = (a^m)^{\frac{p}{q}}$

$$= \sqrt[q]{(a^m)^p}. \text{ by def. of a fractional index}$$
$$= \sqrt[q]{(a^{mp})}, \text{ by law III, since } p \text{ is positive integer}$$
$$= a^{\frac{pm}{q}}, \text{ by def. of a fractional index}$$
$$= a^{mn}, \text{ substituting for } \frac{p}{q}.$$

Case III. Let n be negative and $= -p$ (where p is positive), m having any value.

Then $(a^m)^n = (a^m)^{-p}$

$$= \frac{1}{(a^m)^p}, \text{ by def. of a negative index}$$
$$= \frac{1}{a^{mp}}, \text{ by cases I and II, since } p \text{ is positive}$$
$$= a^{-mp}, \text{ by def. of a negative index}$$
$$= a^{mn}, \text{ substituting for } -p.$$

To prove that $(ab)^m = a^m b^m$ for all values of m.

Case I. Let m be a positive integer. This case needs no consideration here. (See art. 2).

Case II. Let m be a positive fraction equal to $\frac{p}{q}$ where p and q are positive integers, and let $(ab)^m = k$.

Then $k = (ab)^m = (ab)^{\frac{p}{q}}$.

$$\therefore k^q = \left\{(ab)^{\frac{p}{q}}\right\}^q = (ab)^p, \text{ by def. of a fractional index,}$$

$$= a^p b^p, \text{ by law IV since } p \text{ is a positive integer,}$$

$$= \left(a^{\frac{p}{q}}\right)^q \left(b^{\frac{p}{q}}\right)^q, \text{ by def. of a fractional index,}$$

$$= \left(a^{\frac{p}{q}} b^{\frac{p}{q}}\right)^q, \text{ by Law IV since } q \text{ is a } \textit{positive} \text{ integer.}$$

Hence extracting the q^{th} root of both sides,

$$k = a^{\frac{p}{q}} b^{\frac{p}{q}} = a^m b^m.$$

$$\therefore (ab)^m = a^m b^m.$$

Thus the law is proved for any *positive* index.

Case III. Let m be negative and equal to $-p$ where p is positive, integral or fractional.

Then $(ab)^m = (ab)^{-p} = \frac{1}{(ab)^p}$ by def. of a negative index

$= \frac{1}{a^p b^p}$, by cases I and II, since p is positive.

$= \frac{1}{a^p} \cdot \frac{1}{b^p} = a^{-p} \cdot b^{-p}$ by def. of a negative index.

$= a^m \cdot b^m$.

The case when $m = 0$ is left for the student.

Thus $(ab)^m = a^m b^m$ universally.

15. Operations with fractional and negative indices.

The ordinary operations of algebra are applicable to expressions involving fractional and negative indices, for the index laws are universally true. The following are added by way of illustration.

Ex. 1. Simplify $\left(a^{-\frac{2}{3}}b^{\frac{3}{4}}\right)^{-\frac{1}{2}} \div \left(a^{\frac{1}{2}}b^{-\frac{4}{3}}\right)^{\frac{3}{4}}$.

$$\text{The expr. } = a^{-\frac{2}{3}\times(-\frac{1}{2})}\, b^{\frac{3}{4}\times(-\frac{1}{2})} \div a^{\frac{1}{2}\times\frac{3}{4}}\, b^{-\frac{4}{3}\times\frac{3}{4}}$$
$$= a^{\frac{1}{3}}b^{-\frac{3}{8}} \div a^{\frac{3}{8}}b^{-1}$$
$$= a^{\frac{1}{3}}b^{-\frac{3}{8}} \times a^{-\frac{3}{8}}b^{1}$$
$$= \left(a^{\frac{1}{3}} \times a^{-\frac{3}{8}}\right) \times \left(b^{-\frac{3}{8}}b^{1}\right) = a^{\frac{1}{3}-\frac{3}{8}} \times b^{-\frac{3}{8}+1}$$
$$= a^{-\frac{1}{24}}b^{\frac{5}{8}}.$$

Ex. 2. Simplify

$$x^{-\frac{11}{3}}y^{\frac{4}{3}}\left[x^{-\frac{2}{7}}y^{\frac{2}{3}}\left\{x^{\frac{1}{3}}y^{\frac{2}{7}}\left(x^{-1}y^{-3}\right)^{\frac{1}{7}}\right\}^{5}\right]^{7}$$

The expression

$$= x^{-\frac{11}{3}}y^{\frac{4}{3}}\left[x^{-\frac{2}{7}}y^{\frac{2}{3}}\left\{x^{\frac{1}{3}}y^{\frac{2}{7}}x^{-\frac{1}{7}}y^{-\frac{3}{7}}\right\}^{5}\right]^{7}$$
$$= x^{-\frac{11}{3}}y^{\frac{4}{3}}\left[x^{-\frac{2}{7}}y^{\frac{2}{3}}x^{\frac{5}{3}}y^{\frac{10}{7}}x^{-\frac{5}{7}}y^{-\frac{15}{7}}\right]^{7}$$
$$= x^{-\frac{11}{3}}y^{\frac{4}{3}}x^{-2}y^{\frac{14}{3}}x^{\frac{35}{3}}y^{10}x^{-5}y^{-15}$$
$$= x^{-\frac{11}{3}-2+\frac{35}{3}-5}\,y^{\frac{4}{3}+\frac{14}{3}+10-15} = xy.$$

Ex. 3. Multiply $x^{\frac{2}{3}}+y^{\frac{1}{2}}+x^{\frac{1}{3}}y^{\frac{1}{4}}$ by $x^{\frac{5}{3}}+xy^{\frac{1}{2}}-x^{\frac{4}{3}}y^{\frac{1}{4}}$.

Arrange the expressions in descending powers of x.

$$\begin{array}{llll}
x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{4}}+y^{\frac{1}{2}} & & & \\
x^{\frac{5}{3}}-x^{\frac{4}{3}}y^{\frac{1}{4}}+xy^{\frac{1}{2}} & & & \\
\hline
x^{\frac{7}{3}}+x^{2}y^{\frac{1}{4}}+x^{\frac{5}{3}}y^{\frac{1}{2}} & & & =\text{product by } x^{\frac{5}{3}}. \\
\quad -x^{2}y^{\frac{1}{4}}-x^{\frac{5}{3}}y^{\frac{1}{2}}-x^{\frac{4}{3}}y^{\frac{3}{4}} & & & =\text{product by } -x^{\frac{4}{3}}y^{\frac{1}{4}}. \\
\qquad\qquad +x^{\frac{5}{3}}y^{\frac{1}{2}}+x^{\frac{4}{3}}y^{\frac{3}{4}}+xy & & & =\text{product by } xy^{\frac{1}{2}} \\
\hline
x^{\frac{7}{3}}\qquad +x^{\frac{5}{3}}y^{\frac{1}{2}}\qquad +xy & & & =\text{complete product.}
\end{array}$$

Note. In multiplication and in division it is advisable to arrange the expressions involved in ascending or descending powers of some common letter.

Ex. 4. Divide $x^2y^{-2}+x^{-2}y^2+2$ by $x^{\frac{2}{3}}y^{-\frac{2}{3}}+x^{-\frac{2}{3}}y^{\frac{2}{3}}-1$.

Arrange the expressions in descending powers of x, observing that $2=2x^0$; so $1=1x^0$.

$$
\begin{array}{l|l}
x^{\frac{2}{3}}y^{-\frac{2}{3}}-1+x^{-\frac{2}{3}}y^{\frac{2}{3}} & x^2y^{-2}+2+x^{-2}y^2 \;\left(x^{\frac{4}{3}}y^{-\frac{4}{3}}+x^{\frac{2}{3}}y^{-\frac{2}{3}}+x^{-\frac{2}{3}}y^{\frac{2}{3}}+x^{-\frac{4}{3}}y^{\frac{4}{3}}\right. \\
 & x^2y^{-2}-x^{\frac{4}{3}}y^{-\frac{4}{3}}+x^{\frac{2}{3}}y^{-\frac{2}{3}} \\
 & x^{\frac{4}{3}}y^{-\frac{4}{3}}-x^{\frac{2}{3}}y^{-\frac{2}{3}}+2+x^{-2}y^2 \\
 & x^{\frac{4}{3}}y^{-\frac{4}{3}}-x^{\frac{2}{3}}y^{-\frac{2}{3}}+1 \\
 & 1+x^{-2}y^2 \\
 & 1-x^{-\frac{2}{3}}y^{\frac{2}{3}}+x^{-\frac{4}{3}}y^{\frac{4}{3}} \\
 & x^{-\frac{2}{3}}y^{\frac{2}{3}}-x^{-\frac{4}{3}}y^{\frac{4}{3}}+x^{-2}y^2 \\
 & x^{-\frac{2}{3}}y^{\frac{2}{3}}-x^{-\frac{4}{3}}y^{\frac{4}{3}}+x^{-2}y^2
\end{array}
$$

$\therefore$ quotient $=x^{\frac{4}{3}}y^{-\frac{4}{3}}+x^{\frac{2}{3}}y^{-\frac{2}{3}}+x^{-\frac{2}{3}}y^{\frac{2}{3}}+x^{-\frac{4}{3}}y^{\frac{4}{3}}$.

Ex. 5 Extract the square root of

$$x^2-x^{\frac{1}{6}}+\tfrac{4}{9}\left(3+x^{-\frac{1}{2}}\right)x^{\frac{3}{2}}-\tfrac{9}{16}x^{-\frac{2}{3}}\left(\tfrac{8}{3}x^{\frac{4}{3}}-1\right).$$

Simplify and arrange in descending powers of x:

$$x^2+\tfrac{4}{3}x^{\frac{3}{2}}+\tfrac{4}{9}x-\tfrac{3}{2}x^{\frac{2}{3}}-x^{\frac{1}{6}}+\tfrac{9}{16}x^{-\frac{2}{3}}\;\Big(\,x+\tfrac{2}{3}x^{\frac{1}{2}}-\tfrac{3}{4}x^{-\frac{1}{3}}$$

$$
\begin{array}{r|l}
 & x^2 \\
2x+\frac{2}{3}x^{\frac{1}{2}} & \frac{4}{3}x^{\frac{3}{2}}+\frac{4}{9}x-\frac{3}{2}x^{\frac{2}{3}} \\
 & \frac{4}{3}x^{\frac{3}{2}}+\frac{4}{9}x \\
2x+\frac{4}{3}x^{\frac{1}{2}}-\frac{3}{4}x^{-\frac{1}{3}} & -\frac{3}{2}x^{\frac{2}{3}}-x^{\frac{1}{6}}+\frac{9}{16}x^{-\frac{2}{3}} \\
 & -\frac{3}{2}x^{\frac{2}{3}}-x^{\frac{1}{6}}+\frac{9}{16}x^{-\frac{2}{3}}
\end{array}
$$

$\therefore$ the required square root $=x+\frac{2}{3}x^{\frac{1}{2}}-\frac{3}{4}x^{-\frac{1}{3}}$.

Ex. 6. Prove that

$$\frac{1}{1+x^{m-n}+x^{m-p}}+\frac{1}{1+x^{n-m}+x^{n-p}}+\frac{1}{1+x^{p-m}+x^{p-n}}=1.$$

We have $\dfrac{1}{1+x^{m-n}+x^{m-p}}=\dfrac{x^{-m}}{x^{-m}(1+x^{m-n}+x^{m-p})}=\dfrac{x^{-m}}{x^{-m}+x^{-n}+x^{-p}}.$

Similarly, $\dfrac{1}{1+x^{n-m}+x^{n-p}}=\dfrac{x^{-n}}{x^{-m}+x^{-n}+x^{-p}}.$

and $\dfrac{1}{1+x^{p-m}+x^{p-n}}=\dfrac{x^{-p}}{x^{-m}+x^{-n}+x^{-p}}.$

Hence the expression

$$=\frac{x^{-m}}{x^{-m}+x^{-n}+x^{-p}}+\frac{x^{-n}}{x^{-m}+x^{-n}+x^{-p}}+\frac{x^{-p}}{x^{-m}+x^{-n}+x^{-p}}$$

$$=\frac{x^{-m}+x^{-n}+x^{-p}}{x^{-m}+x^{-n}+x^{-p}}=1.$$

Ex. 7. Divide $x^{2^n}-y^{2^n}$ by $x^{2^{n-1}}+y^{2^{n-1}}$.

Let $x^{2^{n-1}}=a$, $y^{2^{n-1}}=b$.

Then $x^{2^n}=x^{2^1\cdot 2^{n-1}}=\left(x^{2^{n-1}}\right)^2=a^2$.

Similarly $y^{2^n}=\left(y^{2^{n-1}}\right)^2=b^2$.

$\therefore$ the quotient $=\dfrac{a^2-b^2}{a+b}=a-b$

$$=x^{2^{n-1}}-y^{2^{n-1}}.$$

16. Exponential Equations. Equations in which the unknown quantities occur in the exponent are called *exponential equations*. We add some typical solutions of such equations.

Ex. 1. Solve the equations.

(i) $5^{2x-3}=625$ (ii) $3^{x+2}-3^{x+1}=162$ (iii) $2^x=1$.

(i) Since $625=5^4$, we have $5^{2x-3}=5^4$; hence $2x-3=4$ or $x=3\frac{1}{2}$.

(ii) Here $3^{x+2}-3^{x+1}=162$ or $3^{x+1}(3-1)=162$.

$\therefore$ $3^{x+1}.2=162$ or $3^{x+1}=81=3^4$.

$\therefore$ $x+1=4$ or $x=3$.

(iii) Since $1=2^0$, we have $2^x=2^0$.

$\therefore$ $x=0$.

Ex. 2. Solve the equations.

$2^{x+1}4^y=64$............(i)

$9^x3^{y+1}=81$............(ii)

From (i) $2^{x+1}(2^2)^y=2^6$ or $2^{x+1}\,2^{2y}=2^6$

$\therefore$ $2^{x+1+2y}=2^6$ whence $x+1+2y=6$.........(iii)

Again, from (ii) $(3^2)^x3^{y+1}=3^4$ or $3^{2x}3^{y+1}=3^4$.

$\therefore$ $3^{2x+y+1}=3^4$ whence $2x+y+1=4$......(iv)

Now from (iii) and (iv)

$x+2y=5$, $2x+y=3$.

$\therefore$ $x=\frac{1}{3}$, $y=2\frac{1}{3}$.

EXERCISE CXIII.

1. Find the value of :—

(1) $27^{\frac{2}{3}}$ (2) $27^{-\frac{2}{3}}$ (3) $(\frac{4}{9})^{\frac{5}{2}}$ (4) $(\frac{4}{9})^{-\frac{5}{2}}$

(5) $8^{\frac{2}{3}} \times \left(\frac{8}{125}\right)^{-\frac{1}{3}}$ (6) $\frac{2^{-2}}{3^{-3}} \times \left(\frac{27}{8}\right)^{-\frac{2}{3}} \times \left(\frac{32}{243}\right)^{\frac{4}{5}}$

(7) $\frac{18 \times 3^{n+1} - 9.3^{n+3}}{14.3^{n+2}}$ (8) $\frac{5^{n} \times (5^{n-2})^{n}}{5^{n+1} \times 5^{n-1}} \times 25^{-n}$

2. Simplify

(1) $\left(a^{-\frac{2}{3}}\right)^{9}$ (3) $\sqrt[4]{(a^{-2}b^{3}c^{3})} \div \sqrt[3]{(a^{4}b^{-3}c^{2})}$

(2) $\sqrt[3]{(a^{-6}b^{9})^{2}}$ (4) $(x^{-\frac{2}{3}}y^{2}z^{-1})^{3} \times (8x^{3}y^{3}z^{3})^{\frac{2}{3}}$

(5) $[a^{-\frac{1}{2}}\{a^{-\frac{1}{4}}b^{-\frac{1}{6}}(a^{4}b^{4})^{\frac{1}{3}}\}^{-\frac{3}{2}}]^{2}$

(6) $\left(\frac{x^{\frac{3}{2}}y^{\frac{2}{3}}}{z^{\frac{1}{2}}} \times \frac{y^{-\frac{2}{3}}z^{\frac{3}{2}}}{x^{-\frac{3}{4}}} \times \frac{z^{-1}x^{-\frac{1}{4}}}{y^{-1}}\right)^{\frac{1}{2}}$

(7) $a^{\frac{1}{6}}\{a^{\frac{1}{4}}b^{-\frac{1}{6}}(a^{4}b^{4})^{\frac{1}{6}}\}^{\frac{1}{2}} \div a^{\frac{1}{3}} \times \{a^{-\frac{1}{2}}b^{-\frac{1}{3}}(a^{2}b^{2})^{\frac{3}{2}}\}^{-\frac{1}{2}}$

(8) $\left(\frac{x^{m}}{x^{n}}\right)^{m+n} \times \left(\frac{x^{n}}{x^{r}}\right)^{n+r} \times \left(\frac{x^{r}}{x^{m}}\right)^{r+m}$

3. Multiply

(1) $x^{\frac{1}{2}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. (2) $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

(3) $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

(4) $7x^{\frac{1}{2}}-3y^{\frac{1}{3}}+2x^{\frac{1}{3}}y^{\frac{2}{3}}$ by $6x^{\frac{1}{3}}-2y^{\frac{2}{3}}+7x^{\frac{2}{3}}y^{\frac{1}{3}}$

(5) $x^{\frac{2}{3}}y^{\frac{2}{3}}+x^{\frac{3}{4}}y^{-\frac{1}{6}}+x^{\frac{5}{4}}y^{\frac{1}{2}}$ by $x^{\frac{1}{3}}y^{\frac{1}{3}}+x^{\frac{1}{4}}y^{-\frac{1}{2}}-x^{-\frac{3}{4}}y^{\frac{3}{4}}$.

4. Find the squares of

(1) $x^{\frac{1}{2}}+y^{\frac{1}{2}}$. (2) $x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}$. (3) $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$.

(4) $(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}$ (5) $1+3x^{-\frac{1}{2}}y+2x^{-\frac{1}{4}}y^{-1}$.

5. Divide

(1) $x+6a^{\frac{1}{5}}x^{\frac{4}{5}}+6a^{\frac{3}{5}}x^{\frac{2}{5}}+a+5a^{\frac{2}{5}}x^{\frac{3}{5}}+7a^{\frac{4}{5}}x^{\frac{1}{5}}$ by $x^{\frac{1}{5}}+a^{\frac{1}{5}}$.

(2) $\left(x^{\frac{2}{3}}-a^{\frac{2}{3}}\right)\left(x^{\frac{2}{3}}+a^{\frac{2}{3}}\right)$ by $x^{\frac{1}{3}}+a^{\frac{1}{3}}$ (C. E. 1859).

(3) $x^{\frac{4}{3}}+a^{\frac{2}{3}}x^{\frac{2}{3}}+a^{\frac{4}{3}}$ by $x^{\frac{2}{3}}+a^{\frac{1}{3}}x^{\frac{1}{3}}+a^{\frac{2}{3}}$ (C. E. 186:).

Divide

(4) $x^2 - 4x^{\frac{4}{3}}y^{-\frac{2}{3}} - 9x^{\frac{2}{3}}y^{-\frac{4}{3}} + 36y^{-2}$ by $x^{\frac{2}{3}} + 3y^{-\frac{2}{3}}$.

(5) $x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{1}{2}} + y$ by $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$. (C. E. 1860).

6. Resolve into factors

(1) $x^{\frac{4}{3}} + y^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{2}{3}}$. (2) $x^{-12} - a^{12}$ (C. E. 1857).

(3) $x + y - 3x^{\frac{1}{3}}y^{\frac{1}{3}} + 1$. (4) $a^{\frac{9}{5}} + b^{\frac{3}{2}}$.

7. Find the H. C. F. of

(1) $x^{\frac{2}{3}} - 4$, $x^{\frac{2}{3}} + 10x^{\frac{1}{3}} + 16$ and $x^{\frac{2}{3}} - 7x^{\frac{1}{3}} - 18$.

(2) $x^{-1} + 6x^{-\frac{2}{3}} + 11x^{-\frac{1}{3}} + 6$ and $x^{-1} + 9x^{-\frac{2}{3}} + 27x^{-\frac{1}{3}} + 27$.

8. Find the L. C. M. of

(1) $6x^{\frac{1}{2}} - 11x^{\frac{1}{4}} + 3$, $6x^{\frac{1}{2}} + 25x^{\frac{1}{4}} - 9$.

(2) $a^{\frac{2}{3}} - b^{\frac{2}{3}}$, $a^{\frac{2}{3}} + b^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}$, $a + b$.

9. Extract the square root of :—

(1) $x^{\frac{8}{5}} - 2a^{-\frac{2}{5}}x^{\frac{11}{5}} + 2a^{\frac{4}{5}}x^{\frac{4}{5}} + a^{-\frac{6}{5}}x^{\frac{14}{5}} - 2a^{\frac{1}{5}}x^{\frac{7}{5}} + a^{\frac{8}{5}}$

(C. E. 1880).

(2) $\left(x^{\frac{2}{3}} + x^{-\frac{2}{3}}\right)^2 - 4\left(x^{\frac{1}{3}} + x^{-\frac{1}{3}}\right)^2 + 12$.

(3) $1 - \{xy + (1 - x^2)^{\frac{1}{2}}(1 - y^2)^{\frac{1}{2}}\}^2$.

10. Divide $x^{2^n} + a^{2^{n-1}}x^{2^{n-1}} + a^{2^n}$ by $x^{2^{n-1}} + a^{2^{n-2}}x^{2^{n-2}} + a^{2^{n-1}}$.

11. Divide $a + b + c + 3(b^{\frac{1}{3}} + c^{\frac{1}{3}})(c^{\frac{1}{3}} + a^{\frac{1}{3}})(a^{\frac{1}{3}} + b^{\frac{1}{3}})$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}$.

12. If $m = a^x$, $n = a^y$ and $a^2 = (m^y n^x)^z$, shew that $xyz = 1$.

13. If $a = b^x$, $b = c^y$, $c = a^z$, prove that $xyz = 1$.

14. If $a^b = b^a$ shew that $(a/b)^{a/b} = a^{(a/b)-1}$ and if $a = 2b$, shew that $b = 2$.

15. Solve.

(1) $3^{x+4} = 243$ (2) $4^{x-1} = 8^{x+3}$ (3) $2^{x+1}3^{x+2} = 648$

(4) $3^{x+2} - 5\cdot 3^x = 36$ (5) $(p/q)^{ax-b} = (q/p)^{cx+d}$

16. Solve

(1) $\left.\begin{aligned} 3^{x+y+3} &= 81 \\ 2^x &= 16^{y-2} \end{aligned}\right\}$ (2) $\begin{aligned} (\sqrt{2})^{3x+1} &= 4^{y+2} \\ 9^{x-2} &= 3^{2y+5}. \end{aligned}$

CHAPTER XXVII.

ELEMENTARY SURDS.

1. When the root of a quantity is not expressible by an integer or a finite fraction, and therefore cannot be exactly obtained, it is called a **surd**.

Thus $\sqrt{2}$, $\sqrt[3]{5}$ are surds. But $\sqrt{4}$, $\sqrt[3]{(27)}$, though in surd forms are not really surds; for they are respectively equal to 2 and 3.

Note. An algebraical expression like $\sqrt{a}$ is called a surd, although for particular values of a such as 4 or 9 it is not a surd arithmetically.

2. A surd is also called an **irrational** quantity while quantities which are not surds are for the sake of distinction called **rational** quantities.

3. The **order** of a surd is denoted by its root-symbol or surd index.

Thus $\sqrt{5}$, $\sqrt[3]{7}$, $\sqrt[n]{a}$ are surds of the second, third and nth orders respectively.

4. A surd of the second order is called **quadratic**, a surd of the third order is called **cubic**, and so on.

Thus $\sqrt{a}$, $\sqrt{a+b}$, are quadratic surds; $\sqrt[3]{7}$, $\sqrt[3]{a}$ are cubic surds.

5. A surd is called a **monomial** when it contains one term, **binomial** when it contains two terms, **trinomial** when it contains three terms; and so on.

Thus $\sqrt{3}$, $4\sqrt[3]{5}$, $\sqrt{2}\times\sqrt{5}$ are monomial surds; $\sqrt{2}+\sqrt{3}$, $\sqrt{a}+\sqrt{b}$ are binomial surds; $\sqrt{5}+\sqrt[3]{7}+\sqrt[4]{11}$, $\sqrt{a}+\sqrt{b}+\sqrt{c}$ are trinomial surds.

Monomial surds are called **simple surds**; while surds other than monomial are called **compound surds**.

6. We shall consider some cases of reduction and transformation of simple surds.

(1) *To reduce a rational quantity to the form of a surd of a given order.* Thus $a=\sqrt{(a^2)}=\sqrt[3]{(a^3)}=\text{etc.}=\sqrt[n]{(a^n)}$,

$$3=\sqrt[4]{(3^4)}=\sqrt[4]{(81)}, \text{ and so on.}$$

(2) *To transform a mixed surd into a complete surd.*

$$\text{Thus } a\sqrt[m]{b}=\sqrt[m]{a^m}\sqrt[m]{b}=\sqrt[m]{(a^mb)}.$$

$$2a\sqrt[3]{(a^2b)}=\sqrt[3]{(2a)^3a^2b}=\sqrt[3]{(8a^5b)}.$$

Conversely, a complete surd may be sometimes expressed in the form of a mixed surd.

Thus $\sqrt{(a^2b^5c^3)} = \sqrt{(ab^2c)^2bc} = ab^2c\sqrt{bc}$,

$$\sqrt[4]{405} = \sqrt[4]{81 \times 5} = \sqrt[4]{81} \times \sqrt[4]{5} = 3\sqrt[4]{5}.$$

(3) *To transform a surd of any order into a surd of a different order.*

$$\sqrt[m]{a} = a^{\frac{1}{m}} = a^{\frac{p}{mp}} = \sqrt[mp]{a^p},\ \text{a surd of the } mp^{\text{th}} \text{ order},$$

$$\sqrt[3]{5} = 5^{\frac{1}{3}} = \left(5^{\frac{4}{3}}\right)^{\frac{1}{4}} = \sqrt[4]{5^{\frac{4}{3}}},\ \text{a surd of the 4th order.}$$

(4) *To transform surds of different orders into surds of the same order.*

Consider the surds $\sqrt[3]{a}$, $\sqrt[4]{b^3}$, $\sqrt[6]{c^5}$.

The *L. C.M.* of the root symbols 3, 4, 6 is 12; hence the surds can all be reduced to those of the 12th order.

Thus $\sqrt[3]{a} = a^{\frac{1}{3}} = a^{\frac{4}{12}} = \sqrt[12]{a^4}$,

$$\sqrt[4]{b^3} = b^{\frac{3}{4}} = b^{\frac{9}{12}} = \sqrt[12]{b^9},$$

$$\sqrt[6]{c^5} = c^{\frac{5}{6}} = c^{\frac{10}{12}} = \sqrt[12]{c^{10}}.$$

This transformation is useful in comparing surds of different orders.

Thus, to compare $\sqrt{5}$, $\sqrt[3]{11}$.

Here $\sqrt{5} = 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{(125)}$,

$$\sqrt[3]{11} = 11^{\frac{1}{3}} = 11^{\frac{2}{6}} = \sqrt[6]{11^2} = \sqrt[6]{121}.$$

Hence $\sqrt{5}$ is greater than $\sqrt[3]{11}$.

(5) *To transform a surd into its simplest form.*

Ex. 1. $\sqrt{72} = \sqrt{(36 \times 2)} = 6\sqrt{2}$.

Ex. 2. $14\sqrt{\frac{1}{2}} = \frac{14}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{14\sqrt{2}}{2} = 7\sqrt{2}$.

Ex. 3. $(\frac{1}{3})^{-\frac{7}{2}} = 3^{\frac{7}{2}} = 3^3 3^{\frac{1}{2}} = 27\sqrt{3}$.

In transforming a surd to the simplest form the quantity under the radical should be integral and as small as possible.

7. Surds are said to be **like** or **similar** when they have the same irrational factors, being reduced to their simplest forms if necessary; otherwise, they are **unlike** or **dissimilar**.

Thus $3\sqrt{2}$, $5\sqrt{2}$, $7\sqrt{2}$, are like surds; so also $\sqrt{(12)}$, $\sqrt{(27)}$ are like surds, for they are equal to $2\sqrt{3}$ and $3\sqrt{3}$ respectively.

8. (1) The sum or difference of any number of like surds may be obtained by the rules for the addition and subtraction of like terms.

Ex. 1. $9\sqrt{2}+3\sqrt{2}-7\sqrt{2}=(9+3-7)\sqrt{2}=5\sqrt{2}$.

Ex. 2. $2\sqrt{125}+3\sqrt{20}-7\sqrt{80}$

$$=2\sqrt{(25\times5)}+3\sqrt{(4\times5)}-7\sqrt{(16\times5)}$$

$$=10\sqrt{5}+6\sqrt{5}-28\sqrt{5}=-12\sqrt{5}.$$

If the surds are unlike, they are simply connected by the signs of addition and subtraction, and cannot be further simplified.

(2) The product of two or more simple surds of the *same order* may be immediately written down from the formula

$$\sqrt[n]{a}\times\sqrt[n]{b}=a^{\frac{1}{n}}\times b^{\frac{1}{n}}=(ab)^{\frac{1}{n}}=\sqrt[n]{ab}.$$

$$\text{Thus } 2\sqrt[3]{5}\times3\sqrt[3]{7}=2\times3\times\sqrt[3]{(5\times7)}=6\sqrt[3]{35}.$$

$$\sqrt{(a+b)}\times\sqrt{(a-b)}=\sqrt{(a^2-b^2)}.$$

If the surds are of different orders they must be first reduced to surds of the same order when the preceding method becomes applicable.

$$\text{Thus } \sqrt[3]{3}\times2\sqrt{5}=\sqrt[6]{3^2}\times2\sqrt[6]{5^3}=2\sqrt[6]{(3^2\times5^3)}=2\sqrt[6]{(1125)}.$$

$$x\sqrt{(2ax-x^2)}=\sqrt{x^2(2ax-x^2)}=\sqrt{(2ax^3-x^4)}.$$

(3) The quotient of one simple surd by another of the *same order* may be immediately written down from the formula

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\left(\frac{a}{b}\right)^{\frac{1}{n}}=\sqrt[n]{\frac{a}{b}}.$$

$$\text{Thus } \frac{5\sqrt{6}}{7\sqrt{3}}=\frac{5}{7}\sqrt{\frac{6}{3}}=\frac{5}{7}\sqrt{2}.$$

$$\frac{\sqrt{(x^2-5x+6)}}{\sqrt{(x^2-6x+8)}}=\sqrt{\frac{(x-2)(x-3)}{(x-2)(x-4)}}=\sqrt{\frac{x-3}{x-4}}.$$

If the surds are of different orders they must be first reduced to surds of the same order when the preceding method becomes applicable.

Thus $\dfrac{\sqrt[3]{5}}{\sqrt{3}}=\dfrac{\sqrt[6]{5^2}}{\sqrt[6]{3^3}}=\sqrt[6]{\dfrac{5^2}{3^3}}=\sqrt[6]{\dfrac{25}{27}}.$

$$\frac{\sqrt[4]{(a+x)}}{\sqrt[3]{(a-x)}}=\frac{\sqrt[12]{(a+x)^3}}{\sqrt[12]{(a-x)^4}}=\sqrt[12]{\frac{(a+x)^3}{(a-x)^4}}.$$

Operations with Compound Surds.

9. The following examples illustrate operations with compound surds which will be performed as in the case of compound rational algebraical expressions.

Ex. 1. $(5\sqrt{2}-2\sqrt{3})(5\sqrt{3}+3\sqrt{2})$
$=25\sqrt{6}+15(\sqrt{2})^2-10(\sqrt{3})^2-6\sqrt{6}$
$=25\sqrt{6}+30-30-6\sqrt{6}=19\sqrt{6}.$

Ex. 2. $(\sqrt{a}+\sqrt{b})^3=(\sqrt{a})^3+3(\sqrt{a})^2\sqrt{b}+3\sqrt{a}(\sqrt{b})^2+(\sqrt{b})^3$
$=a\sqrt{a}+3a\sqrt{b}+3b\sqrt{a}+b\sqrt{b}.$

EXERCISE CXIV.

1. Express : (1) a^2b^3 in the form $\sqrt[6]{P}$ (2) a^3b^2c in the form $\sqrt[4]{\ }$

2 Transform into complete surds

(1) $2\sqrt{5}$. (2) $\frac{2}{3}\sqrt[3]{\frac{3}{4}}$. (3) $a^2bc^{\frac{3}{2}}$. (4) $\dfrac{1-x}{1+x}\sqrt{\dfrac{1+x}{1-x}}$

3. Express :—(1) $\sqrt{(ab^3c^5)}$ as a surd of the third order ;

(2) $a^{\frac{3}{2}}b^{-\frac{1}{2}}c^3$ as a surd of the fourth order.

4. Express :—(1) $a^{\frac{1}{4}}b^{\frac{1}{2}}c^{\frac{3}{4}}$ as a surd with $\frac{2}{3}$ as coefficient.

(2) $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{2}}$ as a surd with $2abc$ as coefficient

5. Transform into surds of the same lowest order

(1) $\sqrt[3]{a}, \sqrt[5]{b}, \sqrt[10]{c}$. (2) $\sqrt[3]{2}, \sqrt[4]{3}, \sqrt[6]{4}$.

(3) $5^{\frac{3}{5}}, 2^{\frac{4}{5}}, 9^{\frac{3}{10}}$. (4) $\sqrt{2}, \sqrt[3]{8}, \sqrt[4]{7}$.

6 Compare :—

(1) $\sqrt{11}, \sqrt[3]{37}$. (2) $\sqrt[3]{5}, \sqrt[4]{8}$. (3) $3\sqrt{2}, 2\sqrt{5}, \sqrt{(21)}$

7. Simplify—

(1) $6\sqrt{12}+4\sqrt{75}-5\sqrt{108}$. (2) $2\sqrt[3]{16}+5\sqrt[3]{54}-3\sqrt[3]{12}$
(3) $7\sqrt{\frac{1}{4}}-2\sqrt{27}+\frac{1}{3}\sqrt{12}-2\sqrt{\frac{3}{16}}$.

(4) $\sqrt{18}+\dfrac{8}{\sqrt{2}}+\dfrac{\sqrt{24}}{3\sqrt{3}}$. (C. F. 1882).

Simplify—

(5) $3\sqrt[4]{80}+4\sqrt[4]{405}-2\sqrt[4]{1280}$.

(6) $4\sqrt{(147)}-\dfrac{10}{\sqrt{3}}+3\sqrt{75}-2\sqrt{[illegible]}$.

8. Prove that $\sqrt[3]{4}$, $\sqrt[6]{([illegible])}$, $\sqrt[3]{([illegible])}$, $\sqrt[4]{(2916)}$ are like surds

9. Perform the operations indicated.

(1) $2\sqrt{3}\times 3\sqrt{5}\times 7\sqrt{8}$. (2) $\sqrt[3]{[illegible]}\times\sqrt{2}$.

(2) $\sqrt[4]{5}\times\sqrt[6]{11}$. (4) $3\sqrt{243}\div 4\sqrt{([illegible])}$.

(5) $5^{\frac{3}{2}}\div 7^{\frac{2}{3}}$. (6) $\sqrt[3]{3}\div\sqrt[4]{5}$.

10. Simplify

(1) $(2+\sqrt{3})(\sqrt{3}-5)$ (2) $(1+\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$

(3) $(2\sqrt{a}-4\sqrt{b})(2\sqrt{b}+3\sqrt{a})$.

(4) $(\sqrt{3}+2\sqrt{5}+\sqrt{7})(\sqrt{3}-2\sqrt{5}+\sqrt{7})$.

(5) $(\sqrt{x+a}-\sqrt{x-a})(\sqrt{x+a}+\sqrt{x-a})$.

(6) $(\sqrt{a+b}+1)(\sqrt{a+b}-3)$.

Rationalization.

10. It is often necessary to multiply one surd by another so that the product may be *rational*. In such a case each is called the **rationalising factor** of the other.

11. Simple surds. A simple surd like $a^{m/n}$ may be rationalised by $a^{p/q}$ where p/q is the defect of the *fractional part of m/n* from unity.

Thus a rationalising factor of $a^{\frac{7}{5}}$ is $a^{1-\frac{2}{5}}$ *i.e.* $a^{\frac{3}{5}}$,

a rationalising factor of $a^{\frac{9}{4}}b^{\frac{12}{5}}$ is $a^{1-\frac{1}{4}}b^{1-\frac{2}{5}}$ *i. e.* $a^{\frac{3}{4}}b^{\frac{3}{5}}$.

12. Binomial quadratic surds. Since we have $(a+\sqrt{b})(a-\sqrt{b})=a^2-b$, the surds $a+\sqrt{b}$, $a-\sqrt{b}$ are each the rationalising factor of the other.

Similarly, since $(a\sqrt{b}+c\sqrt{d})(a\sqrt{b}-c\sqrt{d})=a^2b-c^2d$, it follows that $a\sqrt{b}+c\sqrt{d}$ may be rationalised by multiplying it by $a\sqrt{b}-c\sqrt{d}$ and *vice versa.*

Two binomial surds like $a+\sqrt{b}$, $a-\sqrt{b}$ differing only in the signs which connect their terms are said to be **Conjugate.** Hence a binomial quadratic surds is rationalised by multiplying it by its conjugate.

13. In dealing with a fraction having a surd denominator it is advisable to **rationalise the denominator** by multiplying top

and bottom of the fraction by the rationalising factor of the denominator. It is always easier to treat a fraction having a rational denominator than one whose denominator is irrational, specially so in the case of numerical calculations.

Ex. 1. Find the value of $\frac{7}{\sqrt{2}}$.

Rationalising the denominator, we get

$$\frac{7}{\sqrt{2}}=\frac{7\sqrt{2}}{2}=\frac{7\times 1\cdot4142}{2}=\frac{9\cdot8994}{2}=4\cdot9497.$$

Ex. 2. Find the value of $\frac{3+2\sqrt{5}}{7+3\sqrt{5}}$.

$$\text{The expression}=\frac{3+2\sqrt{5}}{7+3\sqrt{5}}\cdot\frac{7-3\sqrt{5}}{7-3\sqrt{5}}=\frac{21-9\sqrt{5}+14\sqrt{5}-30}{49-45}$$

$$=\frac{-9+5\sqrt{5}}{4}=\frac{-9+5\times 2\cdot2360}{4}=\frac{2\cdot1800}{4}=\cdot545.$$

Ex. 3. Find the value of

$$\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{\sqrt{(1+x)}+\sqrt{(1-x)}} \text{ when } x=\frac{\sqrt{3}}{2}. \qquad \text{(B. P. 1883.)}$$

$$\text{The expression}=\frac{\{\sqrt{(1+x)}-\sqrt{(1-x)}\}^2}{\{\sqrt{(1+x)}+\sqrt{(1-x)}\}\{\sqrt{(1+x)}-\sqrt{(1-x)}\}}$$

$$=\frac{1+x+1-x-2\sqrt{(1-x^2)}}{(1+x)-(1-x)}=\frac{2-2\sqrt{(1-x^2)}}{2x}=\frac{1-\sqrt{(1-x^2)}}{x}$$

$$=\frac{1-\sqrt{(1-\frac{3}{4})}}{\frac{\sqrt{3}}{2}}=\frac{1-\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{2}\times\frac{2}{\sqrt{3}}.$$

$$=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}=\frac{1\cdot73205}{3}=\cdot57735.$$

EXERCISE CXV.

1. Find the rationalising factor of—

(1) $\sqrt[3]{a^5b^2c^7}$. (2) $3^{\frac{4}{5}}.\ 2^{\frac{7}{3}}.7^{\frac{0}{4}}$.

2. Rationalize the denominator of and simplify

(1) $\frac{5}{\sqrt{2}+1}$. (2) $\frac{3+5\sqrt{3}}{5+3\sqrt{3}}$.

(3) $\frac{3\sqrt{5}+5\sqrt{3}}{7\sqrt{3}+2\sqrt{5}}$. (4) $\frac{\sqrt{(x+a)}-\sqrt{(x-a)}}{\sqrt{(x+a)}+\sqrt{(x-a)}}$

3. Given $\sqrt{2}=1·4142$, $\sqrt{3}=1·7321$, $\sqrt{5}=2·2360$, $\sqrt{6}=2·4494$ calculate to 3 places of decimals the value of

(1) $\dfrac{2}{\sqrt{5}}$. (2) $\dfrac{2\sqrt{2}}{\sqrt{3}}$. (3) $\dfrac{\sqrt{8}+\sqrt{27}}{5}$.

(4) $\dfrac{1}{\sqrt{12}}$. (5) $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$. (6) $\dfrac{1}{2\sqrt{3}+\sqrt{\ }}$ [illegible].

4. Prove that $\dfrac{3\sqrt{8}+2\sqrt{7}}{\sqrt{8}+\sqrt{7}}=2·5$. ... (C. F. 1877)

5. Simplify

(1) $\dfrac{1}{\sqrt{2}+1}+\dfrac{1}{\sqrt{3}+\sqrt{2}}+\dfrac{1}{2+\sqrt{3}}$.

(2) $\dfrac{3\sqrt{2}}{\sqrt{6}-\sqrt{3}}-\dfrac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}+\dfrac{2\sqrt{3}}{\sqrt{6}+2}$. (C. F. 1877)

(3) $\dfrac{x+\sqrt{(x^2-a^2)}}{x-\sqrt{(x^2-a^2)}}-\dfrac{x-\sqrt{(x^2-a^2)}}{x+\sqrt{(x^2-a^2)}}$.

(4) $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}+\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$.

Properties of Quadratic Surds.

14. **A quadratic surd cannot be equal to the sum or difference of a rational quantity and a quadratic surd.**

If possible, let $\sqrt{x}=a\pm\sqrt{b}$, squaring, $x=a^2+b\pm 2a\sqrt{b}$;

$$\therefore x-a^2-b=\pm 2a\sqrt{b} \text{ or, } \sqrt{b}=\pm\frac{x-a^2-b}{2a}.$$

Thus an irrational quantity is equal to a rational quantity which is impossible.

Hence $\sqrt{x}$ cannot be equal to $a\pm\sqrt{b}$.

15. If $a+\sqrt{b}=c+\sqrt{d}$, then $a=c$, $b=d$.

For, if a is not *equal* to c, let $a=c+x$, x being necessarily rational.

Then $c+x+\sqrt{b}=c+\sqrt{d}$, or $x+\sqrt{b}=\sqrt{d}$.

That is, a quadratic surd is equal to the sum of a rational quantity and a quadratic surd; which is impossible (Art. 14).

Therefore we must have $a=c$ and hence from the given relation also $b=d$.

16. If $\sqrt{(a+\sqrt{b})}=\sqrt{c}+\sqrt{d}$, then $\sqrt{(a-\sqrt{b})}=\sqrt{c}-\sqrt{d}$.

From the given relation by squaring,

$$a+\sqrt{b}=c+d+2\sqrt{cd}.$$

Hence equating the rational and irrational parts,

$$a=c+d,\ \sqrt{b}=2\sqrt{cd}.$$

$\therefore\ a-\sqrt{b}=c+d-2\sqrt{cd}=(\sqrt{c}-\sqrt{d})^2$. Hence etc.

17. To find the square root of $a+\sqrt{b}$.

Let $\sqrt{(a+\sqrt{b})}=\sqrt{x}+\sqrt{y}$ where x and y are rational and positive.

Then, squaring, $a+\sqrt{b}=x+y+2\sqrt{xy}$

$$\therefore\ \left.\begin{aligned} +\ a&=x+y\ldots\ldots(1)\\ +\sqrt{b}&=2\sqrt{xy}\ldots\ldots(2)\end{aligned}\right\}\quad \text{Art. 16.}$$

To solve (1) and (2) we have $(x-y)^2=(x+y)^2-4xy=a^2-b$.

$$\therefore\ x-y=+\sqrt{(a^2-b)}\ldots\ldots(3)$$

$$\text{or } x-y=-\sqrt{(a^2-b)}\ldots\ldots(4)$$

Taking (1) and (3) we have

$$x=\tfrac{1}{2}\{a+\sqrt{(a^2-b)}\},\ y=\tfrac{1}{2}\{a-\sqrt{(a^2-b)}\}.$$

$$\sqrt{x}=\pm\sqrt{[\tfrac{1}{2}\{a+\sqrt{a^2-b)}\}]},\qquad \sqrt{y}=\pm\sqrt{[\tfrac{1}{2}\{a-\sqrt{(a^2-b)}\}]};$$

and since from (2) $\sqrt{x}\times\sqrt{y}$ is positive, we must take *like* signs together before the radicals.

$$\therefore\ \text{the required square root}=\sqrt{x}+\sqrt{y}$$

$$=\pm[\sqrt{\{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)}\}}+\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)}\}}].$$

If we had taken (1) and (4) we would have got the same result.

Similarly by assuming $\sqrt{(a-\sqrt{b})}=\sqrt{x}-\sqrt{y}$, or, from art, 16 we get $\sqrt{(a-\sqrt{b})}=\pm[\sqrt{\{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)}\}}-\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)}\}}]$

Note. From (1) we must have a positive and if $\sqrt{(a^2-b)}$ is rational *i. e.* a^2-b a positive perfect square, the value of x and y are rational, and the square root $\sqrt{x}+\sqrt{y}$ becomes a quadratic surd. If a^2-b be not a perfect square, the square root as found by this method becomes complicated. In such cases we may sometimes proceed as in ex. 2 below.

Ex. 1. Extract the square root of $11+6\sqrt{2}$.

(Here $a^2-b=121-72=49$, a perfect square).

Assume $\sqrt{(11+6\sqrt{2})}=\sqrt{x}+\sqrt{y}$,

squaring, $11+6\sqrt{2}=x+y+2\sqrt{xy}$.

$\therefore\ x+y=11,\ 2\sqrt{xy}=6\sqrt{2}$, or, $4xy=72$.

Hence $(x-y)^2=(x+y)^2-4xy=121-72=49$.

$\therefore\ x-y=\pm7$. Also $x+y=11$; $\therefore\ x=9$ or 2, $y=2$ or 9.

$\therefore$ the required square root $=\sqrt{x}+\sqrt{y}=\pm(3+\sqrt{2})$.

Ex. 2. Extract the square root of $10+6\sqrt{5}$.

Here, $a^2-b=100-180=-80$ which is not a perfect square; so the preceding method is not suitable. We may proceed thus:

$10+6\sqrt{5}=\sqrt{5}(2\sqrt{5}+6)$; and since $6^2-(2\sqrt{5})^2=16$, a perfect square, we can find the square root of $2\sqrt{5}+6$ by the preceding method, and it is found to be $\pm(1+\sqrt{5})$. Hence the required square root $=\pm\sqrt[4]{5}(1+\sqrt{5})=\pm(\sqrt[4]{5}+\sqrt[4]{125})$.

EXERCISE CXVI.

1. The product and quotient of two similar quadratic surds are rational; and conversely.

2. Prove that the sum or difference of two dissimilar surds cannot be rational.

3 Prove that if two simple quadratic surds cannot be reduced to two others which have the same rational part, their product is irrational. (C. F. 1882).

4. Extract the square root of

(1) $14+6\sqrt{5}$. (2) $7-2\sqrt{6}$. (3) $28-6\sqrt{3}$.

(4) $5-2\sqrt{6}$. (5) $5+4\sqrt{5}$. (6) $7\sqrt{10}-20$

Irrational Equations.

18. In solving equations involving surds we are to rationalize them i.e. transform them so that they may be free from surd quantities. The following examples will illustrate the method of rationalisation.

Ex. 1. Rationalize the equation $\sqrt{a}+\sqrt{b}+\sqrt{c}=0$.

By transposition, $\sqrt{a}+\sqrt{b}=-\sqrt{c}$.

Squaring, $a+b+2\sqrt{(ab)}=c$

Transposing so as to bring the radical on one side,

$$2\sqrt{(ab)}=c-a-b.$$

Squaring, $4ab=c^2+a^2+b^2-2ca+2ab-2bc$.

$$\therefore\ a^2+b^2+c^2-2bc-2ca-2ab=0.$$

Similarly, each of $\sqrt{a}+\sqrt{b}-\sqrt{c}=0$, $\sqrt{a}-\sqrt{b}+\sqrt{c}=0$, $-\sqrt{a}+\sqrt{b}+\sqrt{c}=0$ being rationalised gives the same result. In fact we have the identity

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})(-\sqrt{a}+\sqrt{b}+\sqrt{c})$$
$$=a^2+b^2+c^2-2bc-2ca-2ab.$$

Hence any one of the four factors on the left being equated to zero gives the same rational result.

Ex. 2. Rationalize the equation $\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}=0$.

By transposition, $\sqrt{a}+\sqrt{b}=-(\sqrt{c}+\sqrt{d})$.

$\therefore$ Squaring, $a+b+2\sqrt{(ab)}=c+d+2\sqrt{(cd)}$.

Transposing, $2(\sqrt{ab}-\sqrt{cd})=c+d-a-b$.

Squaring, $4(ab+cd-2\sqrt{abcd})=(c+d-a-b)^2$.

$\therefore$ Transposing, $-8\sqrt{(abcd)}=(c+d-a-b)^2-4(ab+cd)$

$=a^2+b^2+c^2+d^2-2ab-2ac-2ad-2bc-2bd-2cd$.

$\therefore$ Squaring finally, $64\,abcd$

$=(a^2+b^2+c^2+d^2-2ab-2ac-2ad-2bc-2bd-2cd)^2$.

19. We now consider some cases of irrational equations which when rationalised lead to simple equations.

Ex. 1. Solve $\sqrt{(x+11)}+\sqrt{(x-16)}=9$.

Transposing, $\sqrt{(x+11)}=9-\sqrt{(x-16)}$.

Squaring, $x+11=81+(x-16)-18\sqrt{(x-16)}$.

Transposing, $18\sqrt{(x-16)}=54$ or $\sqrt{(x-16)}=3$.

$\therefore x-16=9$ whence $x=9+16=25$.

Ex. 2. Solve $\sqrt{(x+1)}+\sqrt{(x-1)}=\sqrt{(4x-1)}$.

Squaring, $x+1+x-1+2\sqrt{(x^2-1)}=4x-1$.

Transposing, $2\sqrt{(x^2-1)}=2x-1$.

$\therefore$ Squaring, $4x^2-4=4x^2-4x+1$,

$\therefore 4x=5$ or $x=\frac{5}{4}$.

Ex. 3. Solve $\sqrt{(x^2+9x+11)}-\sqrt{(x^2+x-1)}=4$.

Transposing, $\sqrt{(x^2+9x+11)}=4+\sqrt{(x^2+x-1)}$.

$\therefore$ Squaring, $x^2+9x+11=16+x^2+x-1+8\sqrt{(x^2+x-1)}$.

$\therefore 8x-4=8\sqrt{(x^2+x-1)}$, or $2x-1=2\sqrt{(x^2+x-1)}$.

Squaring, $4x^2-4x+1=4x^2+4x-4$, whence $x=\frac{5}{8}$

20. In some irrational equations special methods may be usefully employed as shewn below.

Ex. 1. Solve $\sqrt{(x+28)}-\sqrt{(x+8)}=2$............(1)

We have identically $(x+28)-(x+8)=20$............(2).

From (1) and (2) by division,

$$\frac{(x+28)-(x+8)}{\sqrt{(x+28)}-\sqrt{(x+8)}}=\frac{20}{2},$$

$\therefore \sqrt{(x+28)}+\sqrt{(x+8)}=10$............................(3)

Adding (1) and (3), $2\sqrt{(x+28)}=12$, or, $\sqrt{(x+28)}=6$

$\therefore x+28=36$, $\therefore x=8$.

Ex. 2. Solve $\sqrt{(2x+7)} - \sqrt{(x+1)} = \sqrt{(5x+1)} - \sqrt{(4x-5)}$ ......(1)

We have identically $(2x+7)-(x+1) = (5x+1)-(4x-5)$ ......(2)

From (1) and (2) by division,

$$\sqrt{(2x+7)} + \sqrt{(x+1)} = \sqrt{(5x+1)} + \sqrt{(4x-5)} \text{ ......(3)}$$

From (1) and (3) by addition,

$$2\sqrt{(2x+7)} = 2\sqrt{(5x+1)}, \text{ or, } \sqrt{(2x+7)} = \sqrt{(5x+1)}$$

$$\therefore\ 2x+7 = 5x+1, \quad \therefore\ x = 2.$$

Ex. 3. Solve $\dfrac{ax-1}{\sqrt{(ax)}+1} = 4 + \dfrac{\sqrt{(ax)}-1}{2}$. (C. E. 1885.)

Since $(ax-1) = (\sqrt{ax}+1)(\sqrt{ax}-1)$, the equation becomes,

$$\frac{(\sqrt{ax}+1)(\sqrt{ax}-1)}{\sqrt{ax}+1} = 4 + \frac{\sqrt{ax}-1}{2}$$

$$\therefore\ (\sqrt{ax}-1) = 4 + \frac{\sqrt{ax}-1}{2}, \text{ or } \tfrac{1}{2}(\sqrt{ax}-1) = 4;$$

$$\therefore\ \sqrt{ax}-1 = 8 \text{ or } \sqrt{ax} = 9$$

$$\therefore\ ax = 81, \text{ hence } x = \frac{81}{a}.$$

Ex. 4. Solve $\dfrac{x-7}{\sqrt{(x+2)}-3} + \dfrac{x-4}{\sqrt{(x-3)}-1} = \dfrac{4x-15}{\sqrt{(4x+1)}-4}$.

Rationalize the denominators, thus

$$\frac{x-7}{\sqrt{(x+2)}-3} = \frac{x-7}{\sqrt{(x+2)}-3} \times \frac{\sqrt{(x+2)}+3}{\sqrt{(x+2)}+3}$$

$$= \frac{(x-7)\{\sqrt{(x+2)}+3\}}{(x+2)-9} = \sqrt{(x+2)}+3$$

Similarly, $\dfrac{x-4}{\sqrt{(x-3)}-1} = \dfrac{(x-4)\{\sqrt{(x-3)}+1\}}{(x-3)-1} = \sqrt{(x-3)}+1.$

$$\frac{4x-15}{\sqrt{(4x+1)}-4} = \frac{(4x-15)\{\sqrt{(4x+1)}+4\}}{(4x+1)-16} = \sqrt{(4x+1)}+4.$$

Hence $\sqrt{(x+2)}+3+\sqrt{(x-3)}+1 = \sqrt{(4x+1)}+4$.

$$\therefore\ \sqrt{(x+2)} + \sqrt{(x-3)} = \sqrt{(4x+1)}.$$

Squaring, $(x+2+x-3) + 2\sqrt{(x+2)(x-3)} = 4x+1$,

$$\therefore\ 2\sqrt{(x+2)(x-3)} = 2(x+1).$$

Removing 2 and squaring,

$$(x+2)(x-3) = (x+1)^2,$$

$$\therefore\ x^2 - x - 6 = x^2 + 2x + 1, \quad \text{hence } x = -\frac{7}{3}.$$

Ex. 5. Solve $\dfrac{\sqrt{(x+a)}+\sqrt{(x-a)}}{\sqrt{(x+a)}-\sqrt{(x-a)}}=\dfrac{b}{c}$.

By componendo and dividendo,

$$\frac{(\sqrt{x+a}+\sqrt{x-a})+(\sqrt{x+a}-\sqrt{x-a})}{(\sqrt{x+a}+\sqrt{x-a})-(\sqrt{x+a}-\sqrt{x-a})}=\frac{b+c}{b-c}.$$

$$\therefore\ \frac{2\sqrt{(x+a)}}{2\sqrt{(x-a)}}=\frac{b+c}{b-c} \text{ or } \frac{\sqrt{(x+a)}}{\sqrt{(x-a)}}=\frac{b+c}{b-c}.$$

Squaring $\dfrac{x+a}{x-a}=\dfrac{b^2+c^2+2bc}{b^2+c^2-2bc}$.

Again, applying componendo and dividendo,

$$\frac{(x+a)+(x-a)}{(x+a)-(x-a)}=\frac{(b^2+c^2+2bc)+(b^2+c^2-2bc)}{(b^2+c^2+2bc)-(b^2+c^2-2bc)}$$

$$\therefore\ \frac{x}{a}=\frac{b^2+c^2}{2bc}, \text{ whence}$$

$$x=\frac{a(b^2+c^2)}{2bc}.$$

Ex. 6. Solve $\dfrac{a+\sqrt{(a^2-x^2)}}{a-\sqrt{(a^2-x^2)}}=c^3\,\dfrac{\sqrt{(a+x)}-\sqrt{(a-x)}}{\sqrt{(a+x)}+\sqrt{(a-x)}}$.

We have $2a+2\sqrt{(a^2-x^2)}=\{\sqrt{(a+x)}+\sqrt{(a-x)}\}^2$,

$\qquad 2a-2\sqrt{(a^2-x^2)}=\{\sqrt{(a+x)}-\sqrt{(a-x)}\}^2$.

$\therefore$ by division $\dfrac{a+\sqrt{(a^2-x^2)}}{a-\sqrt{(a^2-x^2)}}=\dfrac{\{\sqrt{(a+x)}+\sqrt{(a-x)}\}^2}{\{\sqrt{(a+x)}-\sqrt{(a-x)}\}^2}$.

Hence the given equation becomes

$$\frac{\{\sqrt{(a+x)}+\sqrt{(a-x)}\}^2}{\{\sqrt{(a+x)}-\sqrt{(a-x)}\}^2}=c^3\,\frac{\sqrt{(a+x)}-\sqrt{(a-x)}}{\sqrt{(a+x)}+\sqrt{(a-x)}}.$$

$$\therefore\ \frac{\{\sqrt{(a+x)}+\sqrt{(a-x)}\}^3}{\{\sqrt{(a+x)}-\sqrt{(a-x)}\}^3}=c^3, \text{ whence}$$

$$\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}=c.$$

Applying componendo and dividendo

$$\frac{\sqrt{(a+x)}}{\sqrt{(a-x)}}=\frac{c+1}{c-1}, \text{ or squaring, } \frac{a+x}{a-x}=\frac{c^2+2c+1}{c^2-2c+1}.$$

Again, by componendo and dividendo, $\dfrac{a}{x}=\dfrac{c^2+1}{2c}$, whence $x=\dfrac{2ac}{c^2+1}$

EXERCISE CXVII.

Solve the following equations :—

1. $\sqrt{(x+7)} = \sqrt{x} + 1$.
2. $\sqrt{(3x+16)} = \sqrt{(3x)} + 2$.
3. $\sqrt{(x+9)} = \sqrt{(x-6)} + 3$.
4. $\sqrt{(2x+3)} = 2 + \sqrt{(2x-5)}$.
5. $3x + 2 = \sqrt{(9x^2+28)}$.
6. $\sqrt{(3x+7)} - \sqrt{(3x-[illegible])} = 3$.
7. $2(x+2) = 1 + \sqrt{(4x^2+9x+14)}$. (C. E. 1877).
8. $\sqrt{\{4x^2+20x+17+\sqrt{(16x^2+11x+10)}\}} = 2(x+2)$.
9. $x + a + \sqrt{(2ax+x^2)} = b$.
10. $x + \sqrt{(x^2+9)} = 9$.
11. $\sqrt{\{(x-a)^2+2ab+b^2\}} = x - a - b$. (M. M. 1886).
12. $\sqrt{(5x-1)} = 1 + \sqrt{(5x-2)}$. (C. E. 1875).
13. $x - k + \sqrt{(k^2+x^2)} = m$. (C. E. 1879).
14. $\sqrt{(x^2+11x+20)} - \sqrt{(x^2+5x-1)} = 3$. (C. E. 1881).
15. $\sqrt{(3x+10)} - \sqrt{(3x-5)} = [illegible]$.
16. $\sqrt{(2x+15)} - \sqrt{(2x-9)} = 4$.
17. $\sqrt{(x+5)} + \sqrt{(x-3)} = 2(2+\sqrt{2})$.
18. $\sqrt{(2x+9)} + \sqrt{(2x-9)} = 9 + 3\sqrt{7}$.
19. $\sqrt{(3x+5)} - \sqrt{(2x+3)} = \sqrt{(5x-9)} - \sqrt{(4x-11)}$.
20. $\sqrt{(4x+9)} + \sqrt{(2x+11)} = \sqrt{(7x+3)} + \sqrt{(9x+1)}$.
21. $\sqrt{x} + \sqrt{\{x - \sqrt{(1-x)}\}} = 1$.
22. $\sqrt{(1+x)} + \sqrt{\{1+x+\sqrt{(1-x)}\}} = \sqrt{(1-[illegible])}$.
23. $\dfrac{3x-1}{\sqrt{(3x)}+1} = 5 + \dfrac{\sqrt{(3x)}-1}{4}$.
24. $\dfrac{x-3}{\sqrt{(x+6)}-3} + \dfrac{x-9}{\sqrt{(x-5)}-2} = 16$.
25. $\dfrac{x-2}{\sqrt{(x+2)}-2} + \dfrac{x-6}{\sqrt{(x+3)}-3} = \dfrac{x}{\sqrt{(x+1)}-1} + \dfrac{x-15}{\sqrt{(x+1)}-4}$.
26. $\dfrac{1}{\sqrt{x}-\sqrt{(x-2)}} + \dfrac{1}{\sqrt{x}+\sqrt{(x+2)}} = 1$. (C. E. 1882).
27. $\dfrac{\sqrt{(x+1)}-\sqrt{(x-1)}}{\sqrt{(x+1)}+\sqrt{(x-1)}} = \dfrac{1}{2}$.
28. $\dfrac{\sqrt{(x+a)}+\sqrt{(x+b)}}{\sqrt{(x+a)}-\sqrt{(x+b)}} = [illegible]$.
29. $\dfrac{a+\sqrt{(a^2-x^2)}}{a-\sqrt{(a^2-x^2)}} = 27\dfrac{\sqrt{(a+x)}-\sqrt{(a-x)}}{\sqrt{(a+x)}+\sqrt{(a-x)}}$.
30. $\dfrac{b^2}{\sqrt{(a+x)}} + \dfrac{c^2}{\sqrt{(a-x)}} = \dfrac{2bc}{\sqrt[4]{(a^2-x^2)}}$.

CHAPTER XXVIII.

QUADRATIC EQUATIONS.

1. An equation which when expressed in a rational integral form, contains the second, and no higher powers of the unknown quantity, is called a **quadratic equation or an equation of the second degree.**

When all the terms of a quadratic equation in x are transposed to one side, it takes the form $ax^2+bx+c=0$ where a, b, c are some constant quantities. This is called the general form of the quadratic equation.

2. A quadratic in which the term containing the first power of the unknown quantity does not occur is called a *pure quadratic*.

The general form of a pure quadratic is $ax^2=c$, or $x^2=\frac{c}{a}$. Hence the values of x are immediately obtained by extracting the square roots of both sides

Thus we get $\pm x=\pm\sqrt{\frac{c}{a}}$, which gives only two different solutions, *viz.*, $x=+\sqrt{\frac{c}{a}}$, and $x=-\sqrt{\frac{c}{a}}$; and this we may write shortly $x=\pm\sqrt{\frac{c}{a}}$.

Hence a pure quadratic has two roots which are equal in magnitude but are of opposite signs.

Ex. 1. Slove $3x^2-19=4(11-x^2)$.

Here $3x^2-19=44-4x^2$.

$\therefore$ transposing, $3x^2+4x^2=44+19$, or, $7x^2=63$.

$\therefore x^2=9$, or, $x=\pm3$.

Ex. 2. Solve $\frac{11}{2x^2+7}=\frac{4}{(x+2)(x-2)}+\frac{3}{(x^2-4)(2x^2+7)}$.

Multiplying by $(2x^2+7)(x^2-4)$, the L.C.M. of the denominators

$11(x^2-4)=4(2x^2+7)+3$.

$\therefore 11x^2-44=8x^2+28+3$, or, $11x^2-8x^2=28+3+44$.

$\therefore 3x^2=75$, or, $x^2=25$. $\qquad\therefore x=\pm5$.

Ex. 3. Solve $\frac{(x+a)(x+mb)}{(x-ma)(x-b)}=\frac{(mx+a)(x+b)}{(x-a)(mx-b)}$. (C. F. 1880.

Taking alternately $\frac{(x+a)(x+mb)}{(mx+a)(x+b)}=\frac{(x-ma)(x-b)}{(x-a)(mx-b)}$,

or $\frac{x^2+x(a+mb)+abm}{mx^2+x(a+mb)+ab}=\frac{x^2-x(b+ma)+abm}{mx^2-x(b+ma)+ab}$

$=\frac{\text{diff. of numerators}}{\text{diff. of denominators}}=\frac{x(a+mb+b+ma)}{x(a+mb+b+ma)}=1,$

supposing x is not zero.

$\therefore x^2+x(a+mb)+abm=mx^2+x(a+mb)+ab.$

$\therefore x^2(1-m)=ab(1-m)$, or, $x^2=ab$. $\quad\therefore x=\pm\sqrt{(ab)}$.

EXERCISE CXVIII.

Solve the following equations :—

1. $8x^2-27=17-3x^2$.
2. $(3x-4)(6+5x)+(x+1)^2=41$.
3. $\frac{x+3}{2x+5}=\frac{7x+1}{11x+4}$.
4. $\frac{6x-1}{2x-1}-\frac{3x-1}{x-1}=\frac{12x-3}{4x-3}-\frac{6x-3}{2x-3}$.
5. $\frac{x+7}{x-7}+\frac{x-7}{x+7}=\frac{100}{x^2-49}$.
6. $\frac{(x+5)^2}{(x-5)^2}-\frac{(x-5)^2}{(x+5)^2}=50$.
7. $\frac{ax+b}{cx+d}=\frac{a+bx}{c+dx}$.
8. $\frac{1}{x+11}+\frac{1}{x+2}+\frac{1}{x-13}=0$.
9. $\frac{4x^2+28}{x^2+3}+\frac{5x^2+10}{x^2+9}=9$.
10. $\frac{x+a}{x-a}+\frac{x-a}{x+a}=b$.
11. $\frac{1+x^3}{(1+x)^3}+\frac{1-x^3}{(1-x)^3}=a$.
12. $\frac{(x+1)^5+(x-1)^5}{(x+1)^4+(x-1)^4}=10$.

---

3. Every complete quadratic equation can be reduced to the form $ax^2+bx+c=0$, or by dividing throughout by the co-efficient of x^2, to the form $x^2+px+q=0$, where p and q may be positive or negative, the co-efficient of x^2 being $+1$.

To solve $x^2+px+q=0$, we factorise the left hand side thus :

x^2+px+q

$=x^2+px+\left(\frac{p}{2}\right)^2-\left(\frac{p}{2}\right)^2+q$ adding $\left(\frac{p}{2}\right)^2$ to get a perfect square and then subtracting it to keep the value unchanged.

$=\left(x+\frac{p}{2}\right)^2-\frac{p^2-4q}{4}$

$$=\left(x+\frac{p}{2}\right)^2-\left(\frac{\sqrt{p^2-4q}}{2}\right)^2$$

$$=\left(x+\frac{p}{2}+\frac{\sqrt{p^2-4q}}{2}\right)\left(x+\frac{p}{2}-\frac{\sqrt{p^2-4q}}{2}\right)$$

$\therefore\ x^2+px+q=0$ gives

either $x+\frac{p}{2}+\frac{\sqrt{p^2-4q}}{2}=0$ or $x+\frac{p}{2}-\frac{\sqrt{p^2-4q}}{2}=0$;

i.e., $x=-\frac{p}{2}-\frac{\sqrt{(p^2-4q)}}{2}$ or $x=-\frac{p}{2}+\frac{\sqrt{(p^2-4q)}}{2}$.

This may be shortly written $x=\dfrac{-p\pm\sqrt{(p^2-4q)}}{2}$.

Ex. Solve $x^2+\frac{7}{6}x-\frac{1}{2}=0$.

We have $x^2+\frac{7}{6}x-\frac{1}{2}$

$=x^2+\frac{7}{6}x+(\frac{7}{12})^2-(\frac{7}{12})^2-\frac{1}{2}$, adding and subtracting $(\frac{7}{12})^2$

$=(x+\frac{7}{12})^2-(\frac{11}{12})^2$

$=(x+\frac{7}{12}+\frac{11}{12})(x+\frac{7}{12}-\frac{11}{12})$

$=(x+\frac{3}{2})(x-\frac{1}{3})$.

$\therefore$ either $x+\frac{3}{2}=0$, or, $x-\frac{1}{3}=0$.

$\therefore\ x=-\frac{3}{2}$ or $\frac{1}{3}$.

4. Let us now consider the general form of a quadratic equation, *viz.*, $ax^2+bx+c=0$.

We have ax^2+bx+c

$$=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$=a\left\{x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right\}$$ adding and subtracting $\left(\frac{b}{2a}\right)^2$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right\}$$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\right\}$$

$$=a\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)$$

$\therefore\ ax^2+bx+c=0$ gives (supposing a is not zero)

$$\text{either } x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}=0, \text{ or, } x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}=0.$$

$$\therefore\ x=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}, \text{ or, } x=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a},$$

$$\text{that is, } x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Ex. Solve $8x^2-2x-3=0$.

We have $8x^2-2x-3$

$=8(x^2-\frac{1}{4}x-\frac{3}{8})$

$=8\{x^2-\frac{1}{4}x+(-\frac{1}{8})^2-(-\frac{1}{8})^2-\frac{3}{8}\}$, adding and subtracting $(-\frac{1}{8})^2$

$=8\{(x-\frac{1}{8})^2-(\frac{5}{8})^2\}$

$=8(x-\frac{1}{8}+\frac{5}{8})(x-\frac{1}{8}-\frac{5}{8})$

$=8(x+\frac{1}{2})(x-\frac{3}{4})$.

$\therefore$ either $x+\frac{1}{2}=0$ or $x-\frac{3}{4}=0$

$\therefore\ x=-\frac{1}{2}$ or $\frac{3}{4}$.

Note 1. From the above we see that the quadratic equation $ax^2+bx+c=0$ has two roots, viz., $\dfrac{-b\pm\sqrt{(b^2-4ac)}}{2a}$. These roots should be committed to memory by the student, and in any particular example he may obtain the solution by substituting the values of a, b, c.

Thus in the above example $a=8$, $b=-2$, $c=-3$.

$$\therefore\ x=\frac{2\pm\sqrt{(-2)^2-4.8.(-3)}}{2.8}$$

$$=\frac{2\pm10}{16}=\frac{3}{4}\text{ or }-\frac{1}{2}, \text{ as before.}$$

Note 2. In the equation $ax^2+bx+c=0$ if we rationalize the numerators of the roots we can write them in the form $\dfrac{2c}{-b\pm\sqrt{b^2-4ac}}$.

Note 3. The student will observe that we have above, the general method of resolving any expression of the form ax^2+bx+c into factors, and this is the process to be used when the method of factorisation by trial fails.

5. The method of solution of a quadratic equation given in the preceding articles may be conveniently presented in a little different form as follows:—

Ex. 1. Solve $6x^2-7x+2=0$.

Transposing, $6x^2-7x=-2$.

Dividing by 6, $x^2-\frac{7}{6}x=-\frac{1}{3}$.

Adding $(\frac{7}{12})^2$ to both sides,

$$x^2-\frac{7}{6}x+(\frac{7}{12})^2=-\frac{1}{3}+(\frac{7}{12})^2$$
$$=\frac{1}{144}.$$

$$\therefore (x-\frac{7}{12})^2=(\frac{1}{12})^2.$$

Extracting the square root, $x-\frac{7}{12}=\pm\frac{1}{12}$

$$\therefore x=\frac{7}{12}\pm\frac{1}{12}=\frac{2}{3} \text{ or } \frac{1}{2}.$$

Ex. 2. Solve $ax^2+bx+c=0$.

Transposing, $ax^2+bx=-c$.

Dividing by a, $x^2+\frac{b}{a}x=-\frac{c}{a}$; adding $\left(\frac{b}{2a}\right)^2$ to both sides,

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2.$$

$$\therefore \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

Extracting the square root, $x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$.

$$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

We may state the method of solution thus :—

(i) Write the equation so that the terms containing the unknown quantity may be on one side and the constant term on the other; (ii) divide both sides by the co-efficient of the square of the unknown quantity; (iii) add to both sides the square of half the co-efficient of the unknown quantity; and (iv) extract the square roots of both sides.

6. Hindu Method. The following is the Hindu method of solving a quadratic equation given in the *Beej Ganita of Bhaskara Acharya*, a famous mathematician who flourished in the beginning of the thirteenth century.

Ex. Solve $ax^2+bx+c=0$.

Transposing, $ax^2+bx=-c$.

Multiply by $4a$, $4a^2x^2+4abx=-4ac$.

Adding b^2 to both sides to make the left-hand side a perfect square, $4a^2x^2+4abx+b^2=b^2-4ac$.

$$\therefore (2ax+b)^2=b^2-4ac.$$

Extracting the square root, $2ax+b=\pm\sqrt{(b^2-4ac)}$.

$$\therefore 2ax=-b\pm\sqrt{(b^2-4ac)}.$$

$$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

We may state the Hindu method of solution thus :—

(i) Write the equation so that the terms containing the unknown quantity may be on one side and the constant term on the other; (ii) multiply both sides by 4 times the co-efficient of the square of the unknown quantity; (iii) add to both sides the square of the co-efficient of the unknown quantity in the original equation; and (iv) extract the square roots of both sides.

7. Nature of the roots. Let $ax^2+bx+c=0$ be a quadratic equation and suppose a, b, c are rational. Then the roots are

$$\frac{-b+\sqrt{(b^2-4ac)}}{2a}, \qquad \frac{-b-\sqrt{(b^2-4ac)}}{2a}.$$

Now the quantity under the radical sign, *viz.*, b^2-4ac may be zero, positive, or negative.

(i) If $b^2-4ac=0$ (or $b^2=4ac$), the roots are *equal in magnitude and of the same sign* $\left(\text{each being } -\frac{b}{2a}\right)$ and are *real* and *rational*. In this case the equation is said to have a *pair of equal roots*.

(ii) If b^2-4ac be positive (or $b^2>4ac$), the roots are *real* and *unequal*. They are in this case *rational* or *irrational* according as b^2-4ac is or is not a perfect square.

(iii) If b^2-4ac be negative (or $b^2<4ac$), the roots are said to be *imaginary*. For, $\sqrt{(b^2-4ac)}$ is a quantity whose square is negative and is therefore called *imaginary* as opposed to *real*.

The quantity b^2-4ac which thus discriminates the nature of the roots of the equation $ax^2+bx+c=0$ is called its **discriminant**.

Ex. Determine the nature of the roots of the following equations :—

(i) $4x^2-12x+9=0$. (ii) $2x^2+3x-5=0$.

(iii) $3x^2-9x+5=0$. (iv) $5x^2+3x+2=0$.

(i) Here the discriminant $b^2-4ac=(-12)^2-4.4.9=0$ Hence the roots are equal, real and rational.

(ii) Here the discriminant $b^2-4ac=3^2-4.2(-5)=49$ which is positive and a perfect square. Hence the roots are unequal, real and rational.

(iii) Here the discriminant $b^2-4ac=(-9)^2-4.3.5=21$ which is positive but not a perfect square. Hence the roots are unequal, real and irrational.

(iv) Here the discriminant $b^2-4ac=3^2-4.5.2=-31$ which is negative. Hence the roots are imaginary.

Note. The student should distinguish between irrational and imaginary roots. In the former case the values of the roots can be found to any degree of accuracy, though they cannot be *exactly* determined ; while in the latter case no real roots can be found to satisfy the equation, for a negative quantity has no square root, whether exact or approximate.

Thus in the equation $x^2-8x+8=0$,

$$x=\frac{8\pm\sqrt{(64-32)}}{2}=\frac{8\pm4\sqrt{2}}{2}$$

$$=4\pm2\sqrt{2}=4\pm2\times1\cdot4142\ldots\ldots$$

$$=6\cdot8284 \text{ or } 1\cdot1716 \text{ to 4 decimal places.}$$

But in the equation $x^2-2x+2=0$,

$$x=\frac{2\pm\sqrt{(4-8)}}{2}=\frac{2\pm\sqrt{(-4)}}{2},$$

and since we cannot extract the square root of -4, the roots are imaginary.

EXERCISE CXIX.

Solve the following equations :—

1. $x^2-7x+12=0$.
2. $x^2-2x-35=0$.
3. $15x^2-8x-63=0$.
4. $14x^2-41x+15=0$.
5. $15x^2+x-2=0$.
6. $8x^2+22x+9=0$.
7. $77(x^2-1)=72x$.
8. $35x^2+x=204$.
9. $10x^2-21x=187$.
10. $57x^2+5x-22=0$.
11. $2x^2+61x-861=0$.
12. $3x^2-104x-1739=0$.
13. $21x^2-58x-143=0$.
14. $22x^2+123x-67=0$.
15. $ax^2+2bx+c=0$.
16. $3(x-1)(x-4)+2(x+2)(x-3)=40$.
17. $(2x+1)(3x+5)+(4x+3)^2=3(x+1)(5x+7)+19$.
18. $(x-\frac{1}{2})(x-\frac{1}{3})+(x-\frac{1}{3})(x-\frac{1}{4})=(x-\frac{1}{4})(x-\frac{1}{5})$. (C. F. 1861.)
19. Resolve into factors :—

    (1) $5x^2+7x-9$. (2) $-7x^2+2x+1$.

    (3) $2x^2+7x+4$. (4) $4x^2-9x+2$.

20. State the nature of the roots of the following equations :—

    (1) $11x^2-46x+39=0$. (2) $3x^2-18x+27=0$.

    (3) $4x^2-7x+1=0$. (4) $2x^2-5x+4=0$.

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8. Irrational equations. We shall now consider some irrational equations which when rationalised lead to quadratics.

Ex. 1. Solve $x+\sqrt{(5x+10)}=8$

Transposing, $\sqrt{(5x+10)}=8-x$

$\therefore$ squaring, $5x+10=64-16x+x^2$.

$\therefore x^2-21x+54=0. \qquad \therefore x=3$ or 18.

On substitution it is found that only the root 3 satisfies the equation, and therefore $x=18$ is an extraneous solution. The latter, however, satisfies the equation $x-\sqrt{(5x+10)}=8$.

Ex. 2. Solve $7-\sqrt{(x^2-16)}=2x$.

Transposing, $-\sqrt{(x^2-16)}=2x-7$.

$\therefore$ squaring, $x^2-16=4x^2-28x+49$.

$\therefore 3x^2-28x+65=0$, whence

$x=5$ or $\frac{13}{3}$.

On substitution it is found that none of the values of x satisfies the equation or there is no solution to the equation in question. These values, however, satisfy the equation $7+\sqrt{(x^2-16)}=2x$ which when rationalised leads to the same quadratic as in the case of the proposed equation, and it is therefore a matter of chance whether one or both the roots of the rationalised quadratic will satisfy the proposed equation. In such cases therefore we are to see by actual substitution whether the values of x so obtained satisfy the given equation or the other form.

Ex. 3. Solve $\sqrt{(x^2+2x-8)}+\sqrt{(x^2+9x-22)}=\sqrt{(8x^2-7x-18)}$.

Here $\sqrt{(x-2)(x+4)}+\sqrt{(x-2)(x+11)}=\sqrt{(x-2)(8x+9)}$.

$\therefore$ Either $\sqrt{(x-2)}=0$, which gives $x=2$,

or $\sqrt{(x+4)}+\sqrt{(x+11)}=\sqrt{(8x+9)}$

To solve this, square both sides ; then

$$x+4+x+11+2\sqrt{(x+4)(x+11)}=8x+9.$$

$$\therefore 2\sqrt{(x+4)(x+11)}=6x-6.$$

Dividing by 2 and squaring, $(x+4)(x+11)=(3x-3)^2$,

or $x^2+15x+44=9x^2-18x+9$.

This gives $8x^2-33x-35=0$ whence $x=5$ or $-\frac{7}{8}$.

By substitution it will be found that the root $-\frac{7}{8}$ does not satisfy the equation ; hence the solutions are 2 and 5.

Note. The student will observe than in this example we have made use of the following principles :—In the equation A.P = B.P having a common factor P on both sides, either P = O, or, A = B.

For, write the equation in the form $A.P - B.P = 0$, or, $P(A - B) = 0$. Then evidently either $P = 0$ or $A - B = 0$ which gives $A = B$. See p. 244.

Ex. 4. Solve $\sqrt{(1+x)^2 - ax} + \sqrt{(1-x)^2 + ax} = x$.

By transposition, $\sqrt{(1+x)^2 - ax} = x - \sqrt{(1-x)^2 + ax}$.

Square both sides; then

$$(1+x)^2 - ax = x^2 + (1-x)^2 + ax - 2x\sqrt{(1-x)^2 + ax}.$$

or $1 + 2x + x^2 - ax = x^2 + 1 - 2x + x^2 + ax - 2x\sqrt{(1 - 2x + x^2 + ax)}$.

$\therefore\ 2x\sqrt{(1 - 2x + x^2 + ax)} = x^2 + 2ax - 4x$.

Divide by x and square, then

$$4 - 8x + 4x^2 + 4ax = x^2 + 4a^2 + 16 + 4ax - 8x - 16a.$$

$\therefore\ 3x^2 = 4a^2 - 16a + 12 = 4(a-1)(a-3)$.

$$\therefore\ x^2 = \frac{4(a-1)(a-3)}{3}.$$

$$\therefore\ x = \pm 2\sqrt{\frac{(a-1)(a-3)}{3}}.$$

Ex. 5. Solve $(a+x)^{\frac{2}{3}} + (a-x)^{\frac{2}{3}} = 3(a^2 - x^2)^{\frac{1}{3}}$.

Transposing, $(a+x)^{\frac{2}{3}} + (a-x)^{\frac{2}{3}} - 3(a^2 - x^2)^{\frac{1}{3}} = 0$.

Now remembering that if $A + B + C = 0$, then $A^3 + B^3 + C^3 = 3ABC$, we have

$$\begin{aligned}(a+x)^2 + (a-x)^2 - 27(a^2 - x^2) &= -9(a+x)^{\frac{2}{3}}(a-x)^{\frac{2}{3}}(a^2 - x^2)^{\frac{1}{3}}.\\ &= -9(a^2 - x^2)^{\frac{2}{3}}(a^2 - x^2)^{\frac{1}{3}}\\ &= -9(a^2 - x^2).\end{aligned}$$

$\therefore\ (a+x)^2 + (a-x)^2 = 18(a^2 - x^2)$.

$\therefore\ 2(a^2 + x^2) = 18(a^2 - x^2)$, or, $a^2 + x^2 = 9a^2 - 9x^2$.

$\therefore\ 10x^2 = 8a^2$ or $x^2 = \dfrac{4a^2}{5}$; hence $x = \pm\dfrac{2a}{\sqrt{5}}$.

9. In some equations involving irrational denominators, it is advisable to begin by rationalising the denominators.

Ex. Solve $\dfrac{1}{\sqrt{(1-x)} + 1} + \dfrac{1}{\sqrt{(1+x)} - 1} = \dfrac{1}{x}$.

Here $$\frac{\sqrt{(1-x)} - 1}{\{\sqrt{(1-x)} + 1\}\{\sqrt{(1-x)} - 1\}} + \frac{\sqrt{(1+x)} + 1}{\{\sqrt{(1+x)} - 1\}\{\sqrt{(1+x)} + 1\}} = \frac{1}{x}$$

$$\therefore\ \frac{\sqrt{(1-x)} - 1}{-x} + \frac{\sqrt{(1+x)} + 1}{x} = \frac{1}{x}.$$

Multiplying both sides by x, $\quad -\sqrt{(1-x)}+1+\sqrt{(1+x)}+1=1$

$\therefore\ \sqrt{(1+x)}+1=\sqrt{(1-x)}.$

Squaring, $1+x+1+2\sqrt{(1+x)}=1-x.$

$\therefore\ 2\sqrt{(1+x)}=-2x-1.$

Squaring again, $4+4x=4x^2+4x+1.$

$\therefore\ 4x^2=3$, whence $x=\pm\dfrac{\sqrt{3}}{2}.$

10. In some irrational equations special methods may be usefully employed.

Ex. 1. Solve $\dfrac{a+x+\sqrt{(a^2-x^2)}}{a+x-\sqrt{(a^2-x^2)}}=\dfrac{bx}{c^2}.$

By comp. and divid., $\dfrac{2(a+x)}{2\sqrt{(a^2-x^2)}}=\dfrac{bx+c^2}{bx-c^2},$

or $\sqrt{\dfrac{a+x}{a-x}}=\dfrac{bx+c^2}{bx-c^2}.$

Squaring $\dfrac{a+x}{a-x}=\dfrac{b^2x^2+2bc^2x+c^4}{b^2x^2-2bc^2x+c^4}.$

Again by comp. and divid., $\dfrac{a}{x}=\dfrac{b^2x^2+c^4}{2bc^2x}.$

Multiplying cross-wise and removing x from both sides,

$b^2x^2+c^4=2abc^2$, or, $b^2x^2=c^2(2ab-c^2)$

$\therefore\ x=\pm\dfrac{c}{b}\sqrt{(2ab-c^2)}.$

Ex. 2. Solve $\quad \sqrt{(6x^2+7x+6)}-\sqrt{(7x^2+7x+2)}=x+2.........(1)$

We have indentically $(6x^2+7x+6)-(7x^2+7x+2)=-(x^2-4)...(2)$

Hence from (1) and (2) by division,

$\sqrt{(6x^2+7x+6)}+\sqrt{(7x^2+7x+2)}=-(x-2)....(3)$

From (1) and (3) by adding, and dividing by 2,

$\sqrt{(6x^2+7x+6)}=2.$

$\therefore\ 6x^2+7x+6=4$, or, $6x^2+7x+2=0.$

This gives $x=-\frac{1}{2}$, or $-\frac{2}{3}$.

Ex. 3. Solve $\sqrt{(10x^2-7x+4)}+\sqrt{(8x^2-3x+2)}$
$=\sqrt{(4x^2-2x+3)}+\sqrt{(2x^2+2x+1)}.$

We have indentically $(10x^2-7x+4)-(8x^2-3x+2)$
$=(4x^2-2x+3)-(2x^2+2x+1)..................(1)$

Hence from (1) and the given equation by division,

$$\sqrt{(10x^2-7x+4)}-\sqrt{(8x^2-3x+2)}$$
$$=\sqrt{(4x^2-2x+3)}-\sqrt{(2x^2+2x+1)}\ldots\ldots\ldots\ldots(2).$$

From (2) and the given equation by adding and dividing by (2),

$$\sqrt{(10x^2-7x+4)}=\sqrt{(4x^2-2x+3)}.$$
$$\therefore\ 10x^2-7x+4=4x^2-2x+3.$$
or $6x^2-5x+1=0$; when $x=\frac{1}{2}$ or $\frac{1}{3}$.

EXERCISE CXX.

Solve the following equations :—

1. $3x+\sqrt{(2x-5)}=10.$

2. $2+\sqrt{(3x^2-11)}=2x.$

3. $\sqrt{(x+6)}+\sqrt{(9x-2)}=8.$

4. $\sqrt{(5-x)}+\sqrt{(x+8)}=5\sqrt{(2-x)}.$ (C. F. 1873).

5. $\sqrt{(x^2-3x-4)}+5\sqrt{(x^2-11x+28)}=4(x-4).$

6. $\sqrt{(x^2-8x+15)}+\sqrt{(x^2+2x-15)}=\sqrt{(4x^2-18x+18)}.$

7. $\sqrt{(1-x+x^2)}-\sqrt{(1+x+x^2)}=a.$

8. $\sqrt[3]{(x+a)}-\sqrt[3]{(x-a)}=b.$

9. $(1+x)^{\frac{1}{3}}+(1-x)^{\frac{1}{3}}=2^{\frac{1}{3}}.$ (C. F. 1885).

10. $\left(\dfrac{2x+3}{2x-3}\right)^{\frac{1}{3}}+\left(\dfrac{2x-3}{2x+3}\right)^{\frac{1}{3}}=\dfrac{8}{13}\cdot\dfrac{4x^2+9}{4x^2-9}.$

11. $\dfrac{\sqrt{(b+x)}+\sqrt{(b-x)}}{\sqrt{(b+x)}-\sqrt{(b-x)}}=ax.$

12. $\sqrt{\dfrac{(1+ax)}{(1-ax)}}=\dfrac{1+bx}{1-bx}$

13. $\dfrac{x+\sqrt{(x^2-1)}}{x-\sqrt{(x^2-1)}}+\dfrac{x-\sqrt{(x^2-1)}}{x+\sqrt{(x^2-1)}}=98.$

14. $\sqrt{(5x^2+7x+3)}+\sqrt{(5x^2+7x+6)}=3.$

15. $\sqrt{(x^2+4ax+4a^2)}-\sqrt{(x^2+4ax-4a^2)}=2a.$

16. $\sqrt{(9x^2+14x-15)}-\sqrt{(8x^2+14x+1)}=x-4.$

17. $\sqrt{(2x-1)}+\sqrt{(3x-2)}=\sqrt{(4x-3)}+\sqrt{(5x-4)}.$

18. $\sqrt{(12x^2-8x-5)}-\sqrt{(7x^2-5x+4)}$
$$=\sqrt{(8x^2+3x-2)}-\sqrt{(3x^2+6x+7)}$$

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11. We append a few illustrative examples on quadratics

Ex. 1. Solve $\dfrac{x+2}{x-2}+\dfrac{2x-3}{2(x-1)}=\dfrac{23}{6}.$ (A. I. 1890).

Multiplying by $6(x-2)(x-1)$, the L. C. M. of the denominators

$$6(x-1)(x+2)+3(x-2)(2x-3)=23(x-2)(x-1).$$

$$\therefore\ 6x^2+6x-12+6x^2-21x+18=23x^2-69x+46,$$
$$\text{or } 11x^2-54x+40=0.$$
$$\therefore\ x=\frac{54\pm\sqrt{(54^2-4.11.40)}}{22}=\frac{54\pm34}{22}$$
$$=4 \text{ or } \frac{10}{11}.$$

Ex. 2. Solve $\frac{1}{2}\cdot\frac{x+5}{x-1}-3\cdot\frac{x+6}{x+2}+\frac{5x-14}{x-3}=\frac{5}{2}$.

Here $\frac{1}{2}\left(1+\frac{6}{x-1}\right)-3\left(1+\frac{4}{x+2}\right)+\left(5+\frac{1}{x-3}\right)=\frac{5}{2}$.

$$\therefore\ (\tfrac{1}{2}-3+5)+\frac{3}{x-1}-\frac{12}{x+2}+\frac{1}{x-3}=\frac{5}{2}.$$
$$\therefore\ \frac{3}{x-1}-\frac{12}{x+2}+\frac{1}{x-3}=0.$$

$\therefore\ 3(x+2)(x-3)-12(x-1)(x-3)+(x-1)(x+2)=0.$

or simplifying, $4x^2-23x+28=0$.

$$\therefore\ x=\frac{23\pm\sqrt{(23^2-4\cdot4\cdot28)}}{8}=\frac{23\pm9}{8}$$
$$=4 \text{ or } \tfrac{7}{4}.$$

Ex. 3. Solve $\frac{x+29}{x+1}+\frac{14x-19}{7(x-2)}=\frac{14x+39}{7(2x+3)}+\frac{2(2x+29)}{2x+1}$.

Here $\left(1+\frac{28}{x+1}\right)+\left(2+\frac{9}{7(x-2)}\right)=\left(1+\frac{18}{7(2x+3)}\right)+\left(2+\frac{56}{2x+1}\right)$.

$$\therefore\ \frac{28}{x+1}+\frac{9}{7(x-2)}=\frac{18}{7(2x+3)}+\frac{56}{2x+1},$$
$$\text{or } 28\left(\frac{1}{x+1}-\frac{2}{2x+1}\right)=\frac{9}{7}\cdot\left(\frac{2}{2x+3}-\frac{1}{x-2}\right).$$
$$\therefore\ \frac{-28}{(x+1)(2x+1)}=-\frac{9}{7}\cdot\frac{7}{(2x+3)(x-2)}.$$
$$\therefore\ 28(2x+3)(x-2)=9(x+1)(2x+1).$$
$$\text{or } 38x^2-55x-177=0.$$

Solving this $x=3$ or $-\frac{59}{38}$.

Ex. 4. Solve $\frac{x^2+2x+2}{x+1}+\frac{x^2+8x+20}{x+4}=\frac{x^2+4x+6}{x+2}+\frac{x^2+6x+12}{x+3}$.

Here $\left(x+1+\frac{1}{x+1}\right)+\left(x+4+\frac{4}{x+4}\right)$

$$=\left(x+2+\frac{2}{x+2}\right)+\left(x+3+\frac{3}{x+3}\right).$$

$$\therefore \frac{1}{x+1}+\frac{4}{x+4}=\frac{2}{x+2}+\frac{3}{x+3}.$$

By transposition $\dfrac{4}{x+4}-\dfrac{3}{x+3}=\dfrac{2}{x+2}-\dfrac{1}{x+1}.$

$$\therefore \frac{x}{(x+4)(x+3)}=\frac{x}{(x+2)(x+1)}.$$

Hence either $x=0$ or $(x+4)(x+3)=(x+2)(x+1)$.

This latter gives $x^2+7x+12=x^2+3x+2$,

or $4x=-10$; $\therefore x=-2\frac{1}{2}$.

Thus the roots are $0, -2\frac{1}{2}$.

Ex. 5. Find the ratio of x to y from the equation

$$35x^2+3xy-2y^2=0.$$

Dividing by y^2, $35\left(\dfrac{x}{y}\right)^2+3\left(\dfrac{x}{y}\right)-2=0.$

$$\therefore \frac{x}{y}=\frac{-3\pm\sqrt{(9+4.2.35)}}{70}=\frac{-3\pm 17}{70}=\tfrac{1}{5} \text{ or } -\tfrac{2}{7}.$$

EXERCISE CXXI.

Solve the following equations :—

1. $bdx^2-x(bc+ad)+ac=0.$ 2. $(a^2-b^2)x^2+2ab^2x-a^2b^2=0.$

3. $(x+a-2b)^3-(x+2a-b)^3=(a+b)^3$

4. $(x+a)(x+b)(x+c)=abc.$ (C. F. 1895).

5. $\dfrac{x+2}{x}+\dfrac{4}{3}=\dfrac{5x+3}{8}-\dfrac{1}{24}.$ 6. $\dfrac{4x-3}{3x+5}+\dfrac{5x-7}{2x+1}=1\frac{11}{14}.$

7. $\dfrac{x+6}{x+7}-\dfrac{x+1}{x+2}=\dfrac{1}{3x+1}.$ 8. $\dfrac{3}{5-x}+\dfrac{2}{4-x}=\dfrac{2}{x+2}.$

9. $\dfrac{5-4x}{3\cdot5x+\cdot125}+\dfrac{3x+2}{\cdot75x+1\cdot375}=5\cdot76.$

10. $\frac{1}{3}\cdot\dfrac{6x-11}{3x-4}+\frac{1}{4}\cdot\dfrac{2x+13}{2x+1}-\frac{1}{12}\cdot\dfrac{20x-9}{4x-3}=\frac{1}{2}.$

11. $\dfrac{x+2}{1+2x}+\dfrac{3x+4}{3+4x}=\dfrac{x-2}{1-2x}+\dfrac{3x-4}{3-4x}.$ (C. F. 1885).

12. $\dfrac{2x^2+5x+3}{x+2}-\dfrac{2x^2+7x+4}{x+3}=\dfrac{4x^2+18x+10}{x+4}-\dfrac{4x^2+22x+12}{x+5}.$

13. $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}+\dfrac{x+c}{x-c}=3.$

Solve the following equations :—

14. $\dfrac{7x+5}{9-2x}+\dfrac{11-9x}{21}=\dfrac{2(6x+\frac{1}{3})}{7}$.

15. $\dfrac{(x-a)^2}{(x-b)(x-c)}+\dfrac{(x-b)^2}{(x-c)(x-a)}+\dfrac{(x-c)^2}{(x-a)(x-b)}=3$.

16. $(1+x+x^2)^2=\dfrac{a+1}{a-1}(1+x^2+x^4)$ (C. F. 1903).

17. Find $x : y$ from

(1) $7x^2-13xy-2y^2=0$. (2) $3x^2-53xy+34y^2=0$.

18 Find $x : y$ from

(1) $\dfrac{x^2+xy-6y^2}{x^2+3xy-10y^2}=\frac{3}{4}$. (2) $\dfrac{2x^2+3xy+y^2}{6x^2+xy-y^2}=\frac{7}{5}$.

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12. **The following are some equations reducible to quadratics.**

Ex. 1. Solve $36x^4-25x^2+4=0$.

Assume $x^2=z$, then $36z^2-25z+4=0$,

or $(9z-4)(4z-1)=0$: whence $z=\frac{4}{9}$ or $\frac{1}{4}$.

$\therefore x^2=\frac{4}{9}$ or $\frac{1}{4}$, whence $x=\pm\frac{2}{3}$, $\pm\frac{1}{2}$.

Ex. 2. Solve $15x^{\frac{1}{3}}+4x^{-\frac{1}{3}}=17$

Multiply both sides by $x^{\frac{1}{3}}$ and transpose,

then $15x^{\frac{2}{3}}-17x^{\frac{1}{3}}+4=0$,

Assume $x^{\frac{1}{3}}=z$, then $15z^2-17z+4=0$,

or $(5z-4)(3z-1)=0$; whence $z=\frac{4}{5}$ or $\frac{1}{3}$.

$\therefore x^{\frac{1}{3}}=\frac{4}{5}$ or $\frac{1}{3}$; $\therefore x=\frac{64}{125}$ or $\frac{1}{27}$.

Ex. 3. Solve $2(x^2-3x+1)^2+5(x^2-3x+1)+3=0$. (C. F. 1873).

Let $x^2-3x+1=z$,

then $2z^2+5z+3=0$, or $(2z+3)(z+1)=0$,

$\therefore z=-\frac{3}{2}$ or -1.

$\therefore x^2-3x+1=-\frac{3}{2}$, or, $x^2-3x+1=-1$.

Solving these quadratics,

$$x=\frac{3\pm\sqrt{-1}}{2}\ ;\ 1,\ 2.$$

Ex. 4. Solve $x^2-5x-7\sqrt{(x^2-5x+6)}+18=0$.

Here $(x^2-5x+6)-7\sqrt{(x^2-5x+6)}+12=0$.

Putting $\sqrt{(x^2-5x+6)}=z$, the equation becomes $z^2-7z+12=0$, whence $z=3$ or 4.

Thus $\sqrt{(x^2-5x+6)}=3$......(i), or, $\sqrt{(x^2-5x+6)}=4$.........(ii)
Taking (i) and squaring, $x^2-5x+6=9$.

$$\therefore \quad x=\frac{5\pm\sqrt{37}}{2}.$$

Taking (ii) and squaring, $x^2-5x+6=16$.

$$\therefore \quad x=\frac{5\pm\sqrt{65}}{2}.$$

Ex. 5. Solve $(a+x)^{\frac{2}{n}}+12(a-x)^{\frac{2}{n}}=7(a^2-x^2)^{\frac{1}{n}}$

Dividing both sides by $(a^2-x^2)^{\frac{1}{n}}$,

$$\left(\frac{a+x}{a-x}\right)^{\frac{1}{n}}+12\left(\frac{a-x}{a+x}\right)^{\frac{1}{n}}=7.$$

$\therefore$ Putting $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{n}}=z$, we have

$$z+\frac{12}{z}=7 \text{ or } z^2-7z+12=0.$$

$$\therefore\ z=3 \text{ or } 4\ ;\ \textit{i.e.,}\ \frac{a+x}{a-x}=3^n \text{ or } 4^n.$$

Hence by comp. and divid.,

$$x=\frac{3^n-1}{3^n+1}\cdot a, \text{ or, } \frac{4^n-1}{4^n+1}\cdot a.$$

Ex. 6. Solve $\dfrac{1}{x^2+11x-8}+\dfrac{1}{x^2+2x-8}+\dfrac{1}{x^2-13x-8}=0.$

Putting $x^2-8=z$, we have $\dfrac{1}{z+11x}+\dfrac{1}{z+2x}+\dfrac{1}{z-13x}=0.$

$\therefore\ (z+2x)(z-13x)+(z-13x)(z+11x)+(z+11x)(z+2x)=0$
whence multiplying and simplifying, $z^2=49x^2$.

$\therefore\ z=\pm 7x.$

Hence $x^2-8=+7x$ and $x^2-8=-7x$.

Solving these $x=\pm 1,\ \pm 8$.

Ex. 7. Solve $(x+1)(x+2)(x+3)(x+4)=24$.

Here $(x^2+5x+4)(x^2+5x+6)=24$,

or $(z+4)(z+6)=24$, where $z=x^2+5x$.

$\therefore\ z^2+10z=0$, whence $z=0$ or -10.

$\therefore\ x^2+5x=0$, or $x^2+5x=-10$.

Solving these quadratics,

$$x=0,-5\ ;\ \frac{-5\pm\sqrt{(-15)}}{2}.$$

Ex. 8. Solve $(x+a)^4+(x+b)^4=2c^4$

Express $x+a$, $x+b$, as the sum and difference of two others, thus :

$$\left.\begin{aligned}\text{let } x+a&=y+h\\ x+b&=y-h\end{aligned}\right\}\text{ so that } y=x+\frac{a+b}{2},\ h=\frac{a-b}{2}.$$

Then the equation becomes $(y+h)^4+(y-h)^4=2c^4$,

or $2(y^4+6h^2y^2+h^4)=2c^4$,

$$\therefore\ y^4+6h^2y^2+(h^4-c^4)=0$$

$$\therefore\ y^2=\frac{-6h^2\pm\sqrt{36h^4-4(h^4-c^4)}}{2}$$

$$=-3h^2\pm\sqrt{(8h^4+c^4)}$$

$$\therefore\ x+\frac{a+b}{2}=\pm\sqrt{-3h^2\pm\sqrt{(8h^4+c^4)}}.$$

$$\therefore\ x=-\frac{a+b}{2}\pm\sqrt{-3h^2\pm\sqrt{(8h^4+c^4)}}.$$

The same method may be applied to the equation

$$(x+a)^5-(x+b)^5=2c^5.$$

Ex. 9. Solve $4x^4-16x^3+23x^2-16x+4=0$. (C. F. 1882).

Here $4(x^4+1)-16x(x^2+1)+23x^2=0$.

Dividing by x^2, $4\left(x^2+\dfrac{1}{x^2}\right)-16\left(x+\dfrac{1}{x}\right)+23=0$.

Put $x+\dfrac{1}{x}=z$, so that $x^2+\dfrac{1}{x^2}=z^2-2$.

Then $4(z^2-2)-16z+23=0$ or $4z^2-16z+15=0$

$$\therefore\ z=\tfrac{5}{2}\text{ or }\tfrac{3}{2},\ i.e.,\ x+\frac{1}{x}=\tfrac{5}{2}\text{ or }x+\frac{1}{x}=\tfrac{3}{2}.$$

Taking $x+\dfrac{1}{x}=\tfrac{5}{2}$, we have $2x^2-5x+2=0$.

$$\therefore\ x=2\text{ or }\tfrac{1}{2}.$$

Taking $x+\dfrac{1}{x}=\tfrac{3}{2}$, we have $2x^2-3x+2=0$.

$$\therefore\ x=\frac{3\pm\sqrt{-7}}{4}.$$

Note. An equation is called **reciprocal** when the co-efficients of terms equidistant from the beginning and the end are equal. The above is a reciprocal equation of the fourth degree and its solution deserves attention.

Ex. 10. Solve $3^x+3^{2-x}=10$.

Multiplying by 3^x, $3^{2x}+3^2=10.3^x$.

$\therefore\ 3^{2x}-10.3^x+9=0,\ (3^x-1)(3^x-9)=0.$

$\therefore\ 3^x=1$ or 9, *i.e.*, 3^0 or 3^2.

$\therefore\ x=0$ or 2.

Ex. 11. Solve $(x+a)^3+(x+b)^3=8x^3+(a+b)^3$.

Here $\{(x+a)+(x+b)\}\{(x+a)^2-(x+a)(x+b)+(x+b)^2\}$

$=\{2x+(a+b)\}\{4x^2-2x(a+b)+(a+b)^2\}$

or, $(2x+a+b)\{x^2+x(a+b)+a^2-ab+b^2\}$

$=(2x+a+b)\{4x^2-2x(a+b)+(a+b)^2\}$.

$\therefore$ either $2x+a+b=0$, which gives $x=-\frac{1}{2}(a+b)$,

or $x^2+x(a+b)+a^2-ab+b^2=4x^2-2x(a+b)+(a+b)^2$,

i.e., $x^2-x(a+b)+ab=0$, which gives $x=a$ or b.

Ex. 12. Solve $\dfrac{ax+b}{cx+b}+\dfrac{bx+a}{cx+a}=\dfrac{(a+b)(x+2)}{cx+a+b}$ (A. I. 1893).

Hence, $\left(\dfrac{ax+b}{cx+b}-1\right)+\left(\dfrac{bx+a}{cx+a}-1\right)=\dfrac{(a+b)(x+2)}{cx+a+b}-2.$

$\therefore\ \dfrac{(a-c)x}{cx+b}+\dfrac{(b-c)x}{cx+a}=\dfrac{(a+b-2c)x}{cx+a+b}.$

This gives $x=0$,

$$\text{or } \frac{a-c}{cx+b}+\frac{b-c}{cx+a}=\frac{a+b-2c}{cx+a+b}$$

$$=\frac{a-c}{cx+a+b}+\frac{b-c}{cx+a+b}.$$

Transposing,

$$(a-c)\left\{\frac{1}{cx+b}-\frac{1}{cx+a+b}\right\}=(b-c)\left\{\frac{1}{cx+a+b}-\frac{1}{cx+a}\right\}.$$

$$\therefore\ \frac{a(a-c)}{(cx+b)(cx+a+b)}=\frac{-b(b-c)}{(cx+a+b)(cx+a)}.$$

$\therefore\ a(a-c)(cx+a)=-b(b-c)(cx+b)$.

$$\text{whence } x=-\frac{a^2(a-c)+b^2(b-c)}{c\{a(a-c)+b(b-c)\}}.$$

EXERCISE CXXII

Solve the equations :—

1. $4x^4-5x^2+1=0$.
2. $9\sqrt{x}-5x=2\frac{1}{4}$.
3. $6x^{-2}+11x^{-1}=10$.
4. $21x^{\frac{1}{2}}+5x^{-\frac{1}{2}}+22=0$.

Solve the equations :—

5. $21x^{-\frac{1}{4}}+6x^{\frac{11}{20}}=25x^{\frac{3}{20}}$. 6. $x^n+abx^{-n}=a+b$.

7. $(x+a)^4+(x-a)^4=2b^4$.

8. $9x^2-6x+1-8\sqrt{(3x^2-2x-1)}=0$.

9. $x(x+1)+3\sqrt{(2x^2+6x+5)}=2(12-x)+1$. (A. I. 1889).

10. $(x+4)(x+1)-\sqrt{(x+5)(x-3)}=3x+31$. (C. F. 1877).

11. $4\sqrt{\left(\dfrac{x^2+x-3}{x^2+2x-2}\right)}+3\sqrt{\left(\dfrac{x^2+2x-2}{x^2+x-3}\right)}=13$.

12. $\dfrac{(x-2)^2}{x^2-4x}+\dfrac{2}{(x-2)^2}=4$. (C. F. 1889).

13. $(x+m)^{\frac{2}{5}}+(x-m)^{\frac{2}{5}}=\left(n+\dfrac{1}{n}\right)(x^2-m^2)^{\frac{1}{5}}$. (C. F. 1884).

14. $\dfrac{2}{x^2-x-14}+\dfrac{4}{x^2-2x-14}=\dfrac{11}{x^2+x-14}$.

15. $(x+a)(x+3a)(x+5a)(x+7a)=384a^4$. (C. F. 1905).

16. $(2x-5)(3x+1)(2x-1)(3x+7)=800$.

17. $(2x-1)^2(4x-1)(4x-3)=3$.

18. $\dfrac{1}{3x^2+11x+10}+\dfrac{1}{6x^2+19x+15}=\dfrac{2x^2+9x+10}{150}$. (M. F. 1889).

19. $(x+1)^4+(x-3)^4=256$. 20. $(x+2)^4+(x+3)^4=17$

21. $10x^3-19x^2-19x+10=0$.

22. $72x^4-306x^3+469x^2-306x+72=0$.

23. (1) $x^3+1=0$. (2) $x^4+1=0$. (3) $x^6-1=0$.

24. $20x^4+12x^3-103x^2-12x+20=0$.

25. $(x-3)(x-5)(x-7)=6.\ 4.\ 2$.

26. $(x-1)(x-2)(x-3)=(a-1)(a-2)(a-3)$.

27. $(5x+8)^3-(3x+5)^3=8x^3+27$.

28. $2^{2x}-3\cdot 2^{x+2}=-32$. 29. $3^x+3^{3-x}=12$.

30. $x+1=x^3(x+2)$. 31. $\dfrac{1-\sqrt{x^2-1}}{1+\sqrt{x^2-1}}=\dfrac{x-\sqrt{x^2+8}}{x+\sqrt{x^2+8}}$.

32. $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}$. (B. P. 1861).

33. $\dfrac{a+c(a+x)}{a+c(a-x)}+\dfrac{a+x}{x}=\dfrac{a}{a-2cx}$.

34. $\dfrac{b+c}{bc-x}+\dfrac{c+a}{ca-x}+\dfrac{a+b}{ab-x}=\dfrac{a+b+c}{x}$.

Solve the equations :—

35. $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{a-c}{x+a-c}+\dfrac{b+c}{x+b+c}$. (P. I. 1892).

36. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}=\dfrac{a}{b}+\dfrac{b}{a}$. (P. I. 1892).

37. $\dfrac{x^2}{3}+\dfrac{48}{x^2}=10\left(\dfrac{x}{3}-\dfrac{4}{x}\right)$.

38. $(x-bc)(x-ca)(x-ab)=abc(b+c)(c+a)(a+b)$.

39. $(a-x)^3+(b-x)^3=(a+b-2x)^3$.

40. $\dfrac{ax+b}{a+bx}+\dfrac{cx+d}{c+dx}=\dfrac{ax-b}{a-bx}+\dfrac{cx-d}{c-dx}$.

Problems leading to quadratics.

13. We shall now consider some problems leading to quadratic equations. The method will be clear from the following examples.

Ex. 1. Divide 30 into 2 parts so that their product may be 125.

Let x=one part, then $30-x$=the other part.

By the question, $x(30-x)=125$.

$\therefore\ x^2-30x+125=0$. $\therefore\ x=5$ or 25.

Thus one part is 5 or 25; the other part is 25 or 5.

Hence the parts are 5 and 25.

Ex. 2. A person buys a number of sheep for Rs. 150. If the prices were 8 annas less per head, he could have bought 10 more. Find the number of sheep bought.

Let x = number of sheep.

Thus $\dfrac{150}{x}$ = price in Rs. of each sheep in one case,

$\dfrac{150}{x+10}$ = price in Rs. of each sheep in the other case.

By the question $\dfrac{150}{x}=\dfrac{150}{x+10}+\dfrac{1}{2}$.

This gives $x^2+10x-3000=0$. $\therefore\ x=50$ or -60.

Neglecting the negative solution, the number of sheep = 50.

Ex. 3. A number consists of 3 digits of which the sum is 15 and the sum of their squares 83; and if 198 is subtracted from the number, the digits are inverted. Find the number.

Let x, y, z be the digits in the unit's, tens' and hundreds' places respectively.

Then the number $=x+10y+100z$.

By the question,

$$x+y+z=15 \quad \ldots\ldots(1)$$
$$x^2+y^2+z^2=83 \quad \ldots\ldots(2)$$
$$x+10y+100z-198=z+10y+100x \quad \ldots\ldots(3)$$

From (3) $99(z-x)=198$ or $z-x=2$.

Also from (1) $z+x=15-y$.

$$\therefore z=\frac{17-y}{2},\ x=\frac{13-y}{2},$$

Substituting in (2)

$$\left(\frac{13-y}{2}\right)^2+y^2+\left(\frac{17-y}{2}\right)^2=83.$$

This gives $y^2-10y+21=0$, whence $y=3$ or 7.

$$\therefore z=7 \text{ or } 5,\ x=5 \text{ or } 3.$$

Thus the number is 735 or 573.

EXERCISE CXXIII.

1. The product of 4 consecutive numbers is 11880. What are the numbers?

2. A takes 5 days more to do a piece of work than B, and if they work together they can finish it in $8\frac{4}{7}$ days. In how many days can A alone do it?

3. A man travels 84 miles and finds that he could have made the journey in 5 hours less, if he had travelled 5 miles an hour faster. At what rate did he travel? (A. I. 1891).

4. A boat's crew can row 18 miles down a river and back again in 8 hours; if the speed of the current is 3 miles per hour, at what rate can the crew row in *still* water?

5. The men in a regiment can be arranged into a hollow square 3 deep; but if 39 are left out, they can be arranged into a solid square having now on each side 10 men less than before. How many men are there in the regiment?

6. A person bought a number of eggs for Rs. 3. 12*as.*; if he had bought 20 more for the same sum, the price of each would have been 3 pies less. How many eggs did he buy?

7. Divide 117 into 3 parts in continued proportion so that the sum of the first and the last may be 90.

8. The area of a rectangular field is $7\frac{1}{2}$ acres, and the sum of the lengths of two adjacent sides exceeds the length of either diagonal by 110 yds. Find the lengths of the sides. (M. F. 1889).

9. Find two numbers such that their sum multiplied by the greater is 140 and their difference multiplied by the less is 24.

10. Find three numbers such that their sum, the sum of their squares, and the sum of their cubes are respectively equal to 8, 26, 92.

11. A person bought 12 seers of sugar and 10 seers of tea for Rs. 8, and it is known that for each rupee he received one seer more sugar than tea. Find the price of each.

12. The sum of the three sides of a right-angled triangle is 12 and their product 60. What are the lengths of the sides ?

CHAPTER XXIX.

QUADRATIC GRAPHS.

1. The student is already familiar with the plotting of points in a system of rectangular co-ordinates, and the drawing of the graphs of linear functions. We shall here consider the graphs of quadratic functions.

2. **Symmetry in graphs.** In an equation like $x^3+2xy^2+5y^4=0$ in which the powers of y are even, if we change y into $-y$ the equation remains unaltered. Hence if (x, y) be a point P on the graph (fig. 34), then $(x, -y)$ is also a point P' on the graph on the other side of OX and equally distant from it. Hence the graph is symmetrical about the axis of x. Similarly in an equation in which the powers of x are even, the graph is symmetrical about the axis of y.

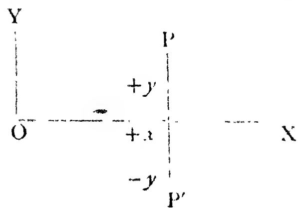


Fig. 34.

Again, if an equation contains no odd power of y and no odd power of x, the graph is symmetrical about both the axes; if (x, y) be a point on the graph then $(x, -y)$, $(-x, y)$ and $(-x, -y)$ are points on the graph. Thus $ax^2+by^2=c$ is symmetrical about both the axes.

Before plotting points on the graph it is useful to note such cases of symmetry. These considerations of symmetry reduce the labour of plotting to a considerable extent, for symmetry about an axis being known we need only plot points on one side of the axis, the points on the other side of the axis being exact reflections of those on the first side.

Graphs of parabola.

3. *Graph of* $y=x^2$.

Take the units of abscissa and ordinate to be each 1″. Put $x=0, \cdot 1, \cdot 2, \cdot 4$............find the corresponding values of y and tabulate as follows :—

x	0	$\pm\cdot 1$	$\pm\cdot 2$	$\pm\cdot 4$	$\pm\cdot 6$	$\pm\cdot 8$	± 1	$\pm 1\cdot 2$	$\pm 1\cdot 4$	$\pm 1\cdot 6$	...	$\pm\infty$
y	0	$\cdot 01$	$\cdot 04$	$\cdot 16$	$\cdot 36$	$\cdot 64$	1	$1\cdot 44$	$1\cdot 96$	$2\cdot 56$	...	∞

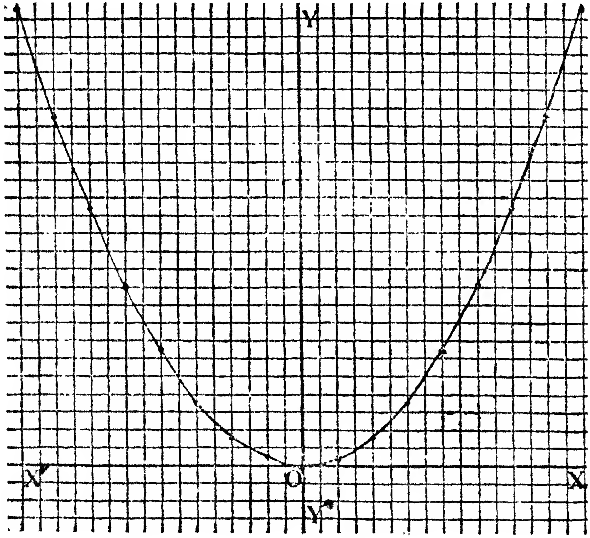


Fig. 35.

Now plot the points (0, 0), ($\cdot 1$, $\cdot 01$), ($\cdot 2$, $\cdot 04$) etc., also the points ($-\cdot 1$, $\cdot 01$), ($-\cdot 2$, $\cdot 04$), etc., and draw a continuous curve freehand through them. This is the graph (fig. 35) of $y=x^2$.

The curve is called **parabola.**

Note. 1. The graph of $y=x^2$ is symmetrical about the axis of y, for there is no odd power of x in the equation. Since x can have any value from $-\infty$ to $+\infty$, the graph consists of an *infinite* branch symmetrically placed with respect of the axis of y.

Again, y cannot be negative for any real value of x; hence the graph lies wholly above the axis of x which is a tangent at the origin.

Note. 2. The graph of $y = -x^2$ will be similarly found: it being the same as that of $y = +x^2$ but turned on the other side of the axis of x, as shown in fig. 36.

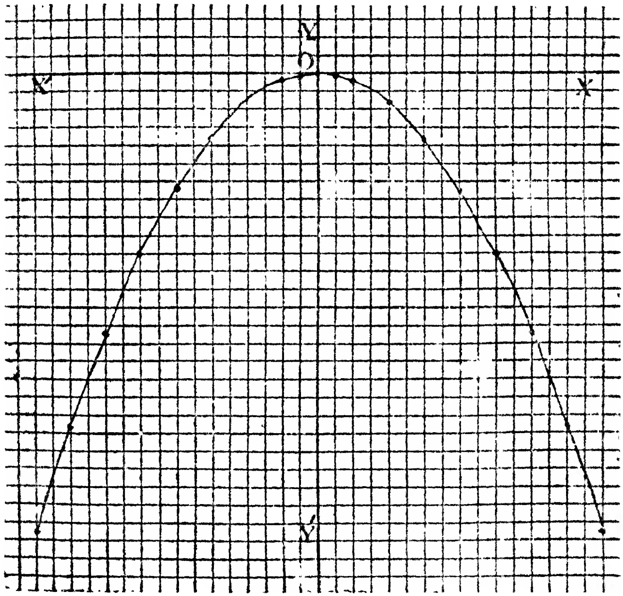


Fig. 36.

Note. 3. The graph of $y = ax^2$ for a positive value of a will be of the same general appearance as that of $y = x^2$, and the graph of $y = ax^2$; for a negative value of a will be of the same general appearance as that of $y = -x^2$.

4. *Graph of* $y = ax^2 + bx + c$. We shall consider two cases according as a is positive or negative.

(i) To draw the graph of $y = 3x^2 - 12x + 8$.

Tabulate the corresponding values of x and y as follows :—

x	$-\infty$	...	$-\cdot 8$	$-\cdot 6$	0	$\cdot 4$	$\cdot 8$	1	1·2	1·4	1·6	1·8
y	∞	...	19·5	16·3	8	3·7	·3	-1	$-2\cdot 1$	$-2\cdot 9$	$-3\cdot 5$	$-3\cdot 9$

x	2	2·2	2·4	2·6	2·8	3	3·2	3·6	4	4·6	...	∞
y	-4	$-3\cdot 9$	$-3\cdot 5$	$-2\cdot 9$	$-2\cdot 1$	-1	·3	3·7	8	16·3	...	∞

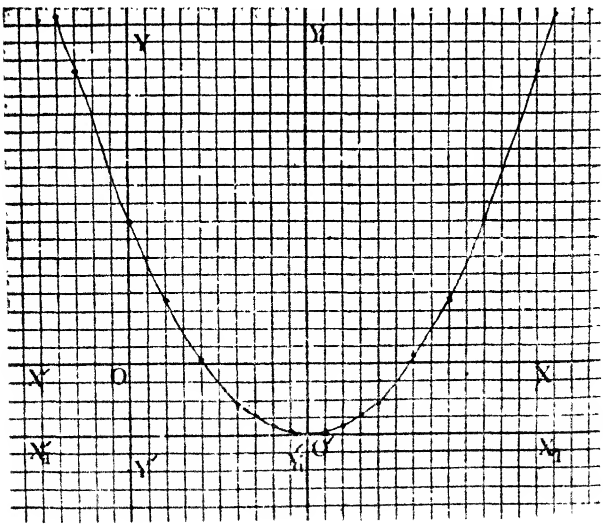


Fig. 37.

Now take the x-unit to be half an inch and the y-unit to be one tenth of an inch *i.e.*, let a side of a small square of the paper

represent ·2 in measuring x and 1 in measuring y. Then plot the points $(-·8, 19·5)$, $(-·6, 16·3)$, $(0,8)$ etc and drawing a curve freehand through them we get the required graph (fig. 37) beginning and ending at an infinite distance *above* the axis of x. At the point O' $(2, -4)$ the graph turns and the point is called a *turning point* on the graph. At a turning point the ordinate is either a maximum or a minimum. In fig. 37 the ordinate is evidently a minimum at the turning point, for it increases on either side of it.

(ii) To draw the graph of $y = -4x^2 + 8x + 7$.

Tabulate the corresponding values of x and y as follows :—

x	$-\infty$	...	-1	$-·8$	$-·6$	$-·4$	0	·2	·4	·6	·8	1
y	$-\infty$	...	-5	$-1·5$	·8	3·2	7	8·4	9·6	10·4	10·8	11

x	1·2	1·4	1·6	1·8	2	2·4	2·6	2·8	3	...	∞
y	10·8	10·4	9·6	8·4	7	3·2	·8	$-1·5$	-5	...	$-\infty$

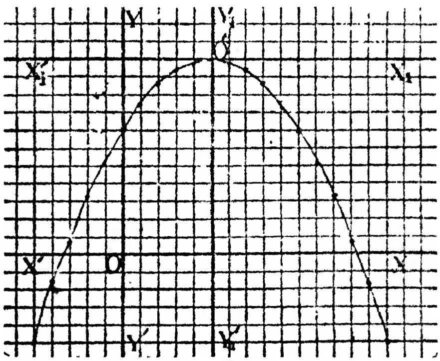


Fig. 38.

Take units in measuring x and y as in the case of (i) above and plot the points and draw the graph as in fig. 38, beginning and

ending at an infinite distance *below* the axis of x. The point O' (1, 11) is a turning point on the graph at which the ordinate is a maximum.

5. Change of origin. Sometimes we can simplify an equation by changing the origin to another point, the axes being parallel to their original directions. Let OX and OY be the original axes and let the point O' be (h,k) with reference to these axes, so that $ON=h$, $O'N=k$. Draw $O'X'$, $O'Y'$ parallel to OX, OY respectively, and take these as new axes of co-ordinates. From any point P draw $PM'M$ parallel to OY to meet $O'X'$ and OX in M' and M respectively. Then the co-ordinates of P in the two systems of axes are connected thus:

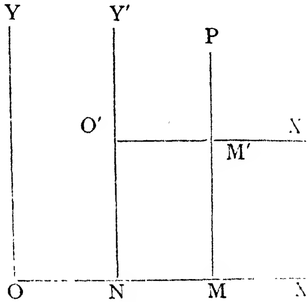


Fig. 39.

we have $OM=ON+NM=ON+O'M'$;

or old $x=h+$new x..........................(1)

Also $PM=M'M+PM'=O'N+PM'$,

or old $y=k+$new y..........................(2)

If any relation between old x and old y are known, we can from (1) and (2) obtain the corresponding relation between new x and new y, *i. e.*, from any equation having O as origin we can obtain the equation having O' as origin.

6. Sometimes the graph of an equation may be conveniently obtained by a change of origin. Consider the equation $y=3x^2-12x+8$ which may be written in the form $y=3(x-2)^2-4$ or $y+4=3(x-2)^2$. Now transfer the origin to the point $(2, -4)$ (point O in fig. 37) so that for the old co-ordinates x and y write $X+2$, $Y-4$ respectively where X and Y respectively denote the new abscissa and new ordinate. Then the equation $y+4=3(x-2)^2$ becomes $Y=3X^2$, which is therefore the equation of the curve referred to the new axes. Thus the graph of $y=3x^2-12x+8$ with O as origin is the same as that of $Y=3X^2$ with O' as origin, and it is easier to draw the graph from the second equation with respect to the second origin.

Similarly the equation $y=-4x^2+8x+7$ may be written $y=-4(x-1)^2+11$, or, $y-11=-4(x-1)^2$.

Hence transferring the origin to the point (1, 11) (O' in fig. 38) and calling the new abscissa and ordinate X, Y respectively, we

have $Y=-4X^2$. Thus the graph of $y=-4x^2+8x+7$ with O as origin is the same as that of $Y=-4X^2$ with O' as origin. The student should draw the graph from this new equation with reference to the new origin (the *turning point.*)

Generally the graph of $y=ax^2+bx+c$ coincides with that of $y=ax^2$, only that they occupy different positions with respect to the co-ordinate axes.

The graph of $y=ax^2+bx+c$ is therefore always a *parabola*, the value of a determining the shape of the curve. When a is positive the curve opens out *upwards* (fig. 37) and when a is negative the curve opens out *downwards* (fig. 38). The equation by a proper change of origin, can be brought to the form $y=ax^2$ (the *new origin being the turning point*), as illustrated above.

Hence an equation of the second degree which contains the second power of either x or y (and not of both) and no xy represents a parabola.

7. Suppose the annexed fig. represents the graph of $y^2=ax$, which is a parabola. The origin O is called the *Vertex* of the parabola. The axis of x is called the *axis* of the parabola.

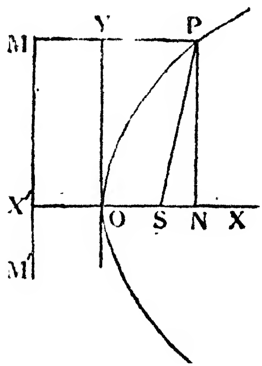


Fig. 40.

If we set off OS along $OX=\frac{a}{4}$ the point S is called the *focus* of the parabola and if we set off $OX'=OS$ along XO produced and draw $MX'M'$ parallel to the axis of y, then $MX'M'$ is called the *directrix* of the parabola.

Take any point P on the parabola and draw PN perpendicular to the axis. Then from the equation $y^2=ax$ we get $PN^2=4OS.ON$, or the *square of the ordinate (the perpendicular from any point on the parabola, upon the axis) is proportional to the abscissa (the portion of the axis between the vertex and the foot of the ordinate).* This is an important property of the parabola.

8. *Graphical solution of the quadratic equation* $ax^2+bx+c=0$ (first method).

As an example, let us solve the equation $9x^2-5x-2=0$. Dividing by 9 so as to make the co-efficient of x^2 equal to $+1$ and transposing, we get $x^2=\cdot55x+\cdot22$.

Now taking the units of abscissa and ordinate to be each 1″ draw the graph of $y=x^2$ (Fig. 41) and on the *same* scale draw

the graph of the straight line $y = \cdot 55x + \cdot 22$. To draw the straight line we observe that $x = 1$ gives $y = 77$ and $y = 0$ gives $x = -\cdot 4$. Hence the points $(1, \cdot 77)$ and $(-\cdot 4, 0)$ are on the straight line. Now at the points A and B *common* to the two graphs $y = x^2$ and $y = \cdot 55x + \cdot 22$, the y is the same; hence the abscissæ are given by $x^2 = \cdot 55x + \cdot 22$, or, in other words, the abscissæ of the points of intersection of the two graphs are the roots of the equation $x^2 = \cdot 55x + \cdot 22$.

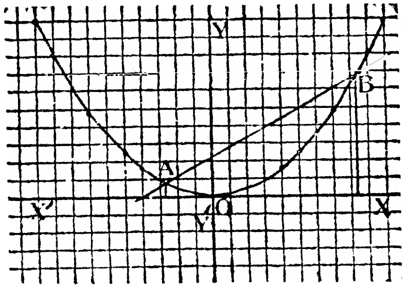


Fig. 41.

The roots are therefore from fig. 41 found to be $-\cdot 27$ and $\cdot 82$ nearly.

Since any quadratic equation can be reduced to the form $x^2 = px + q$, if we have a well-drawn graph of $y = x^2$, then by drawing the straight line $y = px + q$ (or by simply placing a ruler in the position of the line) we can read off the roots of any quadratic equation. (Observe that *scale must be same for both the graphs and same scale must be taken along both the axes*).

9. *Graphical solution of the quadratic equation* $ax^2 + bx + c = 0$ (second method).

Draw the graph of $y = ax^2 + bx + c$. Then for points on the graph for which y is zero *i.e.* which lie on the axis of x, the abscissæ are given by $ax^2 + bx + c = 0$, in other words, *the roots of the equation* $ax^2 + bx + c = 0$ *are the abscissæ of the points where the axis of x meets the graph of* $y = ax^2 + bx + c$. The roots are real and unequal, real and equal, or imaginary according as the x-axis cuts the graph in two points, in one point, or, does not cut it at all.

Ex. 1. Solve the equation $3x^2 - 12x + 8 = 0$.

Draw the graph of $y = 3x^2 - 12x + 8$ (see fig. 37); then the abscissæ of the points where the graph cuts the x-axis are the roots required. Remembering that in measuring abscissa in the figure, the side of a small square of the paper represents $\cdot 2$, the roots are read as $\cdot 85$ and $3 \cdot 15$.

Ex. 2. Trace the changes in the sign and magnitude of the function $7+8x-4x^2$ as x increases from $-\infty$ to $+\infty$.

Draw the graph of $y=7+8x-4x^2$ (see fig. 38). Now observe that in measuring abscissa the side of a small square in the fig. represents $\cdot 2$ and in measuring ordinate it represents 1. Then it is seen that the function $7+8x-4x^2$ is zero when $x=-\cdot 7$ or $2\cdot 7$ and that its maximum value $=11$ which occurs when $x=1$. Also we have the following :—

(i) As x increases from $-\infty$ to $-\cdot 7$ the function is negative and increases algebraically from $-\infty$ to 0.

(ii) As x increases from $-\cdot 7$ to $+1$ the function is positive and increases from 0 to 11 (maximum value).

(iii) As x increases from $+1$ to $+2\cdot 7$ the function is positive and diminishes from 11 to 0.

(iv) As x increases from $+2\cdot 7$ to $+\infty$, the function is negative and decreases algebraically from 0 to $-\infty$.

EXERCISE CXXIV.

1. Draw the graphs of :—(i) $y=x^2+2$. (ii) $y=x^2-2$. (iii) $y=-x^2+2$. (iv) $y=-x^2-2$.

2. Draw the graphs :— (i) $y=5x^2$. (ii) $y=-5x^2$. (iii) $y=5x^2+7$. (iv) $y=-5x^2+7$.

3. Draw the graphs of :—(i) $y=\frac{1}{5}x^2$ (ii) $y=-\frac{1}{5}x^2$. (iii) $y=\frac{1}{5}x^2+7$. (iv) $y=-\frac{1}{5}x^2+7$.

4. Draw the graphs of :—(i) $x=3y^2$. (ii) $x=-3y^2$. (iii) $x=3y^2+4$ (iv) $x=-3y^2+4$.

5. Draw the graphs of :—(i) $x=\frac{1}{4}y^2$ (ii) $x=-\frac{1}{4}y^2$ (iii) $x=\frac{1}{4}y^2+6$ (iv) $x=-\frac{1}{4}y^2+6$.

6. Draw the graphs of :—
(i) $y=x+2x^2$. (ii) $y=x-2x^2$. (iii) $y=-x+2x^2$.

7. Draw the graphs of :—
(i) $y=2x^2-5x+2$. (ii) $y=-3x^2-4x+10$.
(iii) $y=2x^2-8x+3$. (iv) $4y=8x+x^2$.

8. How would you reduce the following equations to the form $y=ax^2$:—
(i) $y=3x^2-5x+8$. (ii) $y=-5x^2+3x-11$.

9. Trace the changes in the sign and magnitude of the following functions as x increases from $-\infty$ to $+\infty$ and find their maximum or minimum values :—
(i) $2x^2-3x+9$. (ii) $-3x^2+x+2$.

10. Solve the equations graphically :—
(i) $6x^2-7x+2=0$. (ii) $15x^2+13x-20=0$.

10. Graphs of a pair of straight lines.

Ex. 1. Draw the graph of $x^2-y^2=0$.

Tabulate as follows :—

x	0	± 1	± 2	± 3	± 4	......
y	0	± 1	± 2	± 3	± 4	......

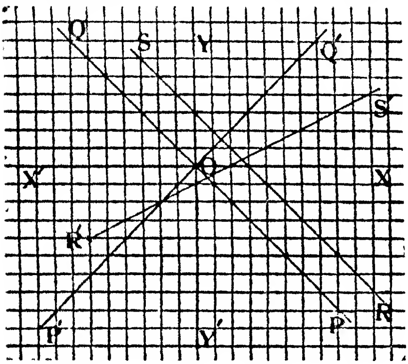


Fig. 42 ($\cdot 1'' = 1$).

By plotting these points as in fig. 42 we find that the graph consists of two straight lines one of which PQ passes through

(0, 0), (1, − 1) (2, − 2)...,
(− 1, 1), (− 2, 2)......

and the other P'Q' passes through

(0, 0), (1, 1), (2, 2),......
(− 1, − 1), (− 2, − 2)...

Otherwise :— since $x^2-y^2=(x-y)(x+y)$, the graph of $x^2-y^2=0$ consists of the graphs : $x+y=0$ and $x-y=0$ which we know to be the straight lines P Q and P'Q' respectively. They intersect at the origin and are perpendicular to each other.

Generally a homogeneous equation of the second degree in x and y represents a pair of straight lines through the origin.

Ex. 2. Draw the graphs of $x^2-xy-2y^2-5x+4y+6=0$.

(i) Tabulate the points thus :—

x	0	1	2	3	4	− 1	− 2	...
y	3 or − 1	2 or − ·5	1 or 0	0 or ·5	− 1 or 1	4 or − 1·5	5 or − 2	...

Plotting the points we find that the graph consists of two straight lines, one RS passing through (0, 3), (1, 2), (2, 1), (3, 0).........and the other R'S' passing through (0, − 1), (1, − ·5), (2, 0), (3, ·5)......

Otherwise : $x^2 - xy - 2y^2 - 5x + 4y + 6 = (x+y-3)(x-2y-2)$.

Hence the graph consists of two straight lines $x+y-3=0$ and $x-2y-2=0$. These are RS and R′S′ in figure 42.

Note. In general, a quadratic equation of the 2nd degree in x and y resolvable into two linear equations in x and y, represents two straight lines.

Ex. 3. Solve graphically the equations $x^2 - 4y^2 = 0$..............(1)

$2x+y=5$..............(2)

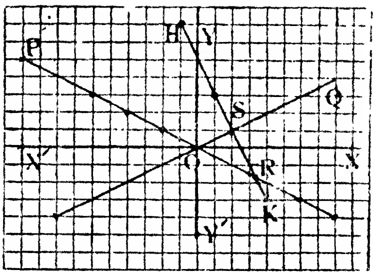


Fig. 43.

Draw the graph of $x^2-4y^2=0$ which consists of two straight lines OP and OQ ($x+2y=0$, $x-2y=0$) and also draw the graph of $2x+y=5$, the straight line HK in the figure. Let HK intersect OP and OQ in R and S respectively. The co-ordinates of R and S give the roots, which are $x=3\cdot35,\ 2$; $y=-1\cdot7,\ 1$.

11. Graphs of circles.

Ex. 1. Draw the graph of $x^2+y^2=1$ or $y=\pm\sqrt{1-x^2}$.

Take the units of abscissa and ordinate to be each 1″. Then putting $x=0,\ \cdot1,\ \cdot2,\ \cdot3$ etc. obtain the *corresponding* values of y, and tabulate as follows :

x	0	$\pm\cdot1$	$\pm\cdot2$	$\pm\cdot3$	$\pm\cdot4$	$\pm\cdot5$	$\pm\cdot6$	$\pm\cdot7$	$\pm\cdot8$	$\pm\cdot9$	±1
y	±1	$\pm\cdot99$	$\pm\cdot98$	$\pm\cdot95$	$\pm\cdot91$	$\pm\cdot87$	$\pm\cdot80$	$\pm\cdot71$	$\pm\cdot60$	$\pm\cdot43$	0

Now plot the four sets of points $(0,1)$, $(\cdot1,\cdot99)$, $(\cdot2,\cdot98)$... ; $(-\cdot1,\cdot99)$, $(-\cdot2,\ \cdot98)$... ; $(0,-1)$, $(\cdot1,-\cdot99)$... ; and $(-\cdot1,-\cdot99)$, $(-\cdot2,-\cdot98)$,... and joining them by a continuous curve we get the graph which is a circle (fig. 44).

Note 1. Observe that we cannot have x positive and greater than 1, nor x negative and numerically greater than 1, for in either case y becomes impossible. Hence the graph lies entirely between the straight lines $x=+1$, $x=-1$. Similarly y must lie between ±1, otherwise x becomes impossible, thus the graph also lies entirely between the straight lines $y=+1$ and $y=-1$.

Note 2. That the graph of $x^2+y^2=1$ is a circle, may be easily seen. For if we take any point (x, y) namely P, on the graph and PM be drawn perpendicular to the axis of x, then $x^2+y^2=1$ $PM^2+OM^2=1$ *i.e.*, $OP^2=1$ which shows that the distance of any point P on the graph from the origin is 1. Hence the graph is a circle with origin as centre and radius$=1$.

Note 3. In general an equation of the form $x^2+y^2=a^2$ is a circle with centre as origin and a as radius (the axes being two perpendicular diameters) for the distance of any point (x, y) on the graph from the origin is $\sqrt{(x^2+y^2)}=a$, a constant.

Conversely the equation of a circle of radius a referred to centre as origin and two diameters at right angles as axes is $x^2+y^2=a^2$.

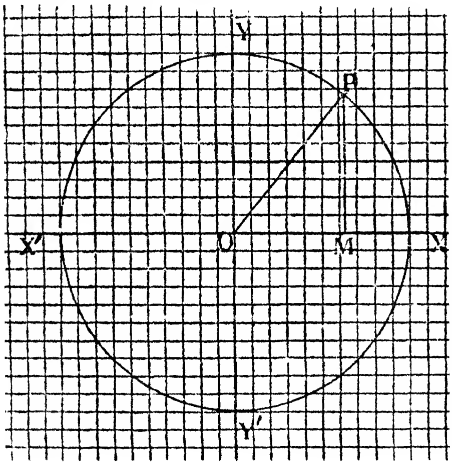


Fig. 44.

Note 4. In the equation $x^2+y^2=a^2$ the powers of x and y are even, hence if we change y into $-y$ and x into $-x$ the equation remains unaltered. Thus if (x, y), be a point on the graph $(x, -y)$, $(-x, y)$, $(-x, -y)$ are also points on the graph. Hence the circle $x^2+y^2=a^2$ is symmetrical about the axes *i.e.* about its diameters.

Ex. 2. Draw the graph of $x^2+y^2-4x-6y=12$.

This equation can be written as $(x-2)^2+(y-3)^2=25$.

If we take any point (x, y) on the graph, the square of its distance from $(2, 3)$ *viz.*, $(x-2)^2+(y-3)^2$ is equal to 25 *i.e.*, 5^2 $\therefore$ The distance of any point on the graph from the point $(2, 3)$ is 5. Hence the graph is a circle with the point $(2, 3)$ as centre and radius$=5$ (fig. 45.)

In general, any equation of the form $(x-h)^2+(y-k)^2=a^2$ represents a circle whose centre is the point (h, k) and radius $=a$.

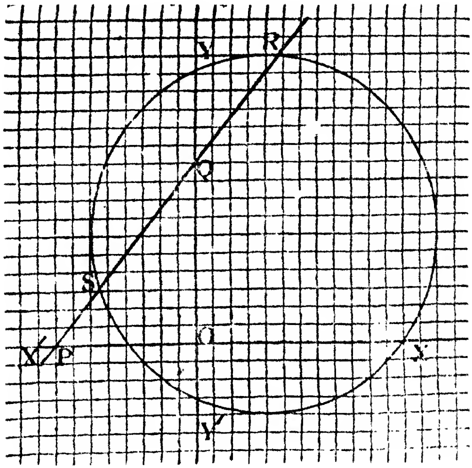


Fig. 45.

Scale $\cdot 2'' = 1$ along both the axes.

Conversely, the equation of a circle with the point (h, k) as centre and radius a is $(x-h)^2+(y-k)^2=a^2$. This may be written in the form $x^2+y^2+gx+fy=c$ in which **the co-efficients of x^2 and y^2 are equal and there is no term containing xy.**

Ex. 3. Solve $x^2+y^2=25$......................(1).

$13x+14y=91$....................(2).

The graph (1) is the circle with origin as centre and radius $=5$ and the graph (2) is the straight line PQ as in the figure 46. (To draw PQ, observe that when $x=0$, $y=6\cdot5$ and when $x=7$, $y=0$).

The circle is intersected by the line PQ at R and S, the co-ordinates of these points as read off from the figure are :—

(2.1, 4.5) and (4.3, 2.6) respectively.

Hence the roots are $\left.\begin{array}{l} x=2\cdot1,\ 4\cdot3 \\ y=4\cdot3,\ 2\cdot1 \end{array}\right\}$

Note. 1. The student should observe that the same scale must b used here (and in similar cases) in drawing the graphs of both (1) and (2) and the unit should be large enough to give distinctly the points R and S.

Scale $\cdot 2'' = 1$ along both the axes.

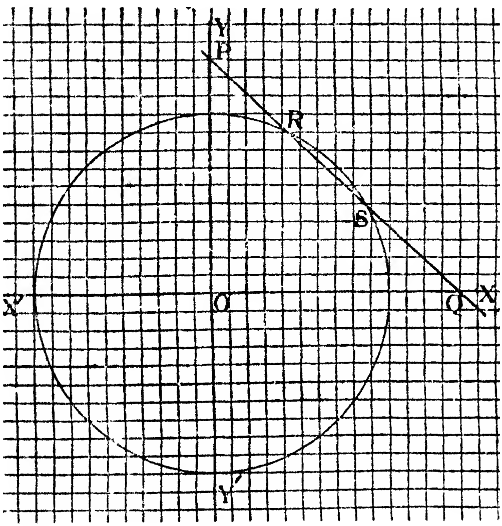


Fig. 46.

Ex. 4. Solve $x^2+y^2-4x-6y=12$.....................(1)

$5x-4y+20=0$.....................(2)

The equation (1) can be written as $(x-2)^2+(y-3)^2=25$. The graph is therefore a circle with the point (2, 3) as centre and radius $=5$. Describe the circle.

Again equation (2) represents a straight line.

To draw it observe that when $x=-4, y=0$ and when $x=0, y=5$ (scale $\cdot 2''=1$). The points of intersection of the graphs are R and S (fig. 45), whose co-ordinates give the roots $x=2\cdot 4$, $y=8$; $x=-2\cdot 8$, $y=1\cdot 5$.

Ex. 5. Find the equation of the circle passing through the points (4,6), (8,10), (16,12).

The general equation of a circle is $x^2+y^2+gx+fy=c$. Since it passes through (4,6), (8,10), (16,12), substituting the co-ordinates of these points we get three simultaneous equations in f, g, c *viz.*

$4g+6f-c=-52$...............(1)
$8g+10f-c=-164$...........(2)
$16g+12f-c=-400$...........(3)

Solving we get $g=-30, f=2, c=-56$.

$\therefore$ the equation of the circle is $x^2+y^2-30x+2y=-56$,

or $(x-15)^2+(y+1)^2=170$. The centre of the circle is the point $(15,-1)$ and radius $=\sqrt{170}=13\cdot04$.

EXERCISE CXXV.

1. Graph the following equations :— (1) $9x^2-16y^2=0$.

(2) $x^2-5xy+6y^2=0$. (3) $x^4-y^4=0$. (4) $x^2+y^2=10$.

(5) $x^2+y^2+6x-14y=-33$. (6) $y=\pm\sqrt{(64-x^2)}$.

(7) $6x^2+6y^2-10x-8y=8$. (8) $16x^2+16y^2-16x-24y=\frac{11}{4}$.

2. What are the equations of the circles in the following cases?

(1) centre (0,4), radius 5, (2) centre (3,2), radius 7,

(3) Passing through three points (2,3), (4,5), (8, 6),

(4) radius 6, passing through (4,6), (−2, 12).

3. Solve graphically :—

(1) $x^2-9y^2=0$, $x+y=4$. (2) $x^2+y^2=25$, $x-2y=10$. (3) $x^2-6x+y^2-4y=12$, $x=2y+11$.

Graphs of Ellipse.

12. Ex. 1. Draw the graph of $4x^2+16y^2=9$ *i.e.*,

$$y=\pm\tfrac{1}{4}\sqrt{(9-4x^2)}\ldots\ldots\ldots(1)$$

In order that y may be real it follows from (1) that $9-4x^2$ or $(3+2x)(3-2x)$ must not be negative. Hence $2x$ must lie between -3 and $+3$, or x must lie within the limits $-1\cdot5$ and $+1\cdot5$. Hence we tabulate as follows taking only the values of x from $-1\cdot5$ to $+1\cdot5$, both inclusive :—

x	$-1\cdot5$	-1	$-\cdot5$	0	$\cdot5$	1	$1\cdot5$
y	0	$\pm\cdot55$	$\pm\cdot70$	$\pm\cdot75$	$\pm\cdot70$	$\pm\cdot55$	0

By plotting these points and joining them by a continuous curve by free-hand drawing we get the figure A B A′ B′ as shown in fig. 47. The curve is called an **ellipse**.

It is oval-shaped and symmetrical about both the axes, as it should be, for its equation $4x^2+16y^2=9$ does not contain any odd power of x or y.

Note 1. The points A, A′, B, B′ where the graph cuts the axes are four prominent points which may be at once found thus :—

Put $y=0$ in the equation $4x^2+16y^2=9$, then the values of x ($\pm 1\cdot 5$) give the intercepts O A, O A′ on the axis of x ; again put $x=0$, then the values of y $y\pm\cdot 75$) give the intercepts OB, OB′ on the axis of y.

Note 2. We have seen from (1) that x must be between $\pm 1\cdot 5$. Similarly putting the equation $4x^2+16y^2=9$ in the from $x=\pm\frac{1}{2}\sqrt{9-16y^2}$, we find that y must lie between $\pm\cdot 75$. Hence the graph must lie within the rectangle contained by the straight lines $x=\pm 1\cdot 5$, $y=\pm\cdot 75$.

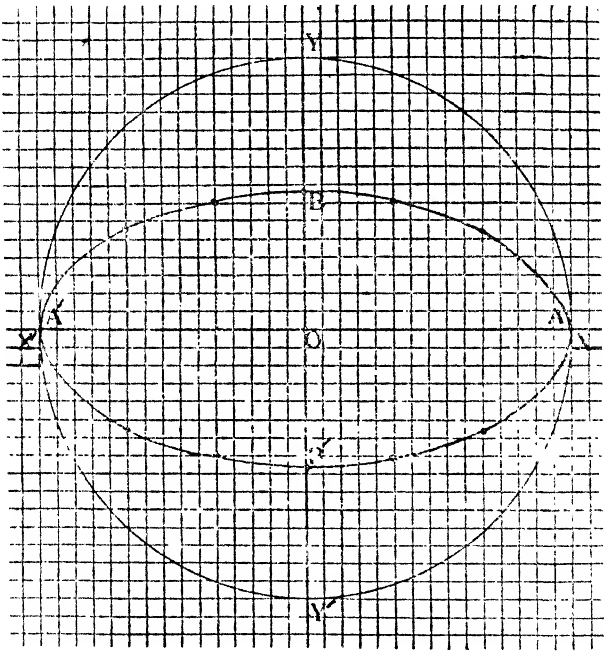


Fig. 47.

Note 3. Generally any equation of the form $Ax^2+By^2=C$ (A, B, C being of the same sign) or of the equivalent form $x^2/a^2+y^2/b^2=1$ represents an ellipse of which origin is the centre.

Note 4. The following method of drawing the above graph is instructive :—

Consider the equation $y=\pm\frac{1}{2}\sqrt{(9-4x^2)}$..............(2)

This on squaring and transposing gives $x^2+y^2=\frac{9}{4}$.

The graph of (2) is therefore a circle of which the centre is the origin and radius equal to $\frac{3}{2}$. Now it follows from (1) and (2) that for the *same* abscissa, the ordinate of the graph of (1) is half the ordinate of the graph of (2). Hence the graph of (1) can be constructed by drawing a curve free-hand through the points of bisection of the ordinates of the graph of (2) *i.e.*, of a circle whose centre is the origin and radius $\frac{3}{2}$. This is shown in fig. 47. The units of ordinate and abscissa are each taken to be 1".

Ex. 2. Draw the graph of $4x^2-32x+y^2-10y+25=0$.

Here $4(x-4)^2+(y-5)^2=64$ or $y-5=\pm\sqrt{64-4(x-4)^2}$......(1).

The values of x and y may be tabulated as follows

x	0	1	2	3	4	5	6	7	8
y	5	10·3, −·3	11·9, −1·9	12·8, −2·8	13, −3	12·8, −2·8	11·9, −1·9	10·3 −·3	5

The curve is as shown in fig. 48
(Scale ·1" = 1)

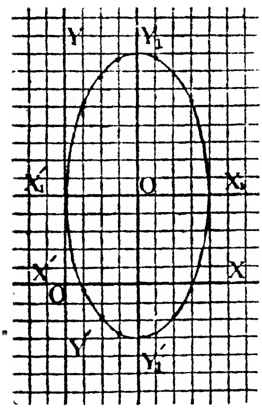


Fig. 48.

Observe from (1) that x lies between 0 and 8, for, all values of x beyond these limits make y impossible. Similarly y lies between 13 and −3, for all values of y beyond these limits make x impossible.

The curve is an ellipse which lies between the st. lines $x=0$, $x=8$, $y=13$, $y=-3$.

It will be easier to draw the curve by transferring the origin to the second point O (4,5) with parallel axes X_1OX_1', Y_1OY_1' for then the equation becomes $4x^2+y^2=64$ (2) and we get the same curve more easily as in ex. 1 by tabulating points with reference to the new axes as follows :

$x = 0, \pm 1, \pm 2, \pm 3, \pm 4.$
$y = \pm 8, \pm 7\cdot 8, \pm 6\cdot 8, \pm 5\cdot 3, 0.$

Generally an equation of the form $ax^2+by^2+gx+fy=c$ where a and b are of the same sign represents an ellipse, for by a change of origin it may be reduced to the simplest form $Ax^2+By^2=C$.

Graphs of Hyperbola.

13. Ex. 1. Graph of $4x^2 - y^2 = 1$, *i.e.*, $y = \pm\sqrt{(4x^2 - 1)}$......(1).

We see from (1) that in order that y may be real $4x^2 - 1$ or $(2x+1) \times (2x-1)$ can not be negative, hence x cannot lie between $\pm\cdot 5$ *i.e.* x can have positive values from $+\cdot 5$ to $+\infty$ and negative values from $-\cdot 5$ to $-\infty$ but can have no value between $-\cdot 5$ and $+\cdot 5$. Confining our attention to the positive values of x, we tabulate thus :—

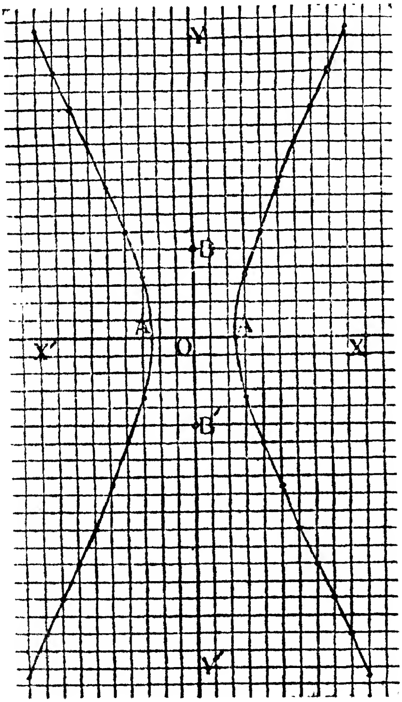


Fig. 49.

x	·5	·6	·8	1	1·2	1·4	1·6	1·8	2	...	∞
y	0	$\pm$·7	$\pm$1·2	$\pm$1·7	$\pm$2·2	$\pm$2·6	$\pm$3	$\pm$3·4	$\pm$3·9	...	$\pm\infty$

Take the units of abscissa and ordinate to be each $\cdot 5''$ *i.e.* let the side of a small square represent $\cdot 2$ in measuring x as well as y. Then plot the point $(\cdot 5, 0)$, $(\cdot 6, \cdot 7)$, $(\cdot 6, -\cdot 7)$ etc. Drawing a curve freehand through them, the right hand part of fig. 49 is obtained.

Again, taking negative values of x corresponding to the positive values of the last table, the values of y remain the same. Plotting these points we get the left hand part of fig. 49.

Thus the graph of $4x^2-y^2=1$ consists of two infinite branches spreading out in opposite directions. The graph is called the *hyperbola.*

It is symmetrical about both axes which is evident from the fact that the equation contains odd power of neither x nor y.

The whole tabulation may be shown as follows :—

x	$\pm\cdot 5$	$\pm\cdot 6$	$\pm\cdot 8$	± 1	$\pm 1\cdot 2$	$\pm 1\cdot 4$	$\pm 1\cdot 6$	$\pm 1\cdot 8$	± 2	...	$\pm\infty$
y	0	$\pm\cdot 7$	$\pm 1\cdot 2$	$\pm 1\cdot 7$	$\pm 2\cdot 2$	$\pm 2\cdot 6$	± 3	$\pm 3\cdot 4$	$\pm 3\cdot 9$	...	$\pm\infty$

Note 1. Since x can not have any value between $+\cdot 5$ and $-\cdot 5$ the curve has no portion of it between the lines $x=+\cdot 5$ and $x=-\cdot 5$. Observe that as x increases y also increases, and that the curve has got two infinite branches spreading out in opposite directions.

Note 2. Generally, any equation of the form $Ax^2-By^2=C$ (where A and B are of the same sign) or of the equivalent form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ represents a hyperbola.

Ex. 2. Draw the graph of :—

$$4x^2-32x-y^2+10y=25.$$

The equation can be written as—

$$4(x-4)^2-(y-5)^2=64.$$

By transferring the origin to the point O_1 (4,5) with parallel axes $X_1O_1X'_1$, and $Y_1O_1Y'_1$, the equation becomes $4x^2-y^2=64$. The tabulation with reference to the new axes is as follows :—

(x can not lie between ± 4, for then y becomes unreal).

x	± 4	± 5	± 6	± 8	......	$\pm\infty$
y	0	± 6	$\pm 8\cdot 9$	$\pm 13\cdot 9$	......	$\pm\infty$

Plotting the points with reference to the new axes, we get the graph as shown in the figure (50).

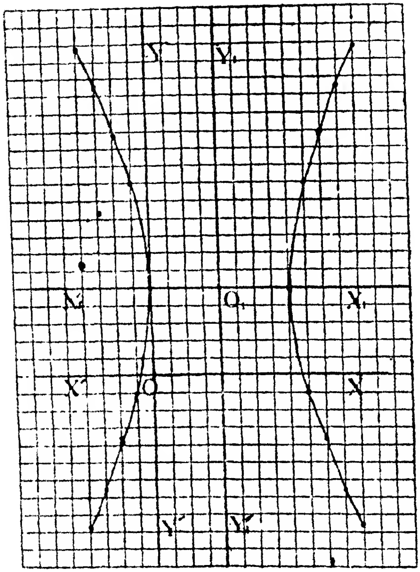


Fig. 50.
(Scale $\cdot 1'' = 1$ along both the axes.)

Generally an equation of the form $ax^2 - by^2 + gx + fy = c$ (where a, b, c are of the same sign) represents a hyperbola, for by change of origin it may be reduced to the simplest form $Ax^2 - By^2 = C$.

14. Rectangular Hyperbola. Graph of $y = k/x$.

Suppose we are to draw the graph $y = 1/x$. Make a table of corresponding values of x and y as follows :—

x	0	...	·1	·2	·4	·6	·8	1	1·2
y	∞	...	10	5	2·5	1·67	1·25	1	·83

x	1·4	1·6	1·8	2	2·2	2·4	2·6	2·8	3	3·2	3·4	...	∞
y	·71	·62	·55	·5	·45	·42	·38	·36	·33	·31	·29	...	0

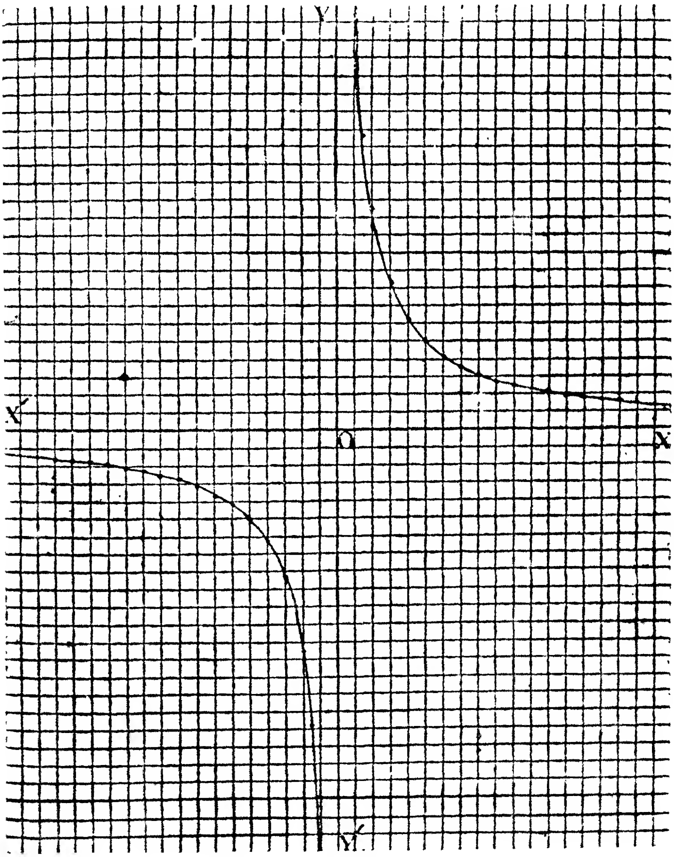


Fig. 51. Scale ·5″ = unit.

If we take $x=-\cdot 1, -\cdot 2 - 4$, etc., we find $y=-10, -5, -2\cdot 5$ etc. Now taking the units of abscissa and ordinate to be each $\cdot 5''$ plot the above points.

The graph is shown in fig. 51. It consists of two infinite branches, one in the first and the other in the third quadrant. It is seen from the figure that the graph approaches the axes of co ordinates more and more. It will never meet them.

Def. If a straight line approaches a curve so that the distance between it and the curve becomes less and less as we proceed towards infinity along the straight line, the straight line is called an *asymptote* to the curve.

The axes of co-ordinates are symptotes to the graph of $y=1/x$ which is called a **rectangular hyperbola.**

In the equation if x is made negative y also becomes negative ; hence the curve has two branches which are symmetrically placed in the 1*st* and 3*rd* quadrants.

15. The graph of $y=(ax+b)/(cx+d)$ can be made to coincide with that of $y=k/x$ by a change of origin.

Ex. 1. Draw the graph of $y=\dfrac{3x+6}{x-2}$.

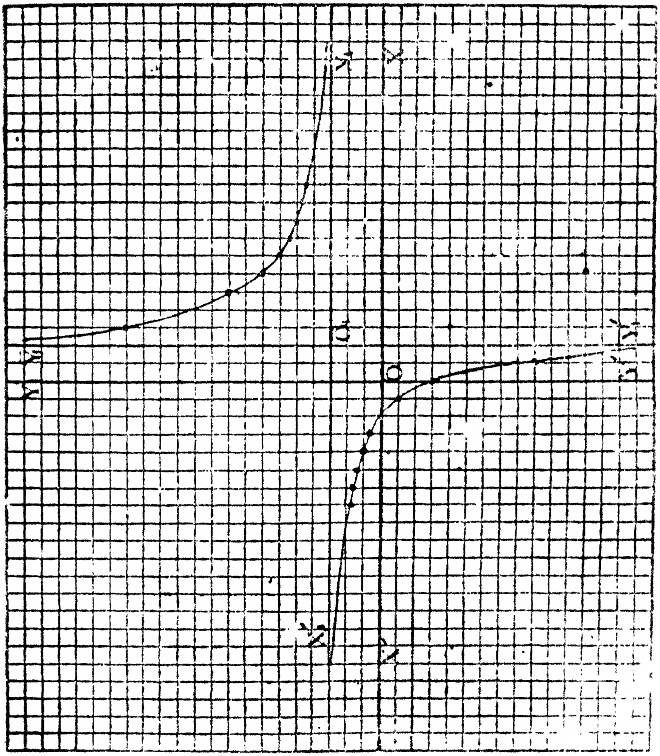
Here $y=3+\dfrac{12}{x-2}$ or $y-3=\dfrac{12}{x-2}$....................(1)

The values of x and y may be tabulated as follows :—

$x=$	0	1	2	3	4	5	6	...	∞
$y=$	-3	-9	∞	15	9	7	6	...	3

	-1	-2	-3	-4	-5	...	...	$-\infty$
	1	0	$\cdot 6$	1	$1\cdot 3$	...	...	3

Plotting these points we get the curve as drawn in figure 52. The curve is a rectangular hyperbola, the point (2, 3) being its centre and the straight lines $x=2$ and $y=3$ its asymptotes.



Scale ·1 = 1

Fig. 52.

Note 1. The equation, (1) on transferring the origin to the point (2, 3) with parallel axes, becomes $xy=12$ and the graph may be drawn from the transformed equation as in Ex. 1.

Note 2. Generally any equation of the form $xy+gx+fy=c$ or of the equivalent form $(x+h)(y+k)=c$ represents a rectangular

hyperbola; for by a change of origin it may be brought to the simplest form $y=\frac{k}{x}$.

16. Solutions of equations.

Ex. 1. Solve graphically

$$y^2-4x^2=64\ldots\ldots\ldots\ldots(1)$$
$$3y-10x=0\ldots\ldots\ldots\ldots(2)$$

The graph of (1) is a hyperbola and that of (2) a straight line, PQ. [Fig. 53].

To solve the equations graphically is to find the co-ordinates of the points (Q, P) of inter-section of the graphs. The roots are therefore $x=3, -3$; $y=10, -10$.

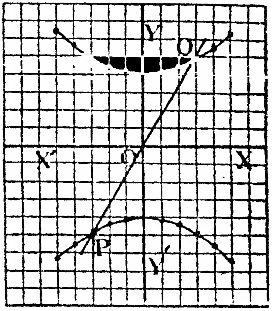


fig. 53. x unit $=\cdot1''$. y ,, $=\cdot05''$

Ex. 2. Solve graphically the equations

$$x^2+y^2=89\ldots\ldots(1).$$
$$xy\quad=40\ldots\ldots(2).$$

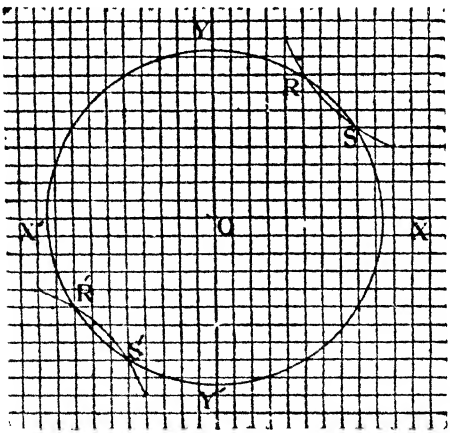


Fig. 54. Scale $\cdot1''=1$.

The graph of equation (1) is a circle and that of (2) a rectangular hyperbola (fig 54). The co ordinates of the points of their intersection R, S, R', S', give the roots which are

$$x = 5,\ 8,\ -8,\ -5.$$
$$y = 8,\ 5,\ -5,\ -8.$$

17. **How to recognise the nature of the curve represented by a given equation, from an inspection of its form.**

I. *A quadratic equation containing the square of either x, or, y and no xy represents a parabola.*

Equations of the forms (1) $y^2 = ax$ (2) $x^2 = ay$ (3) $x = ay^2 + by + c$ (4) $y = ax^2 + bx + c$, *represent parabolas*; a, b, c *being constants.*

II. *A quadratic expression equated to zero represents two straight lines if it can be resolved into two linear factors.*

III. *A quadratic equation containing the squares of both x and y and no xy represents* (1) *a circle when the co-efficients of x^2 and y^2 are equal and of the same sign* (2) *an ellipse when the co-efficients are unequal and of the same sign* (3) *a hyperbola when the co-efficients are of opposite signs.*

Thus (1) $ax^2 + by^2 = c$ *or* (2) $ax^2 + by^2 + gx + fy = c$ *will represent* (i) *a circle if $a = b$ and they are of the same sign* (ii) *an ellipse if a and b are unequal but of the same sign* (iii) *hyperbola if a and b are of opposite signs.*

If an equation be of the form $ax^2 + by^2 = 0$, and a, b are of the same sign then $x = 0$, $y = 0$, or, the equation represents the origin. Observe that if a and b are of opposite signs, then $ax^2 + by^2 = 0$ represents two straight lines (See II above).

IV. *Equations of the form* (1) $xy = k$, *or* (2) $xy + ax + by = c$ *represent rectangular hyperbolas.*

EXERCISE CXXVI.

1. Graph the following equations :—

(i) $x^2 + y^2 = 4$. (ii) $x^2 + y^2 = 9$. (iii) $x^2 - y^2 = 4$. (iv) $x^2 - y^2 = 9$.

(v) $4x^2 + 9y^2 = 1$. (vi) $4x^2 - 9y^2 = 1$. (vii) $x^2 - y^2 = 0$.

(viii) $4x^2 - 9y^2 = 0$. (ix) $y = \dfrac{x-4}{x-1}$. (x) $xy + 3x - 2y + 4 = 0$.

(xi) $4x^2 + 3y^2 - 8x = 96$. (xii) $4x^2 - 3y^2 - 8x = 96$.

2. How would you reduce the equations (i) $y = \dfrac{4x-9}{2x-5}$, and (ii) $xy = 4x - 5y + 9$ to the form $y = \dfrac{k}{x}$?

3. Draw a graph which gives the square roots of numbers from 0 to 2.

4. Solve :—(1) $x=2y+3, \quad 2x^2-3y^2+6y=0.$

(2) $x^2+y^2=41, \; 2y^2-3x^2=2.$

(3) $x^2+y^2=25, \; xy=12.$

(4) $(x+1)(y+2)=8, \; 2x^2+3y^2=14.$

5. State by inspection the nature of the curves represented by the following :—

(1) $25x^2-36y^2=0.$ (2) $4x^2+4y^2=81.$ (3) $y^2=4x.$

(4) $3x^2+4y^2=48.$ (5) $4x^2-3y^2=40.$ (6) $y=\frac{1}{5}\sqrt{4-x}$.

(7) $5x^2+6y^2=0.$ (8) $2x^2+2y^2=0.$ (9) $xy=6.$

(10) $x^2+5xy+6y^2+x+5y-6=0.$ (11) $y=3x^2+6x+2.$

(12) $x^2+y^2+4x+6y=45.$ (13) $2x^2+3y^2+x+2y=8.$

(14) $xy+2x+3y=5.$ (15) $y=\frac{1}{2}\sqrt{3x^2+4}$ (16) $2x^2-3y^2+x-2y=8$

CHAPTER XXX.

PROGRESSIONS.

1. A number of quantities formed according to some definite law forms a **series**. The quantities constitute the **terms** of the series.

I. Arithmetical Progression.

2. A series is said to be an **Arithmetical Progression** or shortly, an A. P. when the difference between any term and the preceding one is constant. This constant quantity is called the *common difference* of the series, which is therefore equal to any term of the series minus the *preceding* term.

Thus the series 5, 8, 11, 14, ... 21, 19, 17, 15, ...

are Arithmetical Progressions.

In the first series the common difference is 3, and in the second it is -2. The first series in which the common difference is positive is an increasing A. P. and the second in which the common difference is negative is a decreasing A. P.

Note. It is evident that if the same quantity is added to, or, subtracted from each term of an A. P., the resulting series is an A. P. with the *same* common difference ; also if each term of an A. P. is multiplied (or divided) by the same quantity, say k, the resulting series is an A. P. with $k \left(\text{or } \frac{1}{k}\right)$ times the previous common difference.

3. The A. P. of which the first term is a and the common difference b is :— $a, (a+b), (a+2b), (a+3b), \ldots\ldots$

Here we observe that in any term the multiple of b is less than the number giving the position of the term in the series by 1 : hence we can write down any term of the series : for example

$$\text{the 10th term} = a+(10-1)b = a+9b,$$

$$\text{the 16th term} = a+(16-1)b = a+15b\ ;$$

and generally **the nth term** $= a+(n-1)b$.

Ex. 1. Find the 15th term of the series 10, 13, 16, 19,.........

$$\text{Here } a = 10,\ b = 3,\ n = 15$$

$$\therefore \text{ the 15th term} = 10+(15-1)3 = 52.$$

Ex. 2. Is 142 a term of the series 7, 11, 15, 19, ... ?

Let 142 be the nth term of the series.

Then $142 = 7+(n-1)4 = 4n+3$

$\therefore\ 4n = 139$, or, $n = 34\frac{3}{4}$.

But n must be a *positive integer* ; hence 142 is not a term of the series. The fractional value of n indicates that 142 lies between the 34th and 35th terms of the series.

4. *If any two terms of an A. P. are given, the series is completely known.*

Let the pth term of an A.P. be P and the qth term Q, then the first term a and the common difference b of the series are obtained from the equations

$$P = a+(p-1)b, \qquad Q = a+(q-1)b.$$

Ex. 3. Find the 16th term of the series of which the 5th term $= 11$ and the 10th term $= 21$.

Let a be the first term and b the common difference of the series.

Then $11 = a+(5-1)b = a+4b$, $\qquad 21 = a+(10-1)b = a+9b$.

$\therefore\ b = 2,\ a = 3$

Hence the 16th term $= 3+(16-1)2 = 3+30 = 33$.

Ex. 4. If the pth term of an A.P. is m and the qth term n, find the $(p+q)$th term and the $(p-q)$th term.

Let a be the first term, b the common difference.

Then $m = a+(p-1)b$.........(1)

$n = a+(q-1)b$.........(2)

$\therefore\ m-n = (p-q)b$, or $b = \dfrac{m-n}{p-q}$...........(3)

Now the $(p+q)$th term $=a+(p+q-1)b$

$$=a+(p-1)b+qb$$

$$=m+\frac{q(m-n)}{p-q} \text{ from (1) and (3).}$$

Also the $(p-q)$th term $=a+(p-q-1)b$

$$=a+(p-1)b-qb$$

$$=m-\frac{q(m-n)}{p-q} \text{ from (1) and (3).}$$

Ex. 5. Find the condition that P, Q, R may be respectively the pth term, the qth term and the rth term of an A.P.

Let a = first term, b = common difference.

Then we have $P=a+(p-1)b$............(1)

$Q=a+(q-1)b$............(2)

$R=a+(r-1)b$............(3)

The required condition is obtained by eliminating a, b from (1), (2), (3).

Multiply (1) by $q-r$, (2) by $r-p$, and (3) by $p-q$, and add, then the right-hand side vanishes and we have

$P(q-r)+Q(r-p)+R(p-q)=0$ for the required condition.

EXERCISE CXXVII.

1. Find

(1) 21st term of the series $-3, 4, 11,$......

(2) nth term of the series $9, 8\frac{1}{3}, 7\frac{2}{3},$......(C. F. 1884).

(3) nth term of $\frac{1}{n}, \frac{n+1}{n}, \frac{2n+1}{n},$.........(C. F. 1886).

2. Are (*i*) 60 (*ii*) 71 (*iii*) 100 terms of the series 7, 11, 15, ...?

3. Determine the A. P. of which (*i*) the 10th term $=9$ and the 15th term $=-7$ (*ii*) the 12th term $=10$ and the 7th term $=15$.

4. What are 11th and the nth terms of the A. P. of which the 6th term is -3 and the 9th term is 7?

5. If the pth term of an A. P. is q and the qth term p, find the mth term.

6. If the $(p+q)$th term of an A.P. is m and the $(p-q)$th term n, find the pth and qth terms.

---

5. *To find the sum of a number of terms of an A.P.*

Let a = first term and b = common difference of the series. Let n be the number of terms and s the required sum. Then

$$s=a+(a+b)+(a+2b)+\ldots+\{a+(n-2)b\}+\{a+(n-1)b\}\cdots\cdots(1)$$

Also putting l for the last or nth term $a+(n-1)b$, and writing the series in the reverse order, we have (common diff. $=-b$)

$$s=l+(l-b)+(l-2b)+\cdots\cdots+\{l-(n-2)b\}+\{l-(n-1)b\}\cdots\cdots(2)$$

Adding (1) and (2) we have $2s=(a+l)+(a+l)+\ldots$to n terms

$$=n(a+l).$$

$$\therefore \quad s=\frac{n}{2}(a+l)\ldots\ldots(i)$$

$=n$ times the average of the first and the last terms

Again, substituting for l its value $a+(n-1)b$, we have

$$s=\frac{n}{2}\{2a+(n-1)b\}\ldots\ldots\ldots\ldots(ii)$$

The formulae (i) and (ii) should be committed to memory by the student: he should use one or the other, as the data in any question may require.

Note 1. *In an A.P. the sum of any two terms equidistant from the beginning and the end is the same.*

For, the rth term of the series (1) above from the beginning $=a+(r-1)b$; and the rth term from the end $=r$th term from the beginning of the series (2) above and $\therefore =l-(r-1)b$.

Hence the sum of the rth terms of the given series from the beginning and the end $=a+(r-1)b+l-(r-1)b$.

$=a+l=$ first term $+$ last term.

Hence the sum is constant for all values of r.

It follows from the above that if the number of terms of an A.P. be odd, this constant sum is twice the middle term; and if the number be even, it is the sum of the two middle terms.

Note 2. If the sum of n terms of an *A.P.* be given and also the sum of n' terms, the series can be completely determined.

For, if $a=$first term, $b=$common difference, they are determined from

$$s=\frac{n}{2}\{2a+(n-1)b\}, \qquad s'=\frac{n'}{2}\{2a+(n'-1)b\}.$$

Ex. 1. Find the sum of the first n natural numbers.

Note. The numbers 1, 2, 3,...are called *natural numbers.*

Here first term $=1$, last term $=n$, number of terms $=n$.

$\therefore$ the sum required $=\frac{1}{2}n(n+1)$ by formula (i).

Ex. 2. Find the sum of the first n odd natural numbers, *i.e.* 1, 3, 5, 7,............to n terms.

Here first term $=1$, common difference $=2$, number of terms $=n$;

$\therefore$ the sum required $=\frac{n}{2}\{2+(n-1)2\}$ by formula (*ii*)

$$=\frac{n}{2}\cdot 2n=n^2.$$

Ex. 3. Sum to 15 terms the series $3+7+11+15+\ldots\ldots$

Here $a=3$, $b=4$, $n=15$.

$\therefore$ the sum required $=\frac{n}{2}\{2a+(n-1)b\}$

$=\frac{15}{2}\{6+(15-1)4\}=\frac{15}{2}.62=465.$

Ex. 4. The pth term of an A.P. is a and the qth term is b; show that the sum of the first $p+q$ terms is

$$\frac{p+q}{2}\left\{a+b+\frac{a-b}{p-q}\right\} \qquad \text{(M.F. 1887).}$$

Let $A=$ first term, $B=$ common difference.

Then we have $a=A+(p-1)B$..............(1)

$b=A+(q-1)B$..............(2)

From (1) and (2) by subtraction, $a-b=(p-q)B$,

or, $B=\frac{a-b}{p-q}$.....................(3)

From (1) and (2) by addition, $a+b=2A+(p+q-2)B$,

or $a+b+B=2A+(p+q-1)B$............. (4)

Now the required sum $=\frac{p+q}{2}\{2A+(p+q-1)B\}$

$=\frac{p+q}{2}\{a+b+B\}$ from (4)

$=\frac{p+q}{2}\left\{a+b+\frac{a-b}{p-q}\right\}$ from (3).

Ex. 5. The sum of p terms of an A.P. is q and the sum of q terms is p, find the sum of $(p+q)$ terms.

Let $a=$ first term and $b=$ common difference.

Then we have $q=\frac{p}{2}\{2a+(p-1)b\}$....................(1)

$p=\frac{q}{2}\{2a+(q-1)b\}$....................(2)

Also sum of $p+q$ terms $=\frac{p+q}{2}\{2a+(p+q-1)b\}$...(3)

Now from (1) and (2)

$$2a+(p-1)b=\frac{2q}{p}\ldots(4) \qquad 2a+(q-1)b=\frac{2p}{q}\ldots\ldots(5)$$

From (4) and (5) by subtraction,

$$(p-q)b=2\left(\frac{q}{p}-\frac{p}{q}\right)=\frac{2(q^2-p^2)}{pq}.$$

$$\therefore\ b=-\frac{2(p+q)}{pq},\quad\text{or,}\quad bq=-\frac{2(p+q)}{p}\ldots\ldots\ldots\ldots\ldots(6)$$

Adding (4) and (6) $2a+(p+q-1)b$

$$=\frac{2q}{p}-\frac{2(p+q)}{p}=-2.$$

Hence from (3) sum of $p+q$ terms

$$=\frac{p+q}{2}\times(-2)=-(p+q).$$

EXERCISE CXXVIII.

1. Sum

(1) $2+4+6+\ldots$to 12 terms.

(2) $5+2-1-\ldots$to 20 terms.

(3) $\frac{2}{3}+\frac{7}{[illegible]}+\frac{5}{[illegible]}+\ldots$to 15 terms.

(4) $\frac{1}{5}+\frac{[illegible]}{[illegible]}+\frac{1[illegible]}{1[illegible]}+\ldots$to 10 terms.

(5) $17\frac{1}{2}+14\frac{1}{6}+10\frac{5}{6}+\ldots$to 24 terms. (B. P. 1883)

(6) $\frac{1}{a}+\frac{2}{a}+\frac{3}{a}+\ldots$to 20 terms.

(7) $(x+y)^2+(x^2+y^2)+(x-y)^2+\ldots$to n terms.

(8) $\frac{4}{\sqrt{2}}+3\sqrt{2}+4\sqrt{2}+\ldots$to 50 terms.

2. The sum of n terms of an A.P. is n times the middle term, or, n times the average of the two middle terms, according as n is odd or even.

3. Sum to 11 terms the A.P. whose 6th term is 45.

4. Sum to 16 terms the A.P. whose 8th and 9th terms are 16 and 24 respectively.

5. Sum to 50 terms the A.P. of which the 13th term is -5 and the 25th term is 7.

6. If the pth term of an A.P. is $1/q$ and the qth term is $1/p$, shew that the sum of pq terms $=\frac{1}{2}(pq+1)$.

7. Determine the A.P. of which the sum to 10 terms is 160 and the sum to 16 terms is 352.

6. Illustrative Examples. We shall sometimes indicate the rth term and the sum to r terms of a series by t_r and s_r respectively.

Ex. 1. Sum to n terms the series of which the rth term is $5r-3$.

Here $t_r=5r-3$ Hence making $r=1,2,3,\ldots\ldots n$,

we have $t_1+t_2+t_3+\ldots+t_n=2+7+12+\ldots\ldots$ to n terms.

$$=\frac{n}{2}\left\{4+(n-1)5\right\}=\tfrac{1}{2}n(5n-1).$$

Ex. 2. Determine the nth term of the series of which the sum to r terms is $2r^2-r$.

Here $s_r=2r^2-r$, $\therefore\ s_{r-1}=2(r-1)^2-(r-1)$.

$$\therefore\ s_r-s_{r-1}=4r-3,\ i.e.\ t_r=4r-3.$$

Hence the nth term $=4n-3$, and the series is the A.P.

$$1+5+9+\ldots\ldots\ldots$$

Ex. 3. Prove that n^2 is the sum of n terms of an Arithmetic series of integers.

Let $n^2=a+(a+b)+(a+2b)+\ldots n$ terms.

$$\text{Then } n^2=\frac{n}{2}\left\{2a+(n-1)b\right\}$$

$$\text{or } 2n=2a+(n-1)b.$$

Since this is true for all values of n, make $n=1$ and also make $n=2$.

Then we get $2=2a$ and $4=2a+b$. Hence $a=1$, $b=2$.

Thus the series is $1+3+5+\ldots+(2n-1)$, the n odd integers beginning with 1.

Ex. 4. How many terms of the series $21+18+15+\ldots$amount to 66?

Let $n=$number of terms.

$$\text{Then } 66=\frac{n}{2}\left\{42+(n-1)(-3)\right\}.$$

$\therefore$ simplifying, $n^2-15n+44=0$ whence $n=4$ or 11.

Hence the sum of 4 terms and also of 11 terms is 66. It thus appears that the sum of the terms from the 5th to the 11th both inclusive is zero, as may be verified by writing down these terms.

Ex. 5. How many terms of the series $19+15+11\ldots$amount to 52?

Let $n=$number of terms.

$$\text{Then } 52=\frac{n}{2}\left\{38+(n-1)(-4)\right\}.$$

$\therefore$ simplifying, $2n^2-21n+52=0$.

$\therefore$ $n=4$ or $6\frac{1}{2}$.

The fractional value $6\frac{1}{2}$ indicates that 52 lies between the sum to 6 terms (which is 54) and the sum to 7 terms (which is 49).

Ex. 6. How many terms of the series $6+8+10+\ldots$ amount to 50?

Let n = number of terms.

$$\text{Then } 50=\frac{n}{2}\left\{12+(n-1)2\right\}.$$

$\therefore$ simplifying, $n^2+5n-50=0$.

$\therefore$ $n=5$ or -10.

The negative value -10 is no solution to the question proposed but we may give a meaning to it thus:—If from the last term of the series given by the positive solution, *i.e.*, from the 5th term we count *backwards* 10 terms, the sum of these 10 terms is also 50.

For the 5th term is 14 and 10 terms reckoned *backwards* give the series

$14+12+10+8+6+4+2+0-2-4$, of which the sum is 50.

The result is true in the general case for which see authors' *Intermediate Algebra*, *pp.* 31–32.

EXERCISE CXXIX.

1. Sum to 10 terms the series whose nth term is $7n+9$.

2. Sum to n terms the series whose rth term is $3r-7$.

3. Determine the series of which the sum of n terms is $5n^2+3n$.

4. Find the rth term of the series of which the sum to n terms is $pn+qn^2$.

5. What is the first term of the A.P. of which the common difference is -3 and the sum of 15 terms is 83?

6. What is the common difference of the A.P. of which the first term is 11 and the sum to 12 terms is 50?

7. How many terms of the series $-60-54-48-\ldots$ amount to -324?

8. Of how many terms of the series $56+64+72+\ldots$ the sum is 360? Explain the double answer.

9. Find how many terms of the series $8+12+16+\ldots$ amount to 140. Explain the double answer.

10. The sum of n terms of an A.P. is $2n^2$; find the first term and the common difference. (C. F. 1878).

7. Arithmetic means. When three quantities are in A.P. the middle one is called the **Arithmetic mean** (A.M.) of the other two.

Let a, b, c, be in A.P.; then b is the Arithmetic Mean between a and c. Now from definition $b-a=c-b$, whence $b=\frac{1}{2}(a+c)$, or *the Arithmetic mean of two quantities is half their sum.*

If any number of quantities are in A.P. all the intermediate quantities are called the *Arithmetic means of the two extremes.*

8. *To insert n Arithmetic means between two given quantities a and c.*

Let $x_1, x_2, x_3, \ldots\ldots\ldots x_n$ be the required means.

Then $a, x_1, x_2, x_3, \ldots\ldots\ldots x_n, c$ are $n+2$ quantities in A.P.

Let b = common difference of the A.P.

Then $c=(n+2)$th term $=a+(n+1)b$.

$$\therefore b=\frac{c-a}{n+1}.$$

$$\therefore x_1=a+b=a+\frac{c-a}{n+1}=\frac{na+c}{n+1};$$

$$x_2=a+2b=a+\frac{2(c-a)}{n+1}=\frac{a(n-1)+2c}{n+1}, \text{ etc.}$$

....

$$x_n=a+nb=a+\frac{n(c-a)}{n+1}=\frac{nc+a}{n+1}.$$

Ex. 1. Insert 4 A.M.'s between 5 and 20.

Let x_1, x_2, x_3, x_4, be the means.

Then $5, x_1, x_2, x_3, x_4, 20$ are in A.P.

Suppose b = common difference of this A.P.

Then $20=5+(6-1)b=5+5b$. $\therefore b=3$.

$\therefore$ the means are 8, 11, 14, 17.

Ex. 2. Find the number of A.M.'s between 1 and 25 when the 3rd mean is to the 5th mean as 5 to 8.

Let n = number of means and b = common difference of the corresponding A.P.

Then $25=1+(n+1)b$..............(1)

Also $\frac{1+3b}{1+5b}=\frac{5}{8}$.........(2)

Solving (1) and (2) $b=3$, $n=7$.

EXERCISE CXXX.

1. Insert (1) 9 A. M.'s between -5 and 21.
(2) 20 A.M.'s between $7\frac{1}{2}$ and 1^{2}.

2. Find the number of A.M's between 2 and 35 such that the ratio of the 4th mean to the 8th mean is as 7 to 13.

3. Find the number of A. M's between 5 and 23 such that the greatest is 3 times the least.

4. Find the number of A.M.'s between 4 and 31 so that their sum is to the greatest of them as 5 to 1.

5. There are n means between 2 and 22 and the 3rd mean : $(n-4)$th mean $= 2 : 3$; find n.

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9. We shall conclude the subject of Arithmetical Progression with a few illustrative examples.

Ex. 1. Find the sum of all numbers of 5 digits which are divisible by 7.

Here the first term $=10003$, last term $=99995$, respectively the least and the greatest number of 5 digits divisible by 7 and common difference $=7$.

Let n = number of terms ; then we have

$$99995 = 10003+(n-1)7 \text{ whence } n=12857.$$

$$\therefore \text{ the sum required} = \frac{12857}{2}(10003+99995)$$

$$=707122143.$$

Ex. 2. Find the sum of the series in the nth group of

$$1+(2+3)+(4+5+6)+(7+8+9+10)+\dots\dots\dots$$

Number of terms in the first n groups $= 1+2+3+\dots+n = \frac{n(n+1)}{2}$

$=\frac{n^2+n}{2} =$ N(suppose), [for the *1st* group contains 1 term, the *2nd* contains 2,...the nth n], for a similar reason number of terms in the first $(n-1)$ groups $=\frac{(n-1)n}{2}=\frac{n^2-n}{2}=$ N$'$ (suppose).

$\therefore$ Sum of the terms in the nth group = sum of the first N terms of the series $(1+2+3+4+\dots\dots)$ − sum of the first N' terms of the same series

$$=\frac{N(N+1)}{2}-\frac{N'(N'+1)}{2}=\tfrac{1}{2}(N^2-N'^2)+\tfrac{1}{2}(N-N')$$

$$=\tfrac{1}{2}(N-N')(N+N'+1)=\left(\frac{n^2+n}{2}-\frac{n^2-n}{2}\right)\times$$

$$\left(\frac{n^2+n}{2}+\frac{n^2-n}{2}+1\right)=\frac{n(n^2+1)}{2}.$$

Ex. 3. Sum to n terms the series $1-7+13-19+25-\ldots\ldots$

(i) Let n be even. Then the series

$$=(1-7)+(13-19)+(25-31)+\ldots\ldots\ldots\text{to } \frac{n}{2} \text{ terms.}$$

$$=-6-6-6-\ldots\text{to } \frac{n}{2} \text{ terms.}$$

$$=-6\times\frac{n}{2}=-3n.$$

(ii) Let n be odd, so that $n-1$ is even. Then the series

$$=1-\{7-13+19-25+31-37+\ldots\text{to } (n-1) \text{ terms}\}$$

$$=1-\left\{(7-13)+(19-25)+(31-37)+\ldots\text{to } \frac{n-1}{2} \text{ terms.}\right\}$$

$$=1-\left\{-6-6-6-\ldots\text{to } \frac{n-1}{2} \text{ terms.}\right\}$$

$$=1+6.\frac{n-1}{2}=1+3(n-1)=3n-2.$$

To put both the results in one form we express them as the sum and difference of two quantities thus :

$$-3n=-1+(1-3n),\ 3n-2=-1-(1-3n).$$

Hence in one expression the sum is $-1+(-1)^n(1-3n)$, whether n is even or odd.

Ex. 4. The sum of the latter half of $2n$ terms of an A.P. is equal to one-third of the sum of $3n$ terms of the same series.

Let the series of $3n$ terms be

$$(t_1+t_2+t_3+\ldots\ldots+t_n)+(t_{n+1}+t_{n+2}+\ldots+t_{2n})+(t_{2n+1}+t_{2n+2}+\ldots+t_{3n})$$

Let the constant sum of any two terms equidistant from the beginning and the end respectively be k.

$$\text{Then } t_{n+1}+t_{n+2}+\ldots+t_{2n}=\frac{n}{2}(t_{n+1}+t_{2n})=\frac{kn}{2}\ldots\ldots(1).$$

$$\text{Also}\quad t_1+t_3+\ldots+t_{3n}=\frac{3n}{2}(t_1+t_{3n})=\frac{3kn}{2}\ldots\ldots\ldots(2).$$

Hence from (1) and (2) by division

$t_{n+1}+t_{n+2}+\ldots+t_{2n}=\frac{1}{3}(t_1+t_2+\ldots+t_{3n})$, which is to be proved.

Ex. 5. If s_n denote the sum of n terms of an A.P., prove that $s_{n+2}-2s_{n+1}+s_n=$ common difference.

Let t_n denote the nth term of the series.

$$\text{Then } s_{n+2}-2s_{n+1}+s_n=(s_{n+2}-s_{n+1})-(s_{n+1}-s_n)$$
$$=t_{n+2}-t_{n+1}=\text{common difference.}$$

Ex. 6. If a^2, b^2, c^2 be in A.P., then $\frac{1}{b+c}$, $\frac{1}{c+a}$, $\frac{1}{a+b}$ are in A.P.

Since a^2, b^2, c^2 are in A. P.,

$\therefore$ $a^2+bc+ca+ab$, $b^2+bc+ca+ab$, $c^2+bc+ca+ab$ are in A. P.

$\therefore$ $(a+b)(a+c)$, $(b+c)(b+a)$, $(c+a)(c+b)$ are in A. P.

Hence dividing each term by $(b+c)(c+a)(a+b)$,

we have $\frac{1}{b+c}$, $\frac{1}{c+a}$, $\frac{1}{a+b}$ in A. P.

Ex. 7. If $\frac{1}{bc}$, $\frac{1}{ca}$, $\frac{1}{ab}$ be in A. P. show that

$\frac{a(b+c)}{bc}$, $\frac{b(c+a)}{ca}$, $\frac{c(a+b)}{ab}$ are in A. P.

Since $\frac{1}{bc}$, $\frac{1}{ca}$, $\frac{1}{ab}$ are in A. P.

$\therefore$ $\frac{bc+ca+ab}{bc}$, $\frac{bc+ca+ab}{ca}$, $\frac{bc+ca+ab}{ab}$ are in A. P.

or $1+\frac{a(b+c)}{bc}$, $1+\frac{b(c+a)}{ca}$, $1+\frac{c(a+b)}{ab}$ are in A. P.

Hence $\frac{a(b+c)}{bc}$, $\frac{b(c+a)}{ca}$, $\frac{c(a+b)}{ab}$ are in A. P.

Ex. 8. Find 3 numbers in *A. P.* whose sum is 27, and product 693.

Let $x-y$, x and $x+y$ be the numbers.

Then $(x-y)+x+(x+y)=27$.........(1)

and $x(x^2-y^2)=693$.......(2)

From (1) $3x=27$ or $x=9$.

$\therefore$ from (2) $9(81-y^2)=693$: $\therefore$ $y^2=4$ or $y=\pm 2$.

Hence the numbers are 7, 9, 11.

Ex. 9. Find 4 numbers in *A. P.* such that the product of the extremes is 10 and the product of the means, 28.

Let $x-3y$, $x-y$, $x+y$, $x+3y$ be the numbers.

Then $x^2-9y^2=10$, $x^2-y^2=28$.

Solving these $x=\pm 11/2$, $y=\pm 3/2$.

Hence the numbers are 1, 4, 7, 10 ; or -1, -4, -7, -10.

Note. The student will note the advantage of writing the terms of an A. P. as in examples 4 and 5. If the number of terms be odd, we put x for the middle term and supposing y to be the common difference, the terms on either side of x can be written down. If the number of terms be even, we put $x-y$, $x+y$ as two middle terms and the common difference being thus $2y$, the other terms are known.

EXERCISE CXXXI.

1. Find the sum of all the even integers of 3 digits.

2. Find the sum of all numbers of 4 digits which are divisible by 6.

3. Find the sum of (i) all even numbers (ii) all odd numbers of 5 digits which are divisible by 9.

4. Find the sum of the consecutive integers from (i) 60 to 150 (ii) m to n.

5. Find the sum of all odd numbers from $2m+1$ to $2n+1$.

6. Find the sum of the series in the nth group of

(1) $3+(5+7)+(9+11+13)+(15+17+19+21)+\ldots\ldots$

(2) $1+(2+3+4)+(5+6+7+8+9)+\ldots\ldots$

7. Find the sum to 20 terms of

$3+4+8+9+13+14+18+19+\ldots\ldots\ldots$ (C. F. 1881).

8. Find the nth term and the sum to n terms of

(1) $1-5+9-13+\ldots\ldots$ (2) $3-8+13-18+\ldots\ldots$

9. The 15th term of an A. P. is 31 and the sum to 20 terms is 440, find the sum of 100 terms of the series.

10. The ratio of the sums to n terms of two A. P.'s is $2n+1 : 4n-5$, find the ratio of their 16th terms.

11. The sums of n terms of two Arithmetic series are as $3n+31 : 5n-3$; shew that their 9th terms are the same.

12. The sides of a rt. angled triangle are in A. P., prove that they are proportional to 3, 4, 5.

13. If s_1, s_2, and s_3 are the sums of n, $2n$, and $3n$ terms respectively of an A. P., shew that $s_3=3(s_2-s_1)$.

14. If the mth term of an A. P. be n and the nth term m; of how many terms is the sum $\frac{1}{2}(m+n)(m+n-1)$, and what is the last term?

15. The sum of the first n terms and the last n terms of an A. P. is equal to twice the sum of the intermediate n terms.

16. If the sum of an A. P. is the same for m terms as for n terms, shew that the sum of $m+n$ terms is zero.

17. The sum of $p+q$ terms of an A. P. is $2p$, and of $p-q$ terms is $2q$, find the sum of p terms and also of q terms.

18. If s_n denote the sum of n terms of an A. P.; prove that

(i) $s_{n+3}-3s_{n+2}+3s_{n+1}-s_n=0$.

(ii) $s_{n+4}-4s_{n+3}+6s_{n+2}-4s_{n+1}+s_n=0$.

19. The sum of three numbers in A. P. is 24 and the sum of their squares is 210, find the numbers.

20. The sum of three numbers in A. P. is 18 and the sum of their cubes 1224; find the numbers.

21. The sum of 5 numbers in A. P. is 25 and the sum of their squares 165; find the numbers.

22. A person is employed to count Rs. 12,000. He counts at the rate of Rs. 150 per minute for an hour, at the end of which time he begins to count at the rate of Rs. 2 less every minute than he did the previous minute. Find when he will finish the task and explain the fact that two solutions occur. (M. F. 1886).

23. A debt can be discharged in a year by paying Rs. 5 in the first week, Rs. 10 in the second week, Rs. 15 in the third week and so on. *Find the amount of the debt.*

II. Geometrical Progression.

10. A series is said to be a **Geometrical Progression** (or shortly G.P.) when the ratio of any term to the *preceding* term is constant. This constant quantity is called the *common ratio* of the series which is therefore equal to any term of the series divided by the *preceding* term.

Thus the series 3, 6, 12, 24, ... $5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$; are Geometrical Progressions.

In the first series the common ratio is 2 and in the second it is $\frac{1}{2}$. The first series in which the common ratio is greater than 1 is an increasing G. P. and the second in which the common ratio is less than 1 is a decreasing G. P.

Note. It easily follows that if each term of a G. P. be multiplied or divided by the same quantity, the resulting series is another G. P. with the same common ratio. Also if all the terms of a G. P. are raised to the same power, the resulting series is a G. P. with a new common ratio.

It is evident that a number of quantities in G. P. are in *continued proportion.*

11. The G. P. of which the first term is a and the common ratio r is $a, ar, ar^2, ar^3, ar^4, \ldots\ldots$

Here we observe that in any term the index of the power of r is less, by 1, than the number giving the position of the term in the series; hence we can write down any term of the series:

for example, the 10th term $= ar^{10-1} = ar^9$,

the 16th term $= ar^{16-1} = ar^{15}$,

and generally, **the** n**th term** $= ar^{n-1}$.

Ex. Find the 12th and nth terms of the series $3, 2, \frac{4}{3}, \ldots$

Here $a = 3$, $r = \frac{2}{3}$;

$\therefore$ 12th term $= ar^{11} = 3.(\frac{2}{3})^{11} = \dfrac{2^{11}}{3^{10}}$.

Also nth term $=ar^{n-1}=3(\frac{2}{3})^{n-1}$.

12. **In a G. P. the product of any two terms equidistant from the beginning and the end is the same.**

Let the series be $a, ar, ar^2 ... ar^{n-1}$, there being n terms in all.

Now the pth term of the series from the beginning ar^{p-1}, and the pth term from the end $=(n-p+1)$th term from the beginning $=ar^{n-p}$.

$\therefore$ the product of the pth terms of the series equidistant from the beginning and the end $=ar^{p-1}.\ ar^{n-p}$

$=a.\ ar^{n-1}$

$=$first term $\times$ last term.

Hence the product is the same for all values of p. It thus follows that if the number of terms of a $G\ P$. be odd, this constant product is the square of the middle term; and if the number be even, it is the product of the two middle terms.

13. **If any two terms of a G.P. are given, the series is completely known.**

Ex. 1. Find the 10th term of the G. P. of which the 4th term is $\frac{27}{2}$ and the 8th term $\frac{2187}{32}$.

Let a=first term and r=common ratio.

Then $ar^3=\frac{27}{2}$, $ar^7=\frac{2187}{32}$.

$\therefore$ by division $r^4=\frac{2187}{32}\times\frac{2}{27}=\frac{81}{16}=(\frac{3}{2})^4$. $\quad\therefore\ r=\frac{3}{2}$.

Hence $a=\frac{27}{2}\div r^3=\frac{27}{2}\times\frac{8}{27}=4$.

$\therefore$ the 10th term $=ar^9=4.\ (\frac{3}{2})^9=\frac{19683}{128}$.

Ex. 2. If the pth term of a G.P. be m and the qth term n, find the $(p+q)$th term.

Let a=first term, r=common ratio.

Then $m=ar^{p-1}$......(1), $n=ar^{q-1}$......(2)

Dividing (1) by (2), $\frac{m}{n}=r^{p-q}$, or, $r=\left(\frac{m}{n}\right)^{\frac{1}{p-q}}$......(3)

Now the $(p+q)$th term $=ar^{p+q-1}$

$=ar^{p-1}.r^q=m\left(\frac{m}{n}\right)^{\frac{q}{p-q}}$ from (1) and (3).

Ex. 3. Find the condition that P, Q, R, may be respectively the pth, the qth and the rth terms of the same G. P.

Let a=first term, k=common ratio.

Then $P=ak^{p-1}$............(1).

$Q=ak^{q-1}$............(2).

$R=ak^{r-1}$............(3).

The required condition is obtained by eliminating a and k from (1), (2) and (3).

Raise (1), (2), (3) to the powers $q-r$, $r-p$, $p-q$ respectively and multiply the results together.

Then $P^{q-r}.\ Q^{r-p}.\ R^{p-q}$

$$=a^{q-r}\,a^{r-p}\,a^{p-q}.\ k^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$$

$$=a^{\circ}.\ k^{\circ}=1.$$

EXERCISE CXXXII.

1. Find

(1) 12th term of the series 4, 12, 36.........

(2) 7th term of the series [illegible], -1, [illegible]............

(3) nth term of the series $\sqrt{3}$, $\frac{1}{\sqrt{3}}$, $\frac{1}{3\sqrt{3}}$, ...

2. Is [illegible] a term of the series [illegible]......?

3. In a G. P. if the $(p+q)$th term $=m$ and the $(p-q)$th term $=n$, find the pth and the qth terms. (B. P. 1888)

4. Determine the G. P. of which

(1) the 10th term $=$ [illegible] and the 14th term [illegible].

(2) the 15th term $=1$ and the 9th [illegible] $=$ [illegible].

5. Prove that the pth term of a G. P. is a mean proportional between the $(p+q)$th and the $(p-q)$th terms.

6. Show that the $2n$th term of a geometrical series is a mean proportional between the nth and $3n$th terms.

---

14. *To find the sum of a number of terms in G. P.*

Let a be the first term and r the common ratio of the G.P. Let n be the number of terms and s the required sum.

$$\text{Then } s=a+ar+ar^2+\ldots\ldots+ar^{n-1}.$$

$$\therefore\ rs=\qquad ar+ar^2+\ldots\ldots+ar^{n-1}+ar^n.$$

Hence by subtraction,

$$s(1-r)=a-ar^n=a(1-r^n).$$

$$\therefore\ s=\frac{a(1-r^n)}{1-r}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{or } =\frac{a(r^n-1)}{r-1}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The formulæ (1) and (2) should be committed to memory by the student. He is advised to use (1) always except when r is positive and greater than unity.

If l denote the last term, so that $l=ar^{n-1}$, we get from (2) $s=\dfrac{rl-a}{r-1}$, a form not so useful as (1) or (2).

Note. It appears from the above that a Geometrical Series can be summed by simply multiplying it by $1-r$.

Ex. 1. Sum to 6 terms the series $\frac{2}{5}+1+\frac{5}{2}+\ldots\ldots\ldots$

Here $a=\frac{2}{5}$, $r=\frac{5}{2}$, $n=6$.

$$\therefore\ s=\frac{a(r^n-1)}{r-1}=\frac{\frac{2}{5}\{(\frac{5}{2})^6-1\}}{\frac{5}{2}-1}=\frac{\frac{2}{5}(\frac{15625}{64}-1)}{\frac{3}{2}}$$

$$=\tfrac{2}{5}\times\tfrac{15561}{64}\times\tfrac{2}{3}=\tfrac{5187}{80}.$$

Ex. 2. Sum to n terms the G. P. whose $3rd$ term $=\frac{12}{25}$ and 6th term $=\frac{96}{3125}$.

Let $a=$ first term, $r=$ common ratio.

Then $ar^2=\frac{12}{25}$, $ar^5=\frac{96}{3125}$. $\therefore$ Solving, $r=\frac{2}{5}$, $a=3$.

Hence the sum $=\dfrac{3\{1-(\frac{2}{5})^n\}}{1-\frac{2}{5}}=5\left(1-\dfrac{2^n}{5^n}\right)$.

EXERCISE CXXXIII.

1. Sum (1) $5+30+180+\ldots$to 10 terms.

(2) $\frac{3}{7}+1+\frac{7}{3}+\ldots$to 7 terms.

(3) $1-\frac{2}{3}+\frac{4}{9}-\ldots$to 6 terms.

(4) $\dfrac{1}{\sqrt{3}}+1+\dfrac{3}{\sqrt{3}}+\ldots$to 18 terms.

(5) $\frac{2}{3}-\sqrt{\frac{2}{3}}+1-\ldots$to n terms.

(6) $2+\sqrt{2}+1+\ldots$to n terms.

(7) $(\sqrt{2}+1)+1+(\sqrt{2}-1)+\ldots$to n terms.

(8) $\dfrac{a+b}{a-b}+1+\dfrac{a-b}{a+b}+\ldots$to n terms.

2. Sum to n terms the series whose nth term $=2.3^n$.

3. Sum to 15 terms the G.P. of which the 5th term $=\frac{8}{81}$ and the 7th term $=\frac{32}{729}$.

4. If the pth term of a G.P. be P and the qth term Q, find the sum of n terms of the series.

5. The sum of the first 6 terms of a G. P. is 9 times the sum of the first 3 terms; find the common ratio.

6. The sum of the first 8 terms of a G.P. is 510 and the sum of the first 4 terms is 30; find the series.

15. Geometrical means. When three quantities are in G. P. the middle one is called the **Geometric mean** (or shortly G.M.) of the other two.

Let a, b, c be in G. P., then b is the Geometric mean between a and c. Now from def. $\frac{b}{a}=\frac{c}{b}$, whence $b^2=ac$ or $b=\sqrt{ac}$, that is, *the geometric mean of two quantities is the square root of their product.*

When any number of quantities are in G.P., all the intermediate quantities are called the **Geometric means** of the two extremes.

16. To insert n geometric means between two given quantities a and c.

Let x_1, x_2, x_3,......x_n be the G. M.'s.

Then a, x_1, x_2,......x_n, c are $(n+2)$ quantities in G. P.

Let r = common ratio of this G.P.

Then $c=(n+2)$th term $=ar^{n+1}$, $\therefore\ r=\left(\frac{c}{a}\right)^{\frac{1}{n+1}}$

$$\therefore\ x_1=ar=a\left(\frac{c}{a}\right)^{\frac{1}{n+1}},\ x_2=ar^2=a\left(\frac{c}{a}\right)^{\frac{2}{n+1}},......x_n=ar^n=a\left(\frac{c}{a}\right)^{\frac{n}{n+1}}.$$

Ex. Insert 3 G. M.'s between 4 and 324 (C. F. 1890).

Let x_1, x_2, x_3, be the means.

Then 4, x_1, x_2, x_3, 324 are in G.P. Let r = common ratio.

Then $324=4r^4$, $\therefore\ r^4=81$ or $r=3$.

Hence $x_1=12$, $x_2=36$, $x_3=108$.

EXERCISE CXXXIV.

1. Insert—(i) 4 G.M.'s between [illegible] and [illegible];

(ii) 3 G.M.'s between [illegible] and [illegible];

(iii) 6 G.M.'s between 5 and [illegible];

(iv) 2 G.M.'s between x and y.

2. What must be added to a, b, c to bring them into G.P.?

3. The geometric mean between a and b is to their Arithmetic mean as m is to n. Shew that $a : b=n+\sqrt{(n^2-m^2)} : n-\sqrt{(n^2-m^2)}$

4. If a, b, c be in G.P. and x, y be the Arithmetic means between a, b and b, c respectively prove that—

$$\frac{2}{b}=\frac{1}{x}+\frac{1}{y}\ \text{and}\ 2=\frac{a}{x}+\frac{c}{y}.\quad \text{(P. I. 1892).}$$

5. If one A. M., A, and two G. M.'s, G_1, G_2 are inserted between two quantities, then $G_1^3+G_2^3=2AG_1G_2$

6. If one G. M., G and two A. M.'s A_1, A_2 are inserted between two quantities, then $G^2=(2A_1-A_2)(2A_2-A_1)$.

---

17. Infinite Geometrical Progression.—If s denote the sum of n terms of the G. P. $a+ar+ar^2+......$we have—

$$s=\frac{a(1-r^n)}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}...........(1).$$

If we suppose r to be less than 1, then evidently each term of $r^1, r^2, r^3,.........$ is less than the preceding, in other words, r goes on diminishing as n increases; and by making n larger and larger we can make r^n smaller, and smaller, and ultimately less than any small quantity which can be named. Hence in this case r^n and ∴ also $\frac{a}{1-r}r^n$ can be made to differ from zero by as small a quantity as we please. Hence from (1) the sum of a sufficiently large number of terms of the G. P. differs from $\frac{a}{1-r}$ by a quantity *viz*, $\frac{a}{1-r}\cdot r^n$ which ultimately vanishes. Thus as we take more and more terms of the series, the sum continually approaches the limit $\frac{a}{1-r}$ but never exceeds it. This is shortly stated thus: "the limiting value of the sum of an infinite number of terms of the G. P. $a+ar+ar^2+...$(where $r<1$) is $\frac{a}{1-r}$, or the limit of the sum to infinity of a decreasing G. P.

$$=\frac{\textbf{first term}}{\textbf{1 - common ratio}}"$$

Note 1. If $r>1$, r^n increases with n and ∴ $\frac{a}{1-r}\cdot r^n$ can be made larger than any large quantity, when n is taken very large. Hence the sum of an infinite number of terms of an *increasing* G. P. is infinitely large.

Note 2. This is perhaps the first occasion on which the student is confronted in his studies with the paradoxical statement that the sum of an infinite number of terms of a series may be a *finite* quantity. The notion of limit introduced here is very important and the student will have more of it in higher Mathematics.

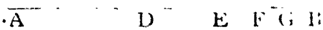
Ex. 1. Sum to infinity $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots$

$$\text{Sum of } n \text{ terms} = \frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 1-\frac{1}{2^n}.$$

Now by increasing n more and more $\frac{1}{2^n}$ can be made as nearly equal to zero as we please. Hence the sum to infinity is 1.

This may be illustrated geometrically :—Take a str. line AB of unit length.

Bisect AB at D, DB at E, EB at F, FB at G, and so on.



Then $AD+DE+EF+FG+\ldots$

$=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots\ldots\ldots\ldots$

But the former sum approaches the limit AB or 1 ; hence also

$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots=1$ in the limit.

Ex. 2. Sum to infinity : $\sqrt{3}+\frac{1}{\sqrt{3}}+\frac{1}{3\sqrt{3}}+\ldots$ .. (C. F. 1886.)

$$\text{Required sum} = \frac{\textit{first term}}{1-\textit{common ratio}} = \frac{\sqrt{3}}{1-\frac{1}{3}} = \frac{3\sqrt{3}}{2}.$$

18. Recurring Decimals. Recurring decimals furnish good illustrations of infinite Geometrical Progressions.

Thus $\cdot 2\dot{3}\dot{7} = \cdot 237373737\ldots\ldots\ldots$

$= \cdot 2$

$+ \cdot 037$

$+ \cdot 00037$

$+ \cdot 0000037$

$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$=\frac{2}{10}+\frac{37}{10^3}+\frac{37}{10^5}+\frac{37}{10^7}+\ldots\ldots\ldots$$

$$=\frac{2}{10}+\frac{\frac{37}{10^3}}{1-\frac{1}{10^2}}=\frac{2}{10}+\frac{37}{990}=\frac{235}{990}.$$

To find in the general case the value of a recurring decimal regarding it as an infinite Geometrical Progression.

Let $\cdot P\dot{Q}$ be the recurring decimal and let the number of digits in the non-recurring part P be p and that in the recurring part Q be q. Let S denote the value of the recurring decimal.

Then $s = \cdot PQQQQ\ldots\ldots\ldots$

$$= \frac{P}{10^p} + \frac{Q}{10^{p+q}} + \frac{Q}{10^{p+2q}} + \frac{Q}{10^{p+3q}} + \ldots\ldots$$

$$= \frac{P}{10^p} + \frac{\dfrac{Q}{10^{p+q}}}{1 - \dfrac{1}{10^q}} = \frac{P}{10^p} + \frac{Q}{10^p(10^q - 1)}.$$

$$= \frac{(P.10^q + Q) - P}{10^p(10^q - 1)} = \frac{PQ - P}{10^p(10^q - 1)}.$$

Now since $10^q - 1$ consists of q nines, we have the Arithmetical rule of reducing a recurring decimal to a vulgar fraction.

19. In a decreasing infinite G. P. each term bears a constant ratio to the sum of all the terms which follow it.

Let the G. P. be $a, ar, ar^2\ldots\ldots\ldots$

Now the nth term $= ar^{n-1}$, and the sum of all terms after it

$$= ar^n + ar^{n+1} + \ldots\ldots = \frac{ar^n}{1-r}.$$

Hence nth term : sum of all the terms following it

$$= ar^{n-1} : \frac{ar^n}{1-r} = \frac{1-r}{r}, \text{ which is independent of } n.$$

Thus the ratio is constant whatever n may be.

Ex. In an infinite G. P. the ratio of any term to the sum of all the terms which follow it, is $\frac{2}{3}$; find the common ratio of the series.

Let the G.P. be $a, ar, ar^2, \ldots\ldots$

Then the nth term : sum of all terms which follow it.

$$= ar^{n-1} : \frac{ar^n}{1-r} = \frac{1-r}{r}.$$

$$\therefore \frac{1-r}{r} = \frac{2}{3}, \text{ whence } r = \frac{3}{5}.$$

EXERCISE CXXXV.

1. Sum to infinity

(1) $1+\frac{1}{3}+\frac{1}{3^2}+\ldots\ldots$ (2) $\frac{5}{8}+\frac{1}{2}+\frac{2}{5}+\frac{8}{25}+\ldots\ldots$

(3) $(2+\sqrt{3})+1+(2-\sqrt{3})+\ldots$ (4) $3-1+\frac{1}{3}-\frac{1}{9}+\ldots\ldots$

2. Find from the principles of G. P. the value of

(1) $\cdot\dot{2}$. (2) $2\cdot\dot{3}4\dot{5}$. (3) $5\cdot721\dot{4}\dot{5}$.

3. In an infinite G. P. if the ratio of any term to the sum of all the terms which follow it is $\frac{3}{7}$, find the common ratio.

4. The first term of a Geometric series continued to infinity is 1, and any term is equal to the sum of all the succeeding terms; find the series. (M. F. 1881).

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20. We shall conclude the subject of Geometrical Progression with a few illustrative examples.

Ex. 1. Find the sum of n groups of

$$(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\ldots\ldots$$

The required sum

$$=\frac{x^2-y^2}{x-y}+\frac{x^3-y^3}{x-y}+\frac{x^4-y^4}{x-y}+\ldots\ldots\text{to } n \text{ terms}$$

$$=\frac{1}{x-y}\left\{(x^2-y^2)+(x^3-y^3)+(x^4-y^4)+\ldots\ldots\text{to } n \text{ terms}\right\}$$

$$=\frac{1}{x-y}\left\{(x^2+x^3+x^4+\ldots\text{to } n \text{ terms})-(y^2+y^3+y^4+\ldots\text{to } n \text{ terms})\right\}$$

$$=\frac{1}{x-y}\left\{\frac{x^2(1-x^n)}{1-x}-\frac{y^2(1-y^n)}{1-y}\right\}.$$

Ex. 2. Find the sum of the series in the nth group of $1+(2+2^2)+(2^3+2^4+2^5)+(2^6+2^7+2^8+2^9)+\ldots\ldots$

The number of terms in the first n groups $=1+2+3+\ldots\ldots+n$ $=\frac{n(n+1)}{2}=N$ (suppose).

The number of terms in the first $(n-1)$ groups $=1+2+3+\ldots\ldots+(n-1)=\frac{(n-1)n}{2}=N'$ (suppose).

Hence the required sum

$=$ sum of N terms of the series $(1+2+2^2+2^3\ldots\ldots)$

$-$ sum of N' terms of the same series

$$=\frac{2^N-1}{2-1}-\frac{2^{N'}-1}{2-1}=2^N-2^{N'}$$

$$=2^{n(n+1)/2}-2^{(n-1)n/2}=2^{(n-1)n/2}(2^n-1).$$

Ex. 2. Find the sum to 10 terms of $4+44+444+4444+\ldots\ldots$

The required sum $=4+44+444+\ldots\ldots$ to 10 terms

$$=4(1+11+111+\ldots\ldots\text{to 10 terms})$$
$$=\tfrac{4}{9}(9+99+999+\ldots\ldots\text{to 10 terms})$$
$$=\tfrac{4}{9}\{(10-1)+(10^2-1)+(10^3-1)+\ldots\text{to 10 terms}\}$$
$$=\tfrac{4}{9}(10+10^2+\ldots\text{to 10 terms}-10)$$
$$=\tfrac{40}{9}\left\{\frac{10^{10}-1}{9}-1\right\}$$
$$=\tfrac{40}{9}\cdot\frac{10^{10}-10}{9}=\frac{400}{9}\cdot\frac{10^9-1}{9}$$
$$=\tfrac{400}{9}\cdot\tfrac{999999999}{9}=4938271600.$$

Note, $\cdot4+\cdot44+\cdot444+\ldots\ldots\ldots$ to n terms

$$=\tfrac{4}{9}(\cdot9+\cdot99+\cdot999+\ldots\ldots\text{to } n \text{ terms})$$
$$=\tfrac{4}{9}\{(1-\cdot1)+(1-\cdot01)+(1-\cdot001)\ldots\ldots\ldots\text{to } n \text{ terms}\}$$
$$=\tfrac{4}{9}[n-\{\cdot1+(\cdot1)^2+(\cdot1)^3\ldots\ldots\ldots\text{to } n \text{ terms}\}]=\text{etc.}$$

Ex. 3. Find 3 numbers in G. P. of which the sum is 26 and product 216.

Let $\frac{x}{y}$, x, xy be the numbers. Then $\frac{x}{y}+x+xy=26\ldots$ (1)

$\frac{x}{y}.x.xy=216\ldots\ldots\ldots\ldots$(2).

From (2) $x^3=216$, or, $x=6$.

Hence from (1) $6/y+6+6y=26$. or $3y^2-10y+3=0$.

$\therefore$ $y=3$, or, $1/3$.

Hence the numbers are 2, 6, 18.

Ex. 4. There are four numbers in *A.P.* and if they are respectively increased by 1, 2, 5, 12 they are in *G.P.* Find the numbers.

Let $x-3y$, $x-y$, $x+y$, $x+3y$ be the numbers.

Then $x-3y+1$, $x-y+2$, $x+y+5$, $x+3y+12$ are in *G.P.*

$$\therefore\ \frac{x-3y+1}{x-y+2}=\frac{x-y+2}{x+y+5}=\frac{x+y+5}{x+3y+12}=k \text{ (suppose).}$$

$$\text{Hence } k=\frac{(x-3y+1)+(x+y+5)-2(x-y+2)}{(x-y+2)+(x+3y+12)-2(x+y+5)}=\tfrac{2}{4}=\tfrac{1}{2}$$

$$\therefore\ \frac{x-3y+1}{x-y+2}=\tfrac{1}{2} \text{ and } \frac{x-y+2}{x+y+5}=\tfrac{1}{2}.$$

Solving these $x=5/2, y=\frac{1}{2}$. Thus the numbers are 1, 2, 3, 4.

EXERCISE CXXXVI.

1. Sum to n groups

$$(1+x)+(1+x+x^2)+(1+x+x^2+x^3)+\dots\dots\dots$$

2. Sum to n terms

$$\left(x-\frac{1}{x}\right)^2+\left(x^2-\frac{1}{x^2}\right)^2+\left(x^3-\frac{1}{x^3}\right)^2+\dots\dots\dots$$

3. Sum to n terms

(i) $7+77+777+\dots$ (ii) $\cdot 3+\cdot 33+\cdot 333+\dots\dots$

4. Sum to 8 terms $\frac{3}{2}+\frac{7}{2^2}+\frac{3}{2^3}+\frac{7}{2^4}+\dots\dots$

5. Sum to 12 terms $\frac{1}{2}+\frac{5}{2^2}+\frac{7}{2^3}+\frac{1}{2^4}+\frac{5}{2^5}+\frac{7}{2^6}+\dots\dots\dots$

6. Find the sum of the series in the nth group of

$$1+\left(\frac{1}{3}+\frac{1}{3^2}\right)+\left(\frac{1}{3^3}+\frac{1}{3^4}+\frac{1}{3^5}\right)+\dots.$$

7. If a, b, c, d be in G. P. then

(i) $(a+b+c+d)^2=(a+b)^2+(c+d)^2+2(b+c)^2$

(ii) $(a-d)^2=(b-c)^2+(c-a)^2+(d-b)^2$ (C. F. 1900.)

8. If s_n denote the sum of n terms of a given series in G. P., find the value of $s_1+s_2+s_3+\dots+s_n$.

9. If the pth, qth and rth terms of an A. P. are in G. P. of which the common ratio is t, then $q(1+t)=r+pt$.

10. If s_1, s_2, s_3, are the sums of n terms, $2n$ terms and to infinity of a G. P., show that $s_1(s_1-s_3)=s_3(s_1-s_2)$. (C. F. 1877).

11. If a, b, c be the pth, qth and rth terms both of an A. P. and a G. P., then $a^{b-c}\, b^{c-a}\, c^{a-b}=1$.

12. If s be the sum, P the product, and R the sum of the reciprocals of the n terms of the series a, ar, ar^2, &c., then $P^2=\left(\frac{s}{R}\right)^n$. (C. F. 1883).

13. There are four numbers in A. P. which being respectively diminished by $\frac{3}{2}$, 3, 4, 4 are in G. P.; find the numbers.

14. There four numbers in G. P.; which being diminished by 1, 1, 3, 9 are in A. P.; find the numbers.

15. The two numbers between which A is the A.M. and G is the G. M. are given by $A \pm \sqrt{(A+G)(A-G)}$.

III. Harmonical Progression.

21. A series is said to be a **Harmonical Progression** (or shortly H. P.) when any of its three consecutive terms x, y, z are such that $x : z = x - y : y - z$.

Thus, a, b, c, d,...form an *H. P.* if $a : c = a - b : b - c$,

$$b : d = b - c : c - d.$$

It follows from the definition that if all the terms of an H. P. are multiplied or divided by the same quantity, the resulting series is an H. P.

22. If a number of quantities form an H. P., their reciprocals form an A. P.

Let a, b, c, d, e...be in *H. P.*

Then from def. $a : c = a - b : b - c$. $\therefore c(a-b) = a(b-c)$.

Hence dividing both sides by abc, $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$.

Similarly, $\frac{1}{c} - \frac{1}{b} = \frac{1}{d} - \frac{1}{c}$,

$\frac{1}{d} - \frac{1}{c} = \frac{1}{e} - \frac{1}{d}$, and so on.

$$\therefore \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b} = \frac{1}{d} - \frac{1}{c} = \frac{1}{e} - \frac{1}{d} = \ldots$$

Hence the series $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{d}$, $\frac{1}{e}$,......is an A. P. which was to be proved.

It follows conversely from the above that if $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$.......form an A. P. then a, b, c,......form an H. P. Hence a Harmonical progression is some times defined thus :—

Reciprocals of the terms of an A. P. are said to form a Harmonical Progression.

Thus $\frac{1}{a}$, $\frac{1}{a+b}$, $\frac{1}{a+2b}$......is the general form of an H. P., the nth term being $\frac{1}{a+(n-1)b}$.

Ex. 1. Find the 10th term of the series 2, $\frac{6}{5}$, $\frac{6}{7}$, $\frac{2}{3}$,.........
Let x = 10th term of the given series which is an H. P.

Then $\frac{1}{x}$ = 10th term of the corresponding A. P. *viz*,

$$\tfrac{1}{2}, \tfrac{5}{6}, \tfrac{7}{6}, \tfrac{3}{2}, \ldots\ldots\ldots,$$

$$\therefore \frac{1}{x} = \tfrac{1}{2} + (10-1)\,\tfrac{1}{3} \text{ whence } x = \tfrac{2}{7}.$$

Ex. 2. Find the 11th term of the H. P. of which the 4th term $=\frac{2}{13}$ and the 7th term $=\frac{1}{8}$.

Let $x=$ 11th term of the H. P.

Then $\frac{1}{x}=$ 11th term of the A. P., of which the 4th term $=\frac{13}{2}$ and the 7th term $=8$.

Let $a=$ first term, $b=$ common diff. of this A. P.

Then $\left.\begin{aligned}\frac{13}{2}&=a+3b\\ 8&=a+6b\end{aligned}\right\}$ whence $a=5$, $b=\frac{1}{2}$.

$$\therefore \frac{1}{x}=a+10b=10. \qquad \therefore x=\frac{1}{10}.$$

Ex. 3. In a Harmonical Progression if the pth term is qr and the qth term pr, prove that the rth term is pq. (A. I. 1892).

Let $x=r$th term of the H. P.; and suppose $a=$ first term and $b=$ common difference of the corresponding A. P.

Then $\frac{1}{qr}=a+(p-1)b$.........(1) $\qquad \frac{1}{pr}=a+(q-1)b$...(2)

Subtracting (2) from (1) $\frac{1}{qr}-\frac{1}{pr}=b(p-q)$;

$$\therefore \frac{p-q}{pqr}=b(p-q), \text{ whence } b=\frac{1}{pqr}. \quad ..(3)$$

$$\text{Now } \frac{1}{x}=a+(r-1)b=a+(p-1)b-(p-r)b$$

$$=\frac{1}{qr}-(p-r)\frac{1}{pqr} \text{ from (1) and (3)}$$

$$=\frac{r}{pqr}=\frac{1}{pq}. \qquad \therefore x=pq.$$

Ex. 4. Find the condition that P, Q, R may be respectively be the pth, qth and rth terms of the same H. P.

Let $a=$ first term, $b=$ common difference of the corresponding A. P.

Then $\frac{1}{P}=a+(p-1)b$...(1), $\frac{1}{Q}=a+(q-1)b$...(2),

$\frac{1}{R}=a+(r-1)b$...(3).

The required condition is obtained by eliminating a, b, c from (1), (2), (3).

Multiply (1), (2), (3) respectively by $q-r$, $r-p$, $p-q$, and add then the right hand side vanishes, and we get

$$\frac{q-r}{P}+\frac{r-p}{Q}+\frac{p-q}{R}=0.$$

or $QR(q-r)+RP(r-p)+PQ(p-q)=0$.........(4)

EXERCISE CXXXVII.

1. Find the (i) 11th term of 2, 3, 6,...... ;

(ii) nth term of 4, $4\frac{2}{7}$, $4\frac{8}{13}$,......(C. F. 1886).

2. Continue the H. P. $\frac{1}{2}$, $\frac{2}{5}$, $\frac{1}{3}$ to 2 terms each way.

3. The first two terms of an H.P. are a, b; continue the series to 3 more terms.

4. Find the H.P. of which the 10th term $=\frac{1}{10}$ and the 15th term $=\frac{6}{85}$.

5. If the mth term of an H.P. is n and the nth term m, then the rth term will be mn/r (All. Inter. 1900), and the mnth term, 1.

6. If the pth term of an H.P. is q and the qth term p, prove that the $(p+q)$th term is $pq/(p+q)$.

---

23. Harmonic means. When three quantities are in H.P., the middle one is called the **Harmonic mean** (H.M.) of the other two.

Let a, b, c, be in H.P., then b is the Hormonic mean, between a and c. Now from def. $a:c=a-b:b-c$, $\therefore a(b-c)=c(a-b)$.

$$b(a+c)=2ac \text{ or } b=\frac{2ac}{a+c}.$$

It thus follows that the **Harmonic mean between two quantities is twice their product divided by their sum.**

If any number of quantities are in H. P., all the intermediate quantities are called the **Harmonic means** of the two extremes.

24. To insert n Harmonic Means between two given quantities a and c.

Let $x_1, x_2, x_3, \ldots x_n$ be the n H.M.'s.

Then $a, x_1, x_2, \ldots x_n, c$ are $n+2$ quantities in H.P.

$\therefore \frac{1}{a}, \frac{1}{x_1}, \frac{1}{x_2} \ldots \frac{1}{x_n}, \frac{1}{c}$ are $n+2$ quantities in A.P.

Let b=common diff. of this A.P.

Then $\frac{1}{c}=(n+2)\text{th term of the A. P.}=\frac{1}{a}+(n+1)b$.

$$\therefore (n+1)\,b=\frac{1}{c}-\frac{1}{a}=\frac{a-c}{ac} \quad \therefore b=\frac{a-c}{ac(n+1)}.$$

Hence $\frac{1}{x_1}=\frac{1}{a}+\frac{a-c}{ac(n+1)}=\frac{a+nc}{ac(n+1)}$,

$$\frac{1}{x_2}=\frac{1}{a}+\frac{2(a-c)}{ac(n+1)}=\frac{2a+(n-1)c}{ac(n+1)},$$

$$\cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot$$

$$\frac{1}{x_n}=\frac{1}{a}+\frac{n(a-c)}{ac(n+1)}=\frac{na+c}{ac(n+1)}.$$

Therefore the required means are

$$\frac{ac(n+1)}{a+nc},\quad \frac{ac(n+1)}{2a+(n-1)c},\quad \cdot\ \cdot\ \cdot\quad \frac{ac(n+1)}{na+c}.$$

25. The Arithmetic, Geometric and Harmonic Means of two unequal positive quantities form a decreasing Geometrical Progression.

Let A be the A.M., G. the G.M., and H. the H.M., between a and b. Then we have

$$A=\frac{a+b}{2},\quad G=\sqrt{ab},\quad H=\frac{2ab}{a+b}.$$

$$\therefore\ A.H.=\frac{a+b}{2}\cdot\frac{2ab}{a+b}=ab=G^2$$

Hence $\frac{A}{G}=\frac{G}{H}$ or A, G, H form a Geometrical Progression

$$\text{Again, } A-G=\frac{a+b}{2}-\sqrt{ab}=\tfrac{1}{2}(a+b-2\sqrt{ab})$$
$$=\tfrac{1}{2}(\sqrt{a}-\sqrt{b})^2.$$

Now $\sqrt{a}$ and $\sqrt{b}$ are both unequal and real, for a and b are both unequal and positive; hence $(\sqrt{a}-\sqrt{b})^2$ is positive.

$$\therefore\ A>G.$$

Also $\because \frac{A}{G}=\frac{G}{H}$, is follows that $G>H$.

Hence $A>G>H$.

$\therefore$ A, G, H are in descending order of magnitude.

Ex. 1. Insert 5 H. M.'s between $\frac{5}{3}$ and $\frac{5}{13}$.

Let x_1, x_2, x_3, x_4, x_5, be the H. M.'s.

Then $\frac{3}{5}$, $\frac{1}{x_1}$, $\frac{1}{x_2}$, $\frac{1}{x_3}$, $\frac{1}{x_4}$, $\frac{1}{x_5}$, $\frac{13}{5}$ are in A.P.

Let b = common diff. of this A.P.

$\therefore \frac{13}{5} = \frac{3}{5} + 6b$, whence $b = \frac{1}{3}$.

Hence $\frac{1}{x_1} = \frac{3}{5} + \frac{1}{3} = \frac{14}{15}$, $\frac{1}{x_2} = \frac{3}{5} + \frac{2}{3} = \frac{19}{15}$, $\frac{1}{x_3} = \frac{3}{5} + \frac{3}{3} = \frac{8}{5}$,

$$\frac{1}{x_4} = \frac{3}{5} + \frac{4}{3} = \frac{29}{15}, \quad \frac{1}{x_5} = \frac{3}{5} + \frac{5}{3} = \frac{34}{15}.$$

Thus the means are $\frac{15}{14}, \frac{15}{19}, \frac{5}{8}, \frac{15}{29}, \frac{15}{34}$.

Ex. 2. If a, b, c be three quantities such that a is the A. M between b and c, and c is the H. M. between a and b, shew that b is the G. M. between a and c, and that $c = a$ or $4a$. (A. I. 1899).

$\because$ a is the A.M. between b and c,

$\therefore 2a = b + c$, or, $b = 2a - c$............(1)

Also $\therefore$ c is the H.M. between a and b,

$$\therefore c = \frac{2ab}{a+b} \text{ whence } b = \frac{ac}{2a-c}\text{...(2)}$$

Hence multiplying (1) and (2) $b^2 = ac$ or b is the G.M. between a and c.

Also we have from (1) and (2) $2a - c = \frac{ac}{2a-c}$,

or $(2a-c)^2 = ac$ or $4a^2 - 5ac + c^2 = 0$.

$\therefore c = a$ or $4a$.

$\therefore c = a = b$ or $c = 4a = -2b$.

EXERCISE CXXXVIII.

1. Find the Harmonic Mean between

(i) 5, 9. (ii) $2\frac{1}{3}$, $\frac{3}{7}$. (iii) $a^2 + b^2$, $a^2 - b^2$.

2. Insert: (1) 5 H.M.'s between $\frac{2}{3}$ and $\frac{2}{7}$.

(2) 6 H.M.'s between $\frac{2}{11}$ and $-\frac{3}{8}$.

3. What must be added to each of three given quantities a, b, c to bring them into H.P.?

4. If A be the A.M. and the H the H.M. between a and b, shew that $\frac{a-A}{a-H} \times \frac{b-A}{b-H} = \frac{A}{H}$.

5. Compare a and b (1) if their A.M. : their H. M. $= m : n$.

(2) if their G.M. : their H. M. $= m : n$.

ELEMENTARY SERIES.

26. **To find the sum of the squares of the first n natural numbers.**

Let s denote the required sum.

We have $x^3-(x-1)^3=3x^2-3x+1$ identically.

In this identity make $x=1, 2, 3 \ldots n$ in succession;

$$
\begin{aligned}
\text{then}\quad 1^3-0^3 &= 3.1^2-3.1+1\\
2^3-3^3 &= 3.2^2-3.2+1\\
3^3-2^3 &= 3.3^2-3.3+1\\
&\ \ \vdots\\
n^3-(n-1)^3 &= 3.n^2-3.n+1
\end{aligned}
$$

Hence by addition we have

$$
\begin{aligned}
n^3 &= 3(1^2+2^2+3^2+\ldots+n^2)-3(1+2+3+\ldots+n)+n\\
&= 3s-\frac{3n(n+1)}{2}+n.
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ 3s=n^3-n+\frac{3n(n+1)}{2} &= n(n+1)(n-1)+\frac{3n(n+1)}{2}\\
&= \frac{n(n+1)(2n+1)}{2}.
\end{aligned}
$$

$$\therefore\ s=\tfrac{1}{6}n(n+1)(2n+1).$$

27. **To find the sum of the cubes of the first n natural numbers.**

Let s denote the required sum.

We have $x^4-(x-1)^4=4x^3-6x^2+4x-1$ identically.

In this identity make $x=1, 2, 3 \ldots\ldots\ldots n$;

$$
\begin{aligned}
\text{then}\ 1^4-0^4 &= 4.1^3-6.1^2+4.1-1\\
2^4-1^4 &= 4.2^3-6.2^2+4.2-1\\
3^4-2^4 &= 4.3^3-6.3^2+4.3-1\\
\ldots\quad &\quad \ldots\ldots\\
n^4-(n-1)^4 &= 4.n^3-6.n^2+4.n-1
\end{aligned}
$$

Hence adding column by column we have

$$
\begin{aligned}
n^4 &= 4(1^3+2^3+3^3+\ldots+n^3)-6(1^2+2^2+3^2+\ldots+n^2)\\
&\qquad +4(1+2+3+\ldots+n)-n.\\
&= 4s-6.\frac{n(n+1)(2n+1)}{6}+4.\frac{n(n+1)}{2}-n
\end{aligned}
$$

$$
\begin{aligned}
\therefore\quad 4s &= (n^4+n)+n(n+1)(2n+1)-2n(n+1)\\
&= n(n+1)(n^2-n+1)+n(n+1)(2n-1)\\
&= n(n+1)(n^2+n)=n^2(n+1)^2.
\end{aligned}
$$

$$\therefore\quad s=\left\{\frac{n(n+1)}{2}\right\}^2.$$

Thus the sum of the cubes of the first n natural numbers is equal to the square of the sum of these numbers.

Note. 1. In the same way we can find the sum of the fourth and higher powers of the first n natural numbers. In the case of the fourth power we should begin with the identity $x^5-(x-1)^5=5x^4-10x^3+10x^2-5x+1$, and the sum will be found to be $\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$.

Note. 2. We shall make use of the following notations :—

$\Sigma(n)$ to denote $1+2+3+\ldots+n$,

$\Sigma(n^2)$ to denote $1^2+2^2+3^2+\ldots+n^2$,

$\Sigma(n^3)$ to denote $1^3+2^3+3^3+\ldots+n^3$

28. We can find the sum of n terms of a series of which the nth term $t_n=An^3+Bn^2+Cn+D$; A, B, C, D being constant quantities not involving n.

$$\text{For, } t_1=A.1^3+B.1^2+C.1+D,$$
$$t_2=A.2^3+B.2^2+C.2+D,$$
$$t_3=A.3^3+B.3^2+C.3+D, \text{ etc. etc.}$$

Hence if s denote the sum of n terms of the series we have by addition

$s=A\Sigma(n^3)+B\Sigma(n^2)+C\Sigma(n)+n.D$, whence putting in the values of $\Sigma(n^3)$, $\Sigma(n^2)$, $\Sigma(n)$, s becomes known.

Ex. 1. Find the sum of n terms of the series $3^2+5^2+7^2+\ldots$

The nth term t_n of the series is the square of the nth term of the A.P. 3, 5, 7,...; hence $t_n=(2n+1)^2=4n^2+4n+1$.

$$\therefore t_1=4.1^2+4.1+1,$$
$$t_2=4.2^2+4.2+1,$$
$$t_3=4.3^2+4.3+1, \quad \text{etc.}$$

$$\therefore \text{required sum}=4\ \Sigma(n^2)+4\ \Sigma(n)+n$$
$$=4.\frac{n(n+1)(2n+1)}{6}+4.\frac{n(n+1)}{2}+n$$
$$=\frac{n}{3}\left\{2(n+1)(2n+1)+6(n+1)+3\right\}$$
$$=\frac{n(4n^2+12n+11)}{3}.$$

Ex. 2. Find the sum of n terms of the series $1.2.3+2.3.4+3.4.5+\ldots$

Here any term of the series is formed by multiplying the corresponding terms of three A.P.'s, *viz.*,

$$1,\ 2,\ 3,\ldots n$$
$$2,\ 3,\ 4,\ldots(n+1)$$
$$3,\ 4,\ 5,\ldots(n+2).$$

Hence if t_n denote the nth term of the given series,

we have $t_n = n(n+1)(n+2) = n^3 + 3n^2 + 2n$.

$$\therefore\ t_1 = 1^3 + 3.1^2 + 2.1,$$

$$t_2 = 2^3 + 3.2^2 + 2.2,$$

$$t_3 = 3^3 + 3.3^2 + 2.3, \qquad \text{etc.}$$

$$\therefore \text{ required sum} = \Sigma(n^3) + 3\Sigma(n^2) + 2\Sigma(n)$$

$$= \frac{n^2(n+1)^2}{4} + 3.\frac{n(n+1)(2n+1)}{6} + 2.\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{4}\left\{n(n+1) + 2(2n+1) + 4\right\}$$

$$= \frac{n(n+1)(n^2+5n+6)}{4}$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}.$$

Ex. 3. Find the sum of the series in the nth group of

$$1^2 + (2^2+3^2) + (4^2+5^2+6^2) + (7^2+8^2+9^2+10^2) + \ldots$$

The number of terms in the first n groups $= 1 + 2 + 3 + \ldots + n$

$$= \frac{n(n+1)}{2} = \frac{n^2+n}{2} = N \text{ (suppose)}.$$

The number of terms in the first $n-1$ groups $= 1 + 2 + 3 + \ldots\ldots\ldots$

$$+ (n-1) = \frac{(n-1)n}{2} = \frac{n^2-n}{2} = N' \text{ (suppose)}.$$

$\therefore$ required sum = Sum of the first N terms of $1^2 + 2^2 + 3^2 + \ldots$
− Sum of the first N' terms of the same series

$$= \tfrac{1}{6}N(N+1)(2N+1) - \tfrac{1}{6}N'(N'+1)(2N'+1)$$

$$= \tfrac{1}{6}(2N^3 + 3N^2 + N) - \tfrac{1}{6}(2N'^3 + 3N'^2 + N')$$

$$= \tfrac{1}{3}(N^3 - N'^3) + \tfrac{1}{2}(N^2 - N'^2) + \tfrac{1}{6}(N - N')$$

$$= \tfrac{1}{24}(6n^5 + 2n^3) + \tfrac{1}{8}.\ 4n^3 + \tfrac{1}{12}.\ 2n$$

$$= \tfrac{1}{12}n(3n^2+1)(n^2+2).$$

29. We shall now consider a class of series of which the differences of successive terms are in A.P. or G. P.

Ex 1. Sum to n terms $4 + 11 + 22 + 37 + \ldots$

[Here the successive differences 7, 11, 15,...are in A.P.]

Let t_n be the nth term and s the sum of the series.

Then $s = 4 + 11 + 22 + 37 + \ldots + t_n$

Also $s = 0 + 4 + 11 + 22 + \ldots + t_{n-1} + t_n$

$\therefore$ by subtraction, $0 = 4 + 7 + 11 + 15 + \ldots\text{to } n \text{ terms} - t_n$

$$\therefore t_n = 4 + (7 + 11 + 15 + \ldots\text{to } n-1 \text{ terms})$$

$$=4+\frac{n-1}{2}\{14+(n-2)4\}=4+(n-1)(2n+3)$$

$$=2n^2+n+1.$$

Hence $t_1=2.1^2+1+1$,

$t_2=2.2^2+2+1$,

$t_3=2.3^2+3+1$, etc.

$$\therefore\ s=2\Sigma(n^2)+\Sigma(n)+n=2.\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}+n$$

$$=\frac{n}{6}(4n^2+9n+11), \text{ on simplification.}$$

Ex. 2. Sum to n terms $1+3+7+15+31+\ldots\ldots$

[Here the successive differences 2, 4, 8, 16...are in G. P.]

Let t_n be nth term and s the sum.

Then $s=1+3+7+15+31+\cdots+t_n$

Also $s=0+1+3+\ 7+15+\cdots+t_{n-1}+t_n$

$\therefore$ by subtraction, $0=(1+2+4+8+16+\cdots$ to n terms$)-t_n$

$\therefore$ $t_n=1+2+4+\cdots$ to n terms $=2^n-1$.

$\therefore$ $t_1=2^1-1$, $t_2=2^2-1$, $t_3=2^3-1$, etc.

$\therefore$ $s=2^1+2^2+2^3+\cdots+2^n-n=2(2^n-1)-n$.

EXERCISE CXXXIX.

1. Sum the series $2^2+6^2+10^2+14^2+\cdots$ to n terms.
2. Sum the series $3.7+5.\ 11+7.\ 15+9.\ 19+\cdots$ to n terms.
3. Sum the series $1^3+3^3+5^3+7^3\ldots$ to n terms.
4. Sum the series $2.\ 1^2+3.\ 2^2+4.\ 3^2+\cdots$ to n terms (C. F. 1887)
5. Sum the series $1.\ 5.\ 9+2.\ 6.\ 10+3.7.\ 11+\cdots$ to n terms.
6. Find the sum of the squares of n terms of an A.P.
7. Find the sum of the cubes of n terms of an A.P. and prove that it is exactly divisible by the sum of the terms.
8. Find the sum of the series in the nth group of
   (i) $1^2+(3^2+5^2)+(7^2+9^2+11^2)+\cdots\cdots$
   (ii) $1^3+(2^3+3^3)+(4^3+5^3+6^3)+\cdots\cdots$
9. Sum the following series to n terms
   (i) $2+5+10+17+\cdots\cdots$ (ii) $2+7+14+23+34\cdots\cdots$
10. Sum the following series to n terms
   (i) $1+4+13+40+121+\cdots$ (ii) $1+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}\cdots$

30. Arithmetico-geometrical series. A series which is formed by multiplying together the corresponding terms of an A.P. and a G.P. is sometimes called an *Arithmetico-geometrical series.*

Thus, a, $(a+b)r$, $(a+2b)r^2, \cdots \{a+(n-1)b\}r^{n-1}$ is an Arithmetico geometrical series ; the terms of the series being formed by multiplying together the corresponding terms of

the A.P. a, $(a+b)$, $(a+2b) \cdots a+(n-1)b$,

and the G. P. 1, r, $r^2, \cdots \cdots r^{n-1}$.

31. To find the sum of n terms of an Arithmetico-geometrical series.

Let the nth term of the series be $\{a+(n-1)b\}r^{n-1}$, and let the sum be denoted by s.

$$\text{Then } s=a+(a+b)r+(a+2b)r^2+\cdots+\{a+(n-1)b\}r^{n-1}$$

$$\therefore \quad rs= \qquad ar+(a+b)r^2+\cdots+\{a+(n-2)b\}r^{n-1} + \{a+(n-1)b\}r^n.$$

$$\therefore \quad (1-r)s=a+br+br^2+\cdots+br^{n-1}-\{a+(n-1)b\}r^n.$$

$$=a+br\frac{1-r^{n-1}}{1-r}-\{a+(n-1)b\}r^n.$$

$$\therefore \quad s=\frac{a-\{a+(n-1)b\}r^n}{1-r}+br\frac{1-r^{n-1}}{(1-r)^2}.$$

Ex. 1. Sum to n terms : $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+\cdots$ (C. F. 1880).

Putting $\frac{1}{3}=r$, the series becomes $r+3r^2+5r^3+\cdots+(2n-1)r^n$.

Denoting the sum by s, we have

$$s=r+3r^2+5r^3+\cdots+(2n-1)r^n$$

$$\therefore \quad rs= \qquad r^2+3r^3+\ldots+(2n-3)r^n+(2n-1)r^{n+1}.$$

$$\therefore \quad (1-r)s=r+2r^2+2r^3+\cdots+2r^n-(2n-1)r^{n+1}$$

$$=r+2r^2(1+r+r^2+\cdots+r^{n-2})-(2n-1)r^{n+1},$$

$$=r+2r^2\frac{1-r^{n-1}}{1-r}-(2n-1)r^{n+1}.$$

$$\therefore \quad \tfrac{2}{3}s=\tfrac{1}{3}+\tfrac{1}{3}\left(1-\frac{1}{3^{n-1}}\right)-\frac{2n-1}{3^{n+1}}=\tfrac{2}{3}-\frac{2n+2}{3^{n+1}}.$$

Hence $s=1-(n+1)/3^n$.

If the sum to infinity is wanted we should proceed as in the next example.

Ex. 2. Sum to infinity $\frac{5}{7}+\frac{7}{21}+\frac{9}{63}+\frac{11}{189}+\cdots$ (B. P. 1883).

Let s denote the sum.

$$\text{Then } s=\tfrac{5}{7}+\tfrac{7}{21}+\tfrac{9}{63}+\tfrac{11}{189}+\cdots\cdots\text{to infinity.}$$

$$\therefore \quad \tfrac{1}{3}s= \qquad \tfrac{5}{21}+\tfrac{7}{63}+\tfrac{9}{189}+\cdots\cdots\text{to infinity.}$$

$$\therefore \quad \tfrac{2}{3}s=\tfrac{5}{7}+\tfrac{2}{21}+\tfrac{2}{63}+\tfrac{2}{189}+\cdots\cdots\text{to infinity.}$$

$$=\frac{5}{7}+\frac{2}{21}\left(1+\frac{1}{3}+\frac{1}{3^2}+\text{.........to infinity}\right).$$

$$=\frac{5}{7}+\frac{2}{21}\cdot\frac{1}{1-\frac{1}{3}}=\frac{5}{7}+\frac{1}{7}=\frac{6}{7}.$$

$$\therefore\ s=1\tfrac{2}{7}.$$

32. If the nth term of a series can be expressed as the difference of two **consecutive** terms of another series, we can immediately find the sum.

Ex. 1. Sum to n terms $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots$

Let t_n be nth term and s the sum.

Then $t_1=\frac{1}{1.2}=1-\frac{1}{2}$,

$t_2=\frac{1}{2.3}=\frac{1}{2}-\frac{1}{3}$,

$t_3=\frac{1}{3.4}=\frac{1}{3}-\frac{1}{4}$,

....

$t_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$.

$\therefore$ by adding $s=1-\frac{1}{n+1}$......... (1)

$$=\frac{n}{n+1}.$$

Note. If n is very large $\frac{1}{n+1}$ is very small; hence from (1) the to infinity $=1$.

Ex. 2. Sum to n terms

$$\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\ldots+\frac{1}{(2n-1)(2n+1)(2n+3)}.$$

We have $\frac{1}{1.3.5}=\frac{1}{4}\left(\frac{1}{1.3}-\frac{1}{3.5}\right)=\frac{1}{4}\{\frac{1}{2}(1-\frac{1}{3})-\frac{1}{2}(\frac{1}{3}-\frac{1}{5})\}$

$=\frac{1}{8}\{(1-\frac{1}{3})-(\frac{1}{3}-\frac{1}{5})\}$

Similarly $\frac{1}{3.5.7}=\frac{1}{8}\{(\frac{1}{3}-\frac{1}{5})-(\frac{1}{5}-\frac{1}{7})\}$

$\frac{1}{5.7.9}=\frac{1}{8}\{(\frac{1}{5}-\frac{1}{7})-(\frac{1}{7}-\frac{1}{9})\}$

....

$$\frac{1}{(2n-1)(2n+1)(2n+3)}=\frac{1}{8}\left\{\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)-\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)\right\}$$

Hence by addition the required sum

$$=\tfrac{1}{8}\left\{\left(1-\frac{1}{2n+1}\right)-\left(\tfrac{1}{3}-\frac{1}{2n+3}\right)\right\}$$

$$=\tfrac{1}{8}\left\{\frac{2n}{2n+1}-\frac{2n}{3(2n+3)}\right\}$$

$$=\frac{n(n+2)}{3(2n+1)(2n+3)}.$$

EXERCISE CXL.

1. Sum to n terms the series

(i) $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots\ldots\ldots$ (ii) $1+\frac{2}{5}+\frac{3}{5^2}+\frac{4}{5^3}+\ldots\ldots$

(iii) $1-2+3-4+\ldots\ldots\ldots$ (iv) $1+3x+5x^2+7x^3+\ldots\ldots$

2 Sum to infinity the following series

(i) $1+4x+7x^2+10x^3+\ldots\ldots$ $(x<1)$.

(ii) $2+\frac{5}{3}+\frac{8}{3^2}+\frac{11}{3^3}+\ldots\ldots$ (iii) $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\ldots\ldots$

Sum to n terms :—

3. $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\ldots\ldots$

4. $\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\ldots\ldots$

5. $\frac{1}{1\cdot3}+\frac{1}{2\cdot4}+\frac{1}{3\cdot5}+\ldots\ldots$

6. $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\ldots$

MISCELLANEOUS EXERCISE PAPERS (V).

Paper I.

1. Solve.—(i) $\dfrac{9x+1}{12x+3}-\dfrac{3x-2}{12x-3}=\frac{5}{9}$.

(ii) $x^2+2(a-b)x+b^2=2ab$

2. Simplify :—(i) $(a+b)^r(a-b)^r\times(a^2+b^2)^r$

(ii) $\left(\dfrac{x^a}{x^b}\right)^{a+b}\div\left(\dfrac{x^a}{x^{2a-b}}\right)^{a^2/b}$ (M. M. 1890).

3. Multiply $a^{\frac{5}{2}}+2a^2b^{\frac{1}{3}}+4a^{\frac{3}{2}}b^{\frac{2}{3}}+8ab+16a^{\frac{1}{2}}b^{\frac{4}{3}}+32b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}-2b^{\frac{1}{3}}$

4. (i) Which term of the series 1.2, 2.4, 4.6...is 3.2^{14}?

(ii) Is 1088 a term of the series 128, 192, 256... ?

5. (i) Find the 24th and nth terms of the series $1, -\frac{1}{4}, \frac{1}{16}\ldots$

(ii) The 6th term of a series in G. P. is 64 and the 9th term -512, find the nth term.

6. If $(b+c)x=(c+a)y=(a+b)z$, shew that

$(x-y)/(a^2-b^2)=(y-z)/(b^2-c^2)=(z-x)/(c^2-a^2)$.

7. Draw the graph of $x^2+y^2-6x-4y=3$

Solve graphically $x^2+y^2=81$ and $x+y=2$.

8. Find three numbers in A.P. whose sum is 21 and product 280.

PAPER II.

1. Solve (i) $\dfrac{1}{x^2-1}-\dfrac{1}{1-x}=\dfrac{7}{8}-\dfrac{1}{x+1}$ (ii) $3x-2=\dfrac{1}{x+1}$.

2. Extract the square root of $16y^3(y-2)-8y(1-3y)+1$.

3. Multiply $\sqrt[4]{a^5}+3a^{-\frac{1}{4}}-2\sqrt{a}$ by $2a^{\frac{1}{2}}-3a^{-\frac{1}{4}}-a^{-1}$.

4. Simplify $\left\{\dfrac{\sqrt[4]{a^{-1}}}{\sqrt[3]{b^4}}\times\left(\dfrac{a^{\frac{1}{3}}}{b^{-\frac{1}{2}}}\right)^2\div\dfrac{b^{-\frac{2}{3}}}{a^{-\frac{1}{2}}}\right\}^{12}$.

5. If $\dfrac{x^2-yz}{a}=\dfrac{y^2-zx}{b}=\dfrac{z^2-xy}{c}$, prove that

$$\frac{a^2-bc}{x}=\frac{b^2-ca}{y}=\frac{c^2-ab}{z}.$$

6. (i) Sum $13\frac{1}{2}+11\frac{1}{6}+8\frac{5}{6}+\ldots$to 34 terms.

(ii) Sum $6+66+666+\ldots$to 10 terms.

7. Evaluate $\dfrac{5+\sqrt{2}}{3-\sqrt{2}}$ to three decimal places.

8. Draw the graph of $y=3x^2-5x+2$; hence solve the equation $3x^2-5x+2=0$.

PAPER III.

1. Solve: (i) $x^2+1=x\left(a+\dfrac{1}{a}\right)$;

(ii) $\dfrac{4x+5}{x+1}+\dfrac{3x+10}{x+3}=\dfrac{7x+15}{x+2}$.

2. Find the value of

$\dfrac{a(b-c)}{a-b}$ when a, b, c are (i) in A.P. (ii) in G.P. (iii) in H.P.

3. Sum the series to n terms :—

$1+2x+3x^2+4x^3+\ldots$; under what condition is the sum to infinity possible and what is the sum?

4. Simplify (1) $\dfrac{2\sqrt{3}}{\sqrt{5}+2\sqrt{3}}+\dfrac{\sqrt{5}}{2\sqrt{3}-\sqrt{5}}$ (2)

5. Solve $a^{x}.a^{y+1}=a^{7}$, $a^{2y}.a^{3x+5}=a^{2c}$.

6. Draw the graphs in one diagram :—

(i) $4x^2+9y^2=144$

(ii) $4x^2-9y^2=144$. (iii) $4x^2-9y^2=0$.

7. If $\dfrac{x+2y}{y+z}=\dfrac{y+2z}{z+x}=\dfrac{z+2x}{x+y}$, then $\dfrac{3x+\ldots+9z}{5x+2y+5z}=\dfrac{3x+4y+5z}{3x+2y+3z}$.

8. If the roots of the equation $(b-c)x^2+(c-a)x+(a-b)=0$ are equal, prove that a, b, c are in A.P.

PAPER IV.

1. Solve (i) $(x-1)^4+(x-4)^4=81$.

(ii) $6x-\dfrac{1}{x}=\frac{1}{2}(\sqrt{5}+7\sqrt{3})$

2. If $\dfrac{a-b}{b-c}=\dfrac{a}{x}$, show that b is the A.M., G.M. or H.M. between a and c according as $x=a$ or b or c. (M. F. 1895).

3. Find the sum of all the numbers between 100 and 500 which are divisible by 3.

4. Simplify.

$$\left\{\frac{1}{\dfrac{4}{(1+t^2)^{\frac{3}{2}}}-\dfrac{3}{(1+t^2)^{\frac{1}{2}}}}\right\}^2-\left(\frac{3t-t^3}{1-3t^2}\right)^2+(1+t^2)^2-2t^2-2. \text{ (C. E. 1895.)}$$

5. Sum the series

(1) $1^3+2^3+3^3+\ldots\ldots\ldots$to n terms.

(2) $1.1+2.3+3.5+\ldots\ldots$to n terms.

(3) $5+55+555\ldots\ldots\ldots$to n terms.

6. Extract the square root of $64+14\sqrt{15}$.

7. Eliminate x from the equations

$ax^2+bx+c=0$, $cx^3+bx^2+a=0$. (M. F. 1909).

8. Plot the graphs :—

(1) $y=x-\dfrac{15}{x}$ and $x=y-\dfrac{10}{y}$ and thus solve the equations ;

(2) $x^2+2y^2+4x-12y=14$; find the limiting values of x and y.

PAPER V.

1. Simplify

$$\left(\frac{x^r}{x^m}\right)^{r^2+rm+m^2} \times \left(\frac{x^m}{x^n}\right)^{m^2+mn+n^2} \times \left(\frac{x^n}{x^r}\right)^{n^2+nr+r^2}$$

2. Solve (i) $(a-b)x^2-(a^2+ab+b^2)x+ab(2a+b)=0.$

(ii) $\frac{1}{x}+\frac{1}{a}-\frac{1}{b}=\left(\frac{1}{x+a-b}\right)$ (B. P. 1904.)

3. Find the sum of a given number of terms in G.P. and when possible sum to infinity.

If $a_1, a_2, a_3 \cdots a_{n+1}$ are in G. P show that

$$\frac{1}{a_1^2-a_2^2}, \frac{1}{a_2^2-a_3^2}, \ldots \frac{1}{a_n^2-a_{n+1}^2} \text{ are also in G.P.}$$

4. Find the sum of the squares of the first n natural numbers.

Sum to n terms the series whose rth term is $\frac{1}{2}(7r^2-5r)$.

5. Find the square root of

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{x}{y}+\frac{y}{x}-1\frac{3}{4}$$ (B. M. 1893)

6. If $(a+b)(b+c)(c+d)(d+a)$

$=(a+b+c+d)(bcd+cda+dab+abc)$, prove that $ac=bd$

(B. M. 1884.)

7. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that

$$(a+c+e)^{\frac{1}{2}}(b+d+f)^{\frac{1}{2}}=a^{\frac{1}{2}}b^{\frac{1}{2}}+c^{\frac{1}{2}}d^{\frac{1}{2}}+e^{\frac{1}{2}}f^{\frac{1}{2}}.$$

8. Trace the changes in the values of $12x^2-7x-18$ as x changes in value from $x=-\infty$ to $x=+\infty$. Find values of x for which it is (a) positive (b) zero (c) negative (d) a minimum, and find the minimum value. Draw the graph of the function. (M. F. 1906).

PAPER VI.

1. Solve (i) $(6x-1)(3x-1)^2(2x-1)=336.$

(ii) $\frac{2x-1}{(x-1)^2}-\frac{2x+1}{(x+1)^2}=4.$

2. Simplify

$$\frac{\left(p^2-\frac{1}{q^2}\right)^p\left(p-\frac{1}{q}\right)^{q-p}}{\left(q^2-\frac{1}{p^2}\right)^q\left(q+\frac{1}{p}\right)^{p-q}}$$ (B. M. 1881).

3. Prove that $\dfrac{6\sqrt{8}+2\sqrt{28}}{\sqrt{8}+\sqrt{7}}=5\cdot02.$

4. If $a=y+z-2x$, $b=z+x-2y$, $c=x+y-2z$, find $b^2+c^2-a^2+2bc$ in terms of x, y, z.

5. How many terms of the series -8, -6, -4, etc. amount to 52? Explain the double answers.

6. If a, b, c, d be in G.P., prove that (i) $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$ (ii) $(a-d)^2=(b-c)^2+(c-a)^2+(d-b)^2$.

7. Sum to 12 terms the series whose nth term $=(2n-1)^3$.

8. Trace the graphs of $y=\dfrac{x+2}{x-2}$ and $y=x^3$; hence solve the equation $x^3=\dfrac{x+2}{x-2}$.

PAPER VII

1. Simplify

(i) $$\frac{\left(p+\frac{1}{q}\right)^p\left(p-\frac{1}{q}\right)^q}{\left(q+\frac{1}{p}\right)^p\left(q-\frac{1}{p}\right)^q}$$ (B. M. 1889)

(ii) $$\frac{1}{(4x^3-3x)^2}-\left\{\frac{\dfrac{3\sqrt{1-x^2}}{x}-\dfrac{(1-x^2)^{\frac{3}{2}}}{x^3}}{1-3\left(\dfrac{1-x^2}{x^2}\right)}\right\}^2$$ (C. E. 1878)

2. Find the square root of $4x^{\frac{3}{2}}-12x^{\frac{3}{4}}+25-24x^{-\frac{3}{4}}+16x^{-\frac{3}{2}}$.

3. Solve (i) $\sqrt{4x+1}-\sqrt{2x}=1$. (ii) $\dfrac{\sqrt{x+2}+\sqrt{x-3}}{\sqrt{x+2}-\sqrt{x-3}}=5$

4. Solve

(i) $6x^4+11x^2=10$

(ii) $x^6=1$.

5. Find the value of $2\cdot\dot{5}$ by regarding it an infinite geometrical progression.

6. Find the sum of (i) $5^2+7^2+9^2+\ldots$to n terms

(ii) $1.3+2.4+3.5+\ldots$to n terms

7. If $(b+c)x=(c+a)y=(a+b)z$,

show that $\dfrac{x(y-z)}{b-c}=\dfrac{y(z-x)}{c-a}=\dfrac{z(x-y)}{a-b}$.

8. Draw the graphs of $y=2x^2$ and $y=3-4x$, hence solve the equation $2x^2+4x-3=0$.

PAPER VIII.

1. Find the square root of

$$\frac{x^3}{16}-\frac{x^2}{6}-\frac{1}{4}x^{\frac{3}{2}}+\frac{x}{9}+\frac{1}{3}x^{\frac{1}{2}}+\frac{1}{4}.$$ (B. M. 1886).

2. Simplify

$$\left\{\sqrt[3]{\frac{x^2}{y^4}}\times\sqrt{\frac{y^3}{x^5}}\right\}^{12}\times x^{22}$$ (M. M. 1894).

3. Prove that the expressions $6x^4-2x^3+9x^2+9x-4$ and $9x^4+80x^2-9$ both vanish for one common value of x and find that value.

4. Solve (i) $\sqrt{\frac{3x}{x+1}}-\sqrt{\frac{x+1}{3x}}=1\frac{1}{2}$.

(ii) $(x+2)^3+(x+3)^3=8x^3+125$.

5. Find the sum of all the even numbers of three digits.

6. The sums of n terms of two Arithmetic series are as $3n+31 : 5n-3$; show that their 9th terms are the same.

(A. I. 1898).

7. Sum to n terms (i) $2+5+10+17+\ldots\ldots$

(ii) $\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\ldots\ldots$

8. Draw the graphs of

(i) $(x-2)^2+(y-3)^2=25$

(ii) $\frac{x^2}{9}-\frac{y^2}{2}=1$.

Calcutta University Papers.

1910.

Compulsory.

1. (1) Find the continued product of
$a+b+c,\ b+c-a,\ c+a-b,\ a+b-c.$
[$2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$

If $x-\frac{1}{x}=p$, find the value of $x^3-\frac{1}{x^3}$ in terms of p. [p^3+3p

(2) Resolve into factors x^3+1 and x^2+x-20.
[$(x+1)(x^2-x+1)$; $(x+5)(x-4)$

2. (1) Find the G. C. M. of $x^2-9,\ (x+3)^2,\ x^2+x-6$; or,
Find the L. C. M. of $x^2-4,\ x^2-x-2,\ x^2+x-2,$
[$(x+3)$, $(x^2-4)\ (x+1)\ (x-1)$

(2) If $\frac{x}{a}=\frac{y}{b}$, prove that $(x^2+y^2)\ (a^2+b^2)=(ax+by)^2$.

3. (1) Solve $\frac{b}{x}=\frac{a}{x-b+a}$ *Or*, $\left.\begin{array}{l}9x-5y=17\\13y-2x=20\end{array}\right\}$ [b ; 3, 2

(2) Draw the graph of $y=x+1$.

Additional.

1. (1) Solve $\frac{x}{x+1}+\frac{x+1}{x}=\frac{25}{12}$. [$-4$, 3.

(2) Draw the graph of $y=4x^2$.

2. (1) Find the square root of $4x^4+20x^2-3-\frac{70}{x^2}+\frac{49}{x^4}$.

[$2x^2+5-\frac{7}{x^2}$

(2) Prove that $a^m\times a^n=a^{m+n}$, for positive integral values of m and n.

3. (1) Find without the assumption of any formula the sum of the first n natural numbers.

(2) If a^2, b^2, c^2 be in A. P., prove that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are also in A. P.

Or, (1) Find, without the assumption of any formula, the sum of n terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$&c. $\left[2\left(1-\frac{1}{2^n}\right)\right.$

(2) If a, b, c be in A. P. and a, b, d be in G. P., show that $a, a-b, d-c$ are in G. P.

4. If $x=b+c-a, y=c+a-b, z=a+b-c$,

find the value of $\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}$. [4

1911.

Compulsory.

1. (*a*) Find the continued product of

$1+x+x^2$, $1-x+x^2$, and $1-x^2+x^4$. $[1+x^4+x^8$

Or, If $x+y+z=13$, and $xy+yz+zx=50$,

find the value of $x^2+y^2+z^2$. [69

(*b*) Resolve into factors

$x^2+2x-143$ and $a^4+2a^3b-2ab^3-b^4$.

$[(x+13)(x-11)$; $(a-b)(a+b)^3$

2. (*a*) Find the G. C. M. of $3x^3+17x^2-62x+14$ and $7x^3+52x^2-46x+8$. $[x^2+8x-2$

Or, Find the L. C. M. of a^2-9b^2, $a^2-ab-6b^2$, $a^2+ab-12b^2$

$[(a-3b)(a+3b)(a+2b)(a+4b)$

(*b*) If $\frac{x}{a+b-c}=\frac{y}{b+c-a}=\frac{z}{c+a-b}$.

show that each of these fractions $=\frac{x+y+z}{a+b+c}$

3 (*a*) Solve $\frac{2x-3}{3}-\frac{3x-5}{5}+\frac{5x+3}{6}-\frac{7x+5}{10}=4$. [20

Or, Solve
$$\left.\begin{aligned} x+y+z&=1\\ 2x+3y+z&=4\\ 4x+9y+z&=16\end{aligned}\right\}$$ $[-3, 3, 1$

(*b*) Draw the graph of $\frac{x}{2}-\frac{y}{3}=1$.

Additional.

1. Solve : (1) $17x^2+19x=1848$; [$-11, 9\frac{15}{17}$

(2) $\frac{x-3}{x+3}-\frac{x+3}{x-3}+6\frac{2}{3}=0.$ [4, $-2\frac{1}{4}$

Or, Draw the graphs of (1) $x^2+y^2=25$, and (2) $3x+4y=25$. Prove that the second graph touches the first and find the co-ordinates of the point of contact. [3, 4

2. (1) Find the square root of

$(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$, [a^2+b^2

$\frac{x^2}{y^2}+\frac{y^2}{x^2}+6\frac{x}{y}-6\frac{y}{x}+7.$ $\left[\frac{x}{y}-\frac{y}{x}+3.\right.$

(2) Prove that $(a^m)^n=a^{mn}$, for positive integral values of m and n.

3. (1) Find without assuming any formula, the sum of n terms of the series $1+3+5+7+\ldots$; [n^2

(2) How many terms of the series $3+5+7+\ldots$ must be taken in order that the sum may be equal to 399 ? [19

Or, Find, without assuming any formula, the sum of n terms of the series $\frac{1}{10}+\frac{6}{100}+\frac{6}{1000}+\frac{6}{10000}+\ldots$

Hence deduce the value of $\cdot 1\dot{6}$.

4. If $x=b+c$, $y=c+a$, $z=a+b$, find the value of

$\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}$. [2

1912.

Compulsory.

1 (1) Multiply $4x^2+9y^2+z^2+3yz-2zx+6xy$ by $2x-3y+z$.
[$8x^3-27y^3+z^3+18xyz$

(2) Divide—

$6x^5-17x^4+42x^3-66x^2+72x-72$ by $2x^2-3x+6$.

Or [$3x^3-4x^2+6x-12$

(1) Find the coefficient of x^4 in the product

$1-2x+4x^2-8x^3+16x^4$ by $1+2x+4x^2+8x^3+16x^4$ [16

(2) Find the L.C.M. of $2x^2-x-1$, $2x^2+3x+1$, x^2-1.

[$(x^2-1)(2x+1)$

2. (1) If $a : b :: b : c$, show $(a+b+c)(a-b+c)=a^2+b^2+c^2$.

(2) Solve $\frac{5x-1}{7}+\frac{9x-5}{11}=\frac{9x-7}{5}$. [3.

Or

(1) Draw the graph of $\frac{x}{4}+\frac{y}{5}=1$.

(2) Solve $\left.\begin{array}{l} x+5y=36 \\ \frac{x+y}{x-y}=\frac{5}{3} \end{array}\right\}$ [16, 4.

3. A man performed a journey of 7 miles in 1 hour 15 minutes. He walked part of the way at 4 miles an hour and rode the rest of the way at 10 miles an hour. How far did he walk? [$3\frac{2}{3}$ miles

Additional.

1. (1) Extract the square root of

$25x^{-2}-12x+16x^{-8}+4x^4-24x^{-5}$. [$2x^2-3x^{-1}+4x^{-4}$

(2) Show that $(x^{2^{n-1}}+a^{2^{n-1}})(x^{2^{n-1}}-a^{2^{n-1}})=x^{2^n}-a^{2^n}$.

Or

(1) Solve $\frac{x+4}{x-4}+\frac{x-4}{x+4}=\frac{10}{3}$. [8, −8

(2) Simplify $\frac{bc}{(b-a)(c-a)}+\frac{ca}{(c-b)(a-b)}+\frac{ab}{(a-c)(b-c)}$. [1.

2. (1) Find, without assuming any formula the sum to n terms of $1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+$&c. $\left[\frac{3}{2}\left(1-\frac{1}{3^n}\right)\right.$

(2) Sum to n terms $1.2+2.3+3.4+4.5+$&c. [$\frac{1}{3}n(n+1)(n+2)$

3. Draw the graphs of $x^2+y^2=25$ and $x+y=7$; and measure the co-ordinates of their points of intersection. [4, 3; 3, 4

4 A horse was sold at a loss for Rs. 840; but if it had been sold for Rs. 1050 the gain would have been three-fourths of the former loss. Find its real value. [Rs. 960

1913.

Compulsory.

1. (1) If $x=b-c$, $y=c-a$, $z=a-b$, find the value of $x^2+y^2-z^2+2xy$. [0

(2) Find the G.C.M. of $2x^3+x^2-5x-3$ and $8x^3+6x^2-21x-18$ [$2x+3$

Or, (1) Simplify :— $\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)\div\left(\frac{1}{\frac{x}{y}-\frac{y}{x}}-\frac{1}{\frac{x}{y}+\frac{y}{x}}\right)$ $\left[\frac{2(x^2+y^2)}{y}\right.$

(2) Divide $6x^4-2x^3-23x^2-5x+20$ by $2x^2-5$. [$3x^2-x-4$

2. Solve the equations :—

(1) $\dfrac{2x+1}{5} - \dfrac{3x-2}{6} = \dfrac{1}{2}$ (2) $2x+y=3y-x=7$. [$\frac{1}{3}$; 2, 3

Or, A man pays 200 Rs. more than one-third of his debt and still owes 210 Rs. more than what he has paid. What was his original debt? [Rs. 1830

3. (1) Draw the graph of $y=x-2$.

(2) If $5(x-y)=3(x+y)$, find the ratio of x to y. [4 : 1

Additional.

1. (*a*) Solve $42x^2-41x-20=0$. [$\frac{4}{3}$, $-\frac{5}{14}$

(*b*) Find the square root of $x^4-x^3-\dfrac{7x^2}{4}+x+1$. [$x^2-\frac{1}{2}x-1$

Or, (*a*) Divide 50 into two parts such that the sum of their reciprocals may be $\frac{1}{12}$. [20, 30

(*b*) Simplify $\dfrac{(b-c)^2+(c-a)^2+(a-b^2)}{(a-b)(a-c)+(b-c)(b-a)+(c-a)(c-b)}$. [$-2$

2. (*a*) The first two terms of an Arithmetical Progression are 3 and 1. Write down the tenth term and the sum of the first ten terms. [-15, -60

(*b*) The first two terms of a Geometrical Progression are 3 and 1. Write down the tenth term and the sum of the first ten terms [$\frac{1}{6561}$, $\frac{29524}{6561}$

3. Draw the graphs of $x^2+y^2=16$ and $x+y=2$, and measure the length of the chord of intersection.

1914.

Compulsory.

1. Find the value of $a^2+b^2+c^2-bc-ca-ab$ when

$a=x+y$, $b=x-y$, $c=x+2y$. [$7y^2$

Or, Divide x^6-6x+5 by x^2-2x+1 ; [$x^4+2x^3+3x^2+4x+5$

2. Factorize :

(1) $x(x-1)(x-2)-3x+3$; [$(x-1)(x+1)(x-3)$

(2) $a^2(b-c)+b^2(c-a)+c^2(a-b)$ [$-(b-c)(c-a)(a-b)$

Or, Simplify $\dfrac{b-c}{a^2-(b-c)^2}+\dfrac{c-a}{b^2-(c-a)^2}+\dfrac{a-b}{c^2-(a-b)^2}$ [0

3. Solve :

(1) $\frac{x-a}{b}+\frac{x-b}{a}+\frac{x-3a-3b}{a+b}=0$; [$a+b$

(2) $\frac{2x+2y-3}{5}=\frac{3x-7y+4}{6}=\frac{8y-x+2}{7}$. [3, 1

Or, Draw the graphs of $3x+4y=25$ and $4x-3y=0$, and measure the co-ordinates of their point of intersection. [3, 4

Additional.

1. Solve :

(1) $6x^2-91x+323=0$; (2) $x+\frac{1}{x}=25\frac{1}{25}$. [$\frac{19}{2}, \frac{17}{3}, 25, \frac{1}{25}$

√ *Or*, Draw the graphs of $4y=x^2$ and $2y=x+4$, between $x=-4$ and $x=+4$, and measure the length of the chord of intersection.

2. (1) Extract the square root of $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)$;

[$x-\frac{1}{x}-2$

(2) Express $(x-a)(x-b)$ as the diffierence of two squares.

[See ex. 3 (*ii*) p. 376

3. Find seven arithmetical means between 1 and 41.

[6, 11, 16, 21, 26, 31, 36

Find three geometrical means between $\frac{1}{9}$ and 9. [$\frac{1}{3}$, 1, 3

1915

Compulsory.

1. (i) Find the product of $(b+c-a)$, $(c+a-b)$, $(-a+b+c)$.

(ii) Divide $x^4-y^4+a^4+2a^2x^2$ by $x^2-y^2+a^2$. [x^2+y^2+a

Or, (i) Find the H. C. F. of x^2-2x-3 and x^3-2x^2-2x-3. [$x-3$.

(ii) Find the L. C. M. of a^2-b^2, a^3-b^3 and a^4-b^4.

[$(a^4-b^4)(a^2+ab+b^2)$

2. Solve the equations :

(i) $\frac{5x+6}{12}+\frac{3x-4}{5}=2(x-9)$, (ii) $3x+4y=27$, $5x-3y=16$.

[18 ; 5, 3.

3. Simplify : $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}$.

[$\frac{1}{abc}$

Or, Draw the graph of $x-y=4$.

Additional.

1. (i) Solve, without assuming the formula, $63x^2 - 62x = 221$. [illegible]

(ii) What number must be added to $x^4 - 6x^3 + 13x^2 - 12x + 1$ to make a perfect square? [3.

Or, (i) Divide unity into two parts such that the sum of their cubes is $\frac{7}{16}$. [illegible]

(ii) Simplify: $\dfrac{\sqrt{2}(2+\sqrt{3})}{\sqrt{3}(\sqrt{3}+1)} - \dfrac{\sqrt{2}(2-\sqrt{3})}{\sqrt{3}(\sqrt{3}-1)}$. $\left[\dfrac{\sqrt{6}}{3}\right.$

2. (i) Find the sum of the squares of the first n natural numbers.

(ii) In a geometrical progression, show that the product of any two terms equidistant from a given term is equal to the square of the given term.

3. Trace the graphs of $y = 2x + 1$, $y = 3x^2$ and determine the points where they intersect. [1, 3; [illegible]

1916.

Compulsory.

4. (*a*) Simplify $2a - 2\{2a - [2(a-b) - b]\}$. [$2a - 6b$.

(*b*) Divide $6 + x^2 - 19x + 6x^3$ by $2 + x$. [$3 - 11x + 6x^2$

Or

(*a*) Find the H.C.F. of $3x^2 - 11x - 4$ and $6x^3 - 25x^2 + 3$. [$3x + 1$.

(*b*) Simplify $\dfrac{x}{x-y} + \dfrac{y}{x+y} + \dfrac{2xy}{y^2 - x^2}$. [1

5. Solve the equations: (*a*) $\dfrac{3x+2}{x-1} + \dfrac{2(x-2)}{x+2} = 5$. [6.

(*b*) $11x + 12y = 58$; $12x + 11y = 57$.

Add 1 to the numerator and denominator of a certain fraction and it reduces to $\frac{4}{5}$: subtract 5 from each and it reduces to $\frac{1}{2}$; required the fraction. [7.

6. Draw the graph of $3y - 2x = 4$, and plot the points on the graph for which $x = -2$, 1, and 3 respectively.

Additional.

3. (*a*) Solve $4x^2 - 65x + 126 = 0$. [14, $\frac{9}{4}$.

(*b*) Find the square root of $x^2 - 6x + 5 + \dfrac{12}{x} + \dfrac{4}{x^2}$

Or

(*a*) Simplify $\left(\frac{x^r}{x^m}\right)^{r+m} \times \left(\frac{x^m}{x^n}\right)^{m+n} \times \left(\frac{x^n}{x^r}\right)^{n+r}$. [1

(*b*) Find two consecutive numbers, the sum of whose squares is 145. [8, 9

4. (*a*) Find, without the assumption of the formula, the sum of 30 terms of the series 1, 3, 5, 7,...... [900

(2) Insert two numbers between 5 and 135 so that the four may form a geometrical progression. [15, 45

5. Trace the graphs of $y=x$, $y=\frac{x^2}{4}$ and determine the points where they intersect. [Origin and (4, 4

1917.

Compulsory.

4. (*a*) Multiply $a^2-ab+a+1$ by $a+b-1$.

Or,

Divide a^4-6a-4 by $a-2$.

(*b*) Find the H.C.F. of x^3-7x+6 and x^3-3x^2+4.

Or,

Find the L.C.M. of x^2+x-6, x^2+2x-3 and x^2-3x+2.

[$(x+3)(x-2)(x-1)$

5. (*a*) Draw the graph of $2x+3y=1$.

(*b*) If $\frac{a}{b}=\frac{b}{c}$, shew that $\frac{a}{c}=\frac{a^2+b^2}{b^2+c^2}$.

6. (*a*) Solve $\frac{x+3}{4}-\frac{x+4}{5}=\frac{x+5}{6}-\frac{x+6}{7}$. [1

Solve $x+2y=3=4x-y$. [1, 1

(*b*) The half of a certain integer exceeds the third of the next greater integer by two. Find the integer. [1

Additional

3. (*a*) Solve, without assuming formula, $x^2-x=1806$. [43, −42

(*b*) Find the square root of $1+2a+2a^2+a^3+\frac{a^4}{4}$. $\left[1+a+\frac{a^2}{2}\right.$

4. (*a*) Shew how to find the sum of *n* terms of an A. P., being given the first term and the common difference.

Or,

Shew how to find the sum of n terms of a G. P., being given the first term and the common ratio.

(*b*) Sum to n terms $1\times2+2\times3+3\times4+4\times5+\ldots\ldots$

$\left[\dfrac{n(n+1)(n+2)}{3}\right.$

Or,

Sum to n terms $1+2x+3x^2+4x^3+\ldots\ldots$

5. Trace the graph of $y=x^2-x$, from $x=-1$ to $x=2$ and there from obtain an approximate solution of the equation $1=x^2-x$.

[1·6 ; −

1918.

Compulsory.

4. (*a*) *Either*, Multiply $1-a+2a^2-3a^3$ by $3a-5+2a^2$.

Or,

Divide $a+a^5+a^9$ by a^2+a+1.

(*b*) *Either*. Find the H. C. F. of x^3+4x^2-5 and x^3-3x+2

Or,

Find the L. C. M. of $x^2-(a-c)x-ac$ and $x^2-(a+c)x+ac$.

5. *Either*, Solve

(*a*) $\dfrac{x+1}{2}+\dfrac{x+2}{3}+\dfrac{x+3}{5}=9.$

(*b*) $y+z=6,\quad z+x=4,\ x+y=2.$ [5, 2,

Or,

A motorist does a journey of 80 miles in 6 hours. During the first part of the journey he travels at 10 miles an hour, and during the latter part at 18 miles an hour. How far does he travel at each rate?

6. Draw the graphs of $x+y=2$, and $x-y=0$, and find the co-ordinates of their point of intersection.

Additional.

3. (*a*) Solve $(x-7)(x-19)=64$. [3, 23

(*b*) Find the square root of $49x^4+36y^4+109x^2y^2-70x^3y-60xy^3$.

4. (*a*) *Either*, Show directly that the sum of n terms of an A.P. equal to n times half the sum of the first term and the last term

Or,

Find directly the sum of n terms of the G. P.,

$$1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots$$

Either, If a be the first term of a G.P., l the n^{th} term, and P the *product* of the first n terms, show that $P=(al)^{\frac{n}{2}}$.

Or,

Find the sum of n terms of the series $1^3+2^3+3^3+\ldots\ldots$

5. Trace the graph of $y=x^2-4x+5$ from $x=0$ to $x=4$, and find the least value of y.

ANSWERS.

PART II.

EXERCISE LXXXIII. (pp. 11-15).

1. 72, 62, 66. **2.** 100, 158. **3.** Rs. 75, Rs. 60, Rs. 30. **4.** £2 15s **5.** 5 days. **6.** Rs. 2. **7.** 40, 60. **8.** 60. **9.** 30 pence **10.** 300, 700. **11.** 57lbs gold, 80 lbs. silver. **12.** Rs. 100000. **13.** 120*s*. **14.** 64 gallons. **15.** Length 102 yds, breadth 51 yds **16.** 10 min. **17.** 26. **18.** $\frac{120}{47}$ days. **19.** $5\frac{5}{11}$ minutes past 7, 27 minutes past 7 and $49\frac{1}{11}$ minutes past 7. **20.** 20 from A and 8 from B. **21.** 6 miles.

22 12 min. past 3. **23.** 12·5 *p.c.* **24.** 9 beggars, 171 *d.*

25. $2\frac{1}{4}$, $1\frac{1}{4}$, $1\frac{1}{2}$ miles per hour. **26.** 800. **27.** 30 years.

28. Hare 1200, Greyhound 960. **29.** 160 min. **30.** $445\frac{5}{7}$ mds

31 Rs. 2000. **32.** $106\frac{2}{3}$, $33\frac{1}{3}$. **33.** 14, 11. **34.** 10, 20.

35. Rs. 600, Rs. 480. **36.** 144 hrs. if they go in the same direction : $13\frac{1}{11}$ hrs. if in the opposite direction.

37. 26 miles. **38.** 1800. **39.** 72 years and 60 years.

40. $\frac{1}{3}(a+4b)$ rooms. **41.** $1\frac{1}{2}$ mds. **42.** 420 *oz.* copper ; 255 *oz.* tin

43. 9 gallons and 6 gallons. **44.** 2600. **45.** $137\frac{1}{7}$, *i.e.* 138 leaps

46. 1600 men. **47.** 10,25,50,75. **48** 10 days.

49. $\frac{20}{3}$, $\frac{40}{3}$. **50.** 6 sec., $8\frac{1}{22}$ sec. **51.** 8 miles per hour.

52. (*a*) (1) $27\frac{3}{11}$ min. past 2, (2) $43\frac{7}{11}$ min. past 2, (3) at 2 and 21 minutes past 2. (*b*) (1) $32\frac{8}{11}$ min. past 9 and at 9 (2) $16\frac{4}{11}$ min past 1. (3) $38\frac{2}{11}$ min. past 9 and at 10. **53.** $\frac{27}{4}$. **54.** Rs. 600

55. 12. **56.** Rs. 78-2*as*. **57.** Rs. 46-8*as*. **58.** 900 men

59. 4800. **60.** $36\frac{7}{13}$ minutes past 3.

In the answers to the following exercises (LXXXIII to LXXXIX the roots are given in the order x, y, z, u, v.

EXERCISE LXXXIV. (p. 18).

1. 2,3. **2.** 1, −1. **3.** 5,6. **4.** 5,3. **5.** 6,2.
6. −3,4. **7.** $a+b$, $a-b$. **8.** a, b.

EXERCISE LXXXV. (p. 19).

1. 3,2. **2.** 7,5. **3.** 3,1. **4.** −4,2. **5.** $-\frac{94}{97}$,

6. 2, −3. **7.** $\frac{mn-l}{m^2+n}, \frac{n^2+lm}{m^2+n}$ **8.** $(m+n)^2, (m-n)^2$.

EXERCISE LXXXVI. (pp. 20-21.)

1. 8,1. **2.** 4,3. **3.** $\frac{4510}{1371}, \frac{6740}{457}$. **4.** 2,6. **5.** $\frac{81}{32}, \frac{125}{32}$.
6. $\frac{813}{5101}, \frac{1117}{434}$. **7.** 40, 16. **8.** 36, 25. **9.** a, b.
10. ab, ac. **11.** $(a^2+b^2)/2ab$, $(b^2-a^2+2ab)/2ab$. **12.** c, c.

EXERCISE LXXXVII (pp, 25-27).

1. 1,2. **2.** $\frac{590}{118}, \frac{202}{59}$. **3.** 4, 5. **4.** $\frac{21}{5}, \frac{4}{5}$. **5.** $\frac{36}{7}, -\frac{3}{7}$. **6.** 12, 6.
7. 12, 6. **8.** 3, 2. **9.** 8, −15. **10.** $\frac{5}{2}, \frac{3}{2}$. **11.** 3, 2. **12.** $\frac{330}{169}, \frac{411}{169}$.
13. 2, 1. **14.** 2, 3. **15** 10, 5. **16.** $a/(a+b)$, $b/(a+b)$. **17.** $a+c$
$(b-a)$. **18.** $\frac{a^2bc}{a^2+b^2}; -\frac{ab^2c}{a^2+b^2}$. **19.** $\frac{12abm}{a+b}, \frac{(a-b)(7b-5a)m}{a+b}$

20. $a(a-b)$, $b(a-b)$. **21.** $\frac{a^2b}{a-b}, \frac{ab^2}{a+b}$ **22.** $\frac{7}{5}, \frac{5}{2}$. **23.** $\frac{1}{a}$, b

24. $\frac{1}{2}, \frac{1}{3}$. **25.** 4, 10. **26.** $\frac{1}{3a}, -\frac{1}{2b}$. **27.** $\frac{1}{2}, \frac{1}{3}$. **28.** $\frac{a^2-b^2}{am-bn}, \frac{a^2-b^2}{an-bm}$

29. 2, 3. **30.** $\frac{mb+an}{cm-da}, \frac{mb+an}{nc+bd}$. **31.** $-\frac{1}{2}, \frac{1}{7}$. **32.** $\frac{1}{6}$, 0. **33.** $\frac{91}{66}, \frac{43}{44}$.
34. $-\frac{11}{5}, \frac{2}{5}$. **35.** $\frac{27}{43}, \frac{58}{43}$.

EXERCISE LXXXVIII (pp. 31-32).

1. 2,1. **2.** −3, 2. **3.** 8, −6. **4.** 5,4. **5.** 2, 11. **6.** 6, 5. **7.** 7, 5
8. 3, 4. **9.** 8, 7. **10.** 2, 3. **11.** 1, 3, 5. **12.** 3, 4, 5. **13.** 2, 3, 4
14. 2, 5, 3. **15.** 4, 3, 2. **16.** bc, ca, ab. **17.** $b-c$, $c-a$, $a-b$.
18. $-\frac{1}{(b-c)(c-a)}, -\frac{1}{(a-b)(b-c)}, -\frac{1}{(a-b)(c-a)}$. **19.** $b-c$, $c-a$,
$a-b$. **20.** $bc(c-b)$, $ca(a-c)$, $ab(b-a)$. **21.** $a(b-c)$, $b(c-a)$, $c(a-b)$
22. $\frac{1}{2}a(b^2-c^2)$, $\frac{1}{2}b(c^2-a^2)$, $\frac{1}{2}c(a^2-b^2)$. **23.** $a+b-2c$, $b+c-2a$, $c+a-2b$.
24. $-\frac{72}{37}, \frac{120}{37}, \frac{108}{37}$. **25.** $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$. **26.** $\frac{3}{61}, \frac{45}{61}, \frac{72}{61}$.
27. $\frac{3}{4}, -\frac{19}{22}, -\frac{19}{44}$.

EXERCISE LXXXIX. (pp. 35-36).

1. 7, 5, 4. **2.** 3, −1, 0. **3.** 2, 3, 5. **4.** 2, 3, 4. **5.** 0, 2, 3
6. 4, 3, 5. **7.** 7, 5, 6, **8.** $-\frac{18}{23}, \frac{150}{23}$, 6. **9.** 2, 5, 4. **10.** 1, 5, 2.
11. 3, 2, 1. **12.** 3, 4, 1. **13.** a, b, c, **14.** $\frac{17}{6}, -\frac{17}{5}, \frac{17}{2}$. **15.** $\frac{1}{2}$, [illegible]
16. $\frac{49}{46}, \frac{49}{51}, \frac{49}{13}$. **17.** 3, 4, 5, 6. **18.** 3, 4, 5, 6. **19.** 1, 2, 3, 4, 5.
20. $1\frac{1}{2}, \frac{7}{3}, \frac{3}{4}$ 3, $2\frac{1}{2}$. **12.** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}$. **22.** $\frac{21}{23}, \frac{7}{12}, -\frac{14}{9}$. **23.** 2, 3, 4.
24. $-\frac{2bc}{b+c}, -\frac{2ca}{c+a}, -\frac{2ab}{a+b}$.

EXERCISE XC. (pp. 41-42).

1. 1, 11, 5. **2** $\frac{1}{3}, \frac{7}{5}, \frac{3}{7}$. **3** $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$.
4. 3, $\frac{5}{2}, \frac{2}{5}$. **5.** a, b, c. **6.** $\frac{48}{7}, \frac{48}{5}$, 48. **7.** $\frac{1}{3}, \frac{1}{2}$, 1. **8.** 3, 3, 3.
9. $6\frac{1}{2}, 5\frac{1}{2}$, 6. **10.** $\frac{27}{7}, \frac{33}{7}$, 5. **11.** $-(bc+ca+ab), a+b+c$, 1. **12.** b, c, a.
13. a, b, c. **14.** 1, 1, 0. **15.** a, b, c. **16.** a^2, b^2, c^2. **17.** a, b, c.
18 $x=y=z=a^2+b^2+c^2-bc-ca-ab$. **19.** $\frac{1}{c^2}, \frac{1}{b^2}, \frac{1}{a^2}$.
20. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$. **21.** 3, 4, 5. **22.** 4, 5, 6. **23.** Yes. **24.** Yes.
25. No. **26.** $k=a$.

EXERCISE XCI. (pp. 48-52).

1. 50, 60. **2.** 32, 23. **3.** Rs. 200, Rs. 100. **4.** Rs. 40, Rs. 5.
5. 18, 35. **6.** $\frac{13}{15}$. **7.** $\frac{11}{19}$. **8.** 12as., 2as. **9.** 75. **10.** 54.
11 646. **12.** 963. **13.** 210, 30. **14.** 5 minutes, $5\frac{1}{3}$ minutes.
15. 2min., 2min. and 10 sec **16.** $4\frac{7}{8}$ miles. **17.** 16 fit, 15 fit.
18. $\frac{165}{18}, \frac{44}{3}, \frac{33}{2}$, **19.** 14. **20.** A 36 days, B 60 days, C 15 days.
21. 60, 30, 12. **22.** 3.
23 Rs. 22, Rs. 24. **24.** Rs, 10, Rs. 16, Rs. 20. **25.** $8\frac{3}{7}$. **26.** [illegible], $\frac{600}{23}, \frac{600}{7}$ days. **27.** 5*s*. 4*d* per lb. **28.** 1*s*., $3\frac{1}{5}$*s*., $\frac{9}{10}$*s*.
29. 3 miles, 5 miles. **30.** 200 miles, $33\frac{1}{3}$ miles per hour.
31. 2 miles per hour. **32.** Rs. 340 in 3 p.c., Rs. 2400 in 4 p.c
33. 45 miles per hour. 35 miles per hoar. **34.** A 11 miles, B 10 miles.
35. $2\frac{1}{2}$ miles per hour. **36.** 63. **37.** 53 or 35. **38.** 253.
39. 30, $7\frac{1}{2}$ and 6. **40.** £4680, £4720.

EXERCISE XCII. (pp. 54-55).

1. (i) 20. (ii) 17. (iii) 101. (iv) 61.
2. (i) 13. (ii) 37. (iii) 25. (iv) 65.
5. (i) 10, 10, 8.9; area 40 sq. units. (ii) 12·8, 10, 16; 64 sq. units (iii) 6, 8·5, 6; 18 sq. units. (iv) 12, 21, 23·7; 126 sq. units.
8. (i) 22. (ii) 78. **9.** $62\frac{1}{2}$.

EXERCISE XCIII. (pp. 65-66).

29. The st. line $x=7$. **30.** The st. line $y=-10$.
31. The graph of $x=y$. **32.** The graph of $x=3y$.
42. 3·3 nearly. **43.** -4; 5·7 nearly. **44.** 4·75. **55.** 14

EXERCISE XCIV. (pp. 68-69).

1. $5y=6x$. **2.** $y=-5x+19$. **3.** $2y=3x$. **4.** $13y=-12x+20$.
5. $\frac{x}{4}+\frac{y}{5}=1$. **6.** $\frac{y}{9}-\frac{x}{7}=1$ **7.** $\frac{x}{12}-\frac{y}{8}=1$.
8. $7x-11y-123=0$. **9.** $6x-y+38=0$. **10.** $7x+3y-9=0$
11. $x+9y-48=0$. **14.** (i) $-\frac{7}{12}$ (ii) $\frac{1}{4}$ (iii) $\frac{1}{7}$ (iv) $\frac{3}{11}$.
15. $3y-4x+24=0$.

EXERCISE XCV. (pp. 78-80).

1. 2,3. **2.** ·76, −3·6 nearly. **3.** 2,1. **4.** −2, 3. **5.** ·74, ·53.
9. (9,2), (5,7), (3,4); lengths of the sides are 6·4, 6·3, 3·6; lengths of the medians are 6·1, 4·1, 4. **10.** 21 oranges.
14. 2·20 in, 8·64 cm. **15.** 4·53 miles; 4·35. **17.** $7\frac{1}{2}$ hours.
22. 21·82 minutes past 4; 8·73 minutes past 4 and also 34·91 minutes past 4

MISCELLANEOUS PAPERS III (pp. 81-87).

PAPER I.

1. $(x-3)(x-1)$. **8.** (i) $-\frac{8}{15}a$. (ii) $\frac{79}{20}$. (iii) $-\frac{5}{2}$. **4** $\frac{2(x-z)}{(a-x)(a-z)}$

6. $a+\frac{1}{a}-2$. **7.** 17 rupees, 34 eight-anna pieces. 68 four-anna pieces **8.** 8 square units.

PAPER II.

1. (i) $(a-x)(a+x)^3$; (ii) $(x+6)(x+2)(x^2+8x+10)$. 2. (i) 1. (ii) 2.
5. $\dfrac{(x-y)^4}{x}$. 6. $3x^4+4x^3-5x+2$. 7. 938. 8. $\frac{3}{2}$, -1 ; 5, $\frac{15}{2}$.

PAPER III.

1. (i) $(x-y-z)(x+y+z+1)$, (ii) $2(ax+by+cz)(ayz+bzx+cxy)$.
2. (i) $-\frac{3}{10}$, (ii) $-\frac{16}{29}$, (iii) $(a+b)^2$, $(a-b)^2$.
4. $3(2x+3)$; $(x+2)(2x-1)(3x+1)$. 5. (i) $\dfrac{1+a-a^2}{2a-1}$, (iii) $\dfrac{2xy}{27x^3+y^3}$.
6. $4a^2-4a+1$. 7. £10. 8. (1) $4x-3y+24=0$. (2) $3x+4y-7=0$.

PAPER IV.

1. (i) $\dfrac{ab}{c}$, $\dfrac{bc}{a}$, $\dfrac{ca}{b}$ (ii) $b+c-a$, $c+a-b$, $a+b-c$.
3. $a(x+y)+b(x-y)$. 4. (i) $(x-a)(x^2+xa+a^2)$, (ii) $(a-c)(pn+qmb)$.
5. $l=6$, $m=-37$. 6. $\dfrac{p+q}{p-q}$. 7. $30\frac{69}{89}$ hrs. 8. 4, 1.

PAPER V.

1. (i) $x=12, y=3$. (ii) 4, 2, -3. 2. $y^2-xy-x-\frac{1}{2}$.
3. (i) $\dfrac{(a-b)(b-c)(c-a)(x^2-1)x}{(a-x)(b-x)(c-x)(1-ax)(1-bx)(1-cx)}$; (ii) a^2.
4. $11y=20x-32$. 5. $\frac{13}{3}$, $\frac{16}{3}$. 7. coeff. of $x^2=\frac{1}{2}(a+b)(a+b-1)$. 8 576.

PAPER VI.

2. x^2-3x+1 ; $(2x^2-x-3)(3x^4-10x^3+8x^2-7x+2)$.
3. (i) $(x-2y)(x+3y)(x+6y)$. (ii) $(a-11b+1)(2a+b-3)$.
(iii) $(x+z+a-y)(x-z-a-y)(z-a-x-y)(z-a+x+y)$.
4. $x^2-x(y+z)-yz$. 5. (i) 1, 2, 3. (ii) 3, -2. 6. area $=92\cdot5$ sq. units
7. 24. 8. (i) $\dfrac{a+b-2c}{bc+ca-2ab}$, (ii) 3, 7, 5.

PAPER VII.

2. (i) 2, $\frac{5}{3}$, (ii) 5, -7, 3. (iii) $\dfrac{ab}{c}$. 3. $x^2+xy-2y^2$. 5. $2\cdot9$, $-\cdot02$.
7. 20 annas nearly. 8. 16, 10.

Paper VIII.

1. (i) $2x^2 - xy - 4y^2$. (ii) $(a-b)(b-c)+(b-c)(c-a)+(c-a)(a-b)$
2. $p = -\frac{8}{3}$, $q = -\frac{28}{3}$. **3.** (i) 1, 1 ; (ii) 0, 0, 0. **4.** $b^2 - ac$, $c^2 - ab$, $a^2 - bc$. **6.** 1·6. **7.** 16·36 and 49·09 minutes past 6.
8. 45 miles per hour, $22\frac{1}{2}$ miles per hour.

Paper IX.

1. $6x+3$. **2.** 1. **3.** 1. **4.** (i) $\dfrac{3b+c-4a}{2a-b-c}$, (ii) $a+b$, $a-c$.
5. $x^2 - 3x + 5$. **6.** $3x^2 - \frac{1}{3}xy + 3y^2$.

Paper X.

1. $2x^3 - x^2 - 3$. **2.** $x^2 + x - 3$; $(x^2+x-3)^2$ $(x^2-x+3)(x^3-x-3)$
3. 1. **4.** $(x^2+pxy-y^2)(x^2-pxy-y^2)$. **5.** $ax+by+cz$.
6. (i) $-c$, (ii) $a+b+c$. **7.** 1, 4 ; 2, -3 ; -1, -2.

EXERCISE XCVI. (pp. 90-91.)

1. (1) 3 : 2. (2) 19 : 6. (3) $bc' - b'c : ca' - c'a$.
2. (1) 13 : 12, (2) 17 : 3, (3) 0. (4) $-\frac{27}{5}$. **3.** $-\frac{31}{60}$.
4. (1) 1 : 22 : 17. (2) $bc(b-c) : ca(c-a) : ab(a-b)$.
5. (1) The former greater. (2) The former greater.
6. Second army. **7.** (1) 4 : 9. (2) $(x-y) : (a-b)$.
8. 5 : 12. **9.** $4\frac{1}{4}$. **10.** 7. **11.** 9. **12.** $\frac{3}{5}$. **13.** 255, 153. **14.** 8, 10
15. 12, 18. **16.** 12, 21 ; 24, 42 ; 36, 63 ; 48, 84.
17. A, 72 years ; B, 63 years. **18.** 8 : 57.

EXERCISE XCVII (p. 95).

1. (1) 6. (2) $(a-b)^2$. **2.** 2. **3.** 2. **4.** 1. **8.** $\dfrac{2(b^4-a^4)}{a^2b^2}$.

EXERCISE XCIX (pp. 101-102).

1. (1) 16. (2) 1. (3) $(a-b)^2/(a+b)$. **2.** (1) 6. (1) a^3b^3
(3) $(a+b)(a^2-ab+b^2)^{\frac{1}{2}}$. **3.** 2·45. **4.** 1. **7.** $\dfrac{b^2-ac}{a+c-2b}$

EXERCISE C. (pp. 105-107).

11. 0. **12.** (1) 0. (2) 0. (3) 0.

EXERCISE CI. (p. 109).

1. 3. **2.** 8. **3.** 6.

EXERCISE CIII. (pp. 116-117).

1. $\frac{3}{20}$. **2.** -4. **3.** $-11\frac{1}{2}$. **4.** $\dfrac{(a+b)cd-(c+d)ab}{ab-cd}$.

5. $\dfrac{cd(c+d)-ab(a+b)}{(a^2+ab+b^2)-(c^2+cd+d^2)}$. **6.** $-2\frac{1}{4}$. **7.** 13. **8.** $-2\frac{1}{3}$. **9.** $2\frac{1}{7}$

10. $-5\frac{1}{2}$. **11.** -7. **12.** $-8\frac{1}{2}$. **13.** $-2\frac{1}{2}$. **14.** -2. **15.** $-2\frac{1}{6}$.

16. $-1\frac{3}{4}$. **17.** $-2\frac{2}{5}$. **18.** $-1\frac{4}{9}$. **19.** $\dfrac{a+1}{a-1}$. **20.** $\dfrac{a-1}{a+1}$.

21. $\dfrac{(c^2+1)(a-b)-2c(a+b)}{4c}$. **22.** $\frac{5}{4}$. **23.** $-\dfrac{a^2+b^2}{2b}$. **24.** $\dfrac{ab}{a-b}$

25. $x=\frac{1}{2}(b+c-a)d,\ y=\frac{1}{2}(c+a-b)d,\ z=\frac{1}{2}(a+b-c)d.$

26. $x=\dfrac{d(ca+ab-bc)}{2abc},\ y=\dfrac{d(ab+bc-ca)}{2abc},\ z=\dfrac{d(bc+ca-ab)}{2abc}$.

27. $x=b+c-a,\ y=c+a-b,\ z=a+b-c.$

28. $x=3a-b-c,\ y=3b-c-a,\ z=3c-a-b.$

EXERCISE CIV. (p. 121).

1. 116. **2.** -4. **3.** 11. **4.** $-\frac{23}{4}$. **5.** $\frac{52}{9}$. **6.** 2. **7.** 12. **8** 16.
9. 4. **10.** 1.

13. $x^4+x^3y+x^2y^2+xy^3+y^4$; $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$;
$x^7+x^6y+x^5y^2+x^4y^3+x^3y^4+x^2y^5+xy^6+y^7$.

14. $x^4-x^3y+x^2y^2-xy^3+y^4$; $x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6$;
$x^8-x^7y+x^6y^2-\ldots\ldots-xy^7+y^8$.

15. $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$; $x^7+x^6y+x^5y^2+\ldots\ldots\ldots\ldots$
$+xy^6+y^7+\dfrac{2y^8}{x-y}$; $x^9+x^8y+x^7y^2+\ldots\ldots+xy^8+y^9+\dfrac{2y^{10}}{x-y}$.

16. $x^6-x^5y+\ldots\ldots-xy^5+y^6-\dfrac{2y^7}{x+y}$; $x^7-x^6y+\ldots\ldots+xy^6-y^7+\dfrac{2y^8}{x+y}$;
$x^6+x^5y+\ldots\ldots+xy^5+y^6+\dfrac{2y^7}{x-y}$.

17. $(x-2)(x+3)(x+4)$. **18.** $(x+3)(x+2)(x-1)$.

19. $(x-4)(x-3)(x-2)$. **20.** $(x+3)(x+1)(x+5)$.

21. $(x-3)(x+5)(x+1)(x+2)$. **22.** $(x-1)(x+1)(x+3)(x+2)$.

EXERCISE CV. (p. 123).

1. $-(2b-c-a)(2c-a-b)(2a-b-c)$.
2. $-(a+b-2c)(b+c-2a)(c+a-2b)$. 3. $(a-b)(c+a)(b+c)$.
4. $(a-b)(b-c)(c-a)$. 5.6.7. $-(b-c)(c-a)(a-b)$.
8. $-p(b-c)(c-a)(a-b)$. 9. $(a-b)(b-c)(a-c)(a+b+c+3)$.
10. 11. $(a-b)(b-c)(a-c)(ab+bc+ca)$.
12. $(a-b)(b-c)(c-a)(a^2b^2+b^2c^2+c^2a^2)$.
13. $-(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$.
14. $-(a-b)(b-c)(c-a)(a^3+b^3+c^3+ab^2+a^2b+bc^2+b^2c+ca^2$
$+c^2a+abc)$.
15. $(a-b)(b-c)(c-a)(ab+bc+ca)$.
16. $-2(b-c)(c-a)(a-b)(a+b+c)$.
17. $2(a-b)(b-c)(c-a)(a+b+c)$. 18. $-x(a-b)(b-c)(c-a)$.
19. $(a-b)(b-c)(c-a)$. 20. $abc(a-b)(b-c)(a-c)$.

EXERCISE CVI. (pp. 124—25.)

1. 2. 3. $(a+b)(b+c)(c+a)$. 4. 5. 6. $(a+b+c)(ab+bc+ca)$.
7. $(a+b)(b-c)(a-c)$. 8. $(a+b-c)(ab-bc-ca)$.
9. $(2b+3c)(3c+a)(a+2b)$. 10. $(2b+3c+a)(6bc+3ca+2ab)$.
11. $(a+b+c)(ab+bc+ca)$. 12. $(a+b)(b+c)(c+a)$. 13. $pq-r$.
14. $p^3-3pq+3r$. 15. $pq-3r$.

EXERCISE CVII. (pp. 130—132.)

1. (i) -31. (ii) -57. (iii) -47. 2. (i) 50. (ii) -150. (iii) -142.
4. (i) $(4x^2-3x+1)^2-(2x^2-4x+4)^2$. (ii) $(x^2+5x+5)^2-(1)^2$
(iii) $(x^2+12ax+31a^2)^2-(4a^2)^2$.

EXERCISE CVIII. (pp. 138—143).

1. $\dfrac{3x-y+2}{x+y-2}$. 2. $\dfrac{x+y-3}{3x-y+5}$. 3. $\dfrac{x-y-z}{x-4y+4z}$. 4. $2(x+y+z)$.
5. $\frac{3}{2}(a+b+c+d)$. 6. $\dfrac{ab+bc+ca}{a+b+c}$. 7. $\dfrac{(x^2+2)(1+x^4)}{x}$. 8. $\dfrac{c}{a}$.
20. $a+b+c+3$. 21. 4. 22. -3. 23. $-(bc+ca+ab)$. 24. 4.
25. 1. 26. $a+b+c$. 27. $-(a+b+c)$. 28. 1. 29. x.
30. x^2. 31. 1. 32. lm.
33. $(m+1)^2$. 34. -1. 35. abc. 36. abc.

37. $(a+b+c)^2$. **38.** 0. **39.** 0. **40.** -1. **41.** 0

42. $\dfrac{a+b+c}{(b+c)(c+a)(a+b)}$. **43.** $\dfrac{1}{(x-a)(x-b)(x-c)}$.

44. $\dfrac{x^2}{(x-a)(x-b)(x-c)}$. **45.** $\dfrac{3x+4}{(x-a)(x-b)(x-c)}$.

46. $\dfrac{x^2+x+1}{(x-a)(x-b)(x-c)}$. **47.** $\dfrac{x^2(a+b+c)-x(bc+ca+ab)+abc}{(x-a)(x-b)(x-c)}$.

48. $\dfrac{4x^2+12x+9}{(x-a)(x-b)(x-c)}$.

49. $\dfrac{hx^2+kx+l}{(x-a)(x-b)(x-c)}$. **64.** $\dfrac{78}{101}$. **65.** $\dfrac{201}{100}$.

EXERCISE CIX. (pp. 146–47).

1. (1) $ad=bc$. (2) $a^3b^2+b^3d+c^2=3abc$. (3) $(ca'-c'a)^3$.
$=(bc'-b'c)(ab'-a'b)^2$.

2. (1) $a^2-b^2=2c^2$. (2) $a^3+2c^3=3ab^3$. (3) $(a^2-b^2)^2.=2(b^4-c^4)$
(4) $(bc'-b'c)^2+(ca'-c'a)^2=d^2(ab'-a'b)^2$

3. (1) $c^3+3ab^2=a^3$. (2) $a^3+2c^3=3ab^2+6d^3$.
(3) $a^3+3d^3=c^3+3ab^2$. (4) $b^3c^3+c^3a^3+a^3b^3=5a^2b^2c^2$.

4. $abc+2fgh-af^2-bg^2-ch^2=0$.

5. (1) $3d^2-2d(a+b+c)+bc+ca+ab=0$. (2) $abc+d^2(a+b+c)=0$.
(3) $2d^3+d^2(a+b+c)=abc$. (4) $d(a+b+c)+bc+ca+ab=0$.

6. $27ay^2=4(x-2a)^3$. **7.** (1) $a^2b^2c^2=d^5$. (2) $a^2+b^2+c^2+2abc=1$.
(3) $a+b+c+3d=0$. (4) $a^2+b^2+c^2=abc+4$.

8 $x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}=c^{\frac{n}{n+1}}$. **9** Add and subtract the equations,
whence $(a+b)^{\frac{2}{5}}-(a-b)^{\frac{2}{5}}=4$. **10.** $(a+b)^{\frac{2}{3}}-(a-b)^{\frac{2}{3}}=(8c)^{\frac{2}{3}}$.

EXERCISE CX. (pp. 149-50).

1. $21x^6-43x^5+36x^4-23x^3+20x^2-14x+3$.

2. $2x^7+3x^6-9x^5+32x^4-35x^3+41x^2-18x+8$.

3. $\frac{3}{4}x^5-\frac{3}{4}x^4+\frac{11}{6}x^3+\frac{61}{9}x^2+\frac{26}{9}x-\frac{16}{3}$.

4 $2x^2+\frac{1}{3}x+\frac{8}{9}-\frac{1}{9}\left(\dfrac{32x-46}{3x^2-2x+1}\right)$.

5. $\frac{3}{2}x^2+\frac{1}{4}x-\frac{21}{8}-\frac{1}{6}\left(\dfrac{53x-164}{\frac{1}{3}x^2-\frac{1}{2}x+2}\right)$.

6. $6x^9+21x^8+31x^7+34x^6+34x^5+10x^4-33x^3-50x^2-30x-7$.

7. -118. **8.** $-\frac{27}{2}$.

MISCELLANEOUS EXERCISE PAPERS (IV.) (pp. 157—63).

Paper I.

1. (i) $x=a(a+b)^2$, $y=a(a-b)^2$. (ii) $2a+b$.
3. $A=1$, $B=-2$, $C=4$. 4. $a^4-2a^3-2a^2-1$.
7. $a+b+c=2abc$. 8. $(a+b+c)^2$.

Paper II.

2. 0. 4. $a^2+b^2+c^2-abc=4$. 5. 34; $x=-7$. 7. $(b-c)(c-a)(a-b)$.

Paper III.

1. D$(-3, 6\cdot5)$, E $(-1, 3\cdot5)$, F $(-2, 11)$; PD, $x-2y+16=0$; QE, $7x+2y=0$; RF, $x+2=0$. 4. $ab=c+1$.
5. $5x^2-7xy+5y^2$. 7. (i) $(a+b)(a-b)^3$; (ii) $27ab^2$. 8. $-24abc$.

Paper IV.

3. $2a^4-3a^2+1$.
4. $abc+2fgh-af^2-bg^2-ch^2=0$.

Paper V.

1. (i) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. (ii) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.
5. $(a^3+b^3+c^3)-3abc=1$. 6. $\frac{(a-b)^2}{(a+b)^2}$ 8. $a+b+c+3d$.

Paper VI.

2. $(a^2+b^2+c^2)(ab+bc+ca)$.
3. $p^3+q^3+r^3=p^2(q+r)+q^2(r+p)+r^2(p+q)$.

Paper VII.

4. $z^3-az^2+bz=c$. 5. $x^4+2x^3+3x^2+4x+5$. 8. 12, 18, 30.

Paper VIII.

4. $\frac{(x-1)^2}{(x-a)(x-b)(x-c)}$. 8. $a^3+b^3+c^3=3abc$.

Paper IX.

2. (i) $-\frac{2ab}{c}$ (ii) b, a. 4. x^2.
8. $(1+a)(1+b)(1+c)=(1-a)(1-b)(1-c)$.

PAPER X.

2. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}$; to find $\sqrt{101}$ put $a=10$, $x=1$.

3. 7. 6. $m=2pn$, where p is any positive integer.

8. $(a^3+b^3)^{\frac{2}{3}}-(a^3-b^3)^{\frac{2}{3}}=4c^3$.

EXERCISE CXIII. (pp. 173-74).

1. (1) 9. (2) $\frac{1}{9}$. (3) $\frac{32}{243}$. (4) $\frac{243}{32}$. (5) 10. (6) $\frac{8}{9}$ (7) $-\frac{3}{2}$. (8) 5^{n^2-5n}.

2. (1) $\frac{1}{a^{\frac{1}{2}}}$. (2) $\frac{b^6}{a^4}$. (3) $\frac{b^{\frac{7}{4}}c^{\frac{1}{12}}}{a^{\frac{11}{6}}}$. (4) $\frac{4y^6}{z}$. (5) $a^{-\frac{17}{4}}b^{-\frac{7}{2}}$.

   (6) $x^{\frac{23}{6}}y^{\frac{3}{2}}z^2$. (7) $\frac{1}{2}a^{\frac{17}{12}}b^{\frac{19}{12}}$. (8) 1.

3. (1) $x-y$. (2) $x^{\frac{3}{2}}-y^{\frac{3}{2}}$. (3) $x-y$. (4) $42x^{[illegible]}-18x^{\frac{1}{3}}y^{\frac{1}{3}}-9x^{\frac{2}{3}}y^{\frac{2}{3}}$ $-14x^{\frac{1}{2}}y^{\frac{2}{3}}+6y-4x^{\frac{1}{3}}y^{\frac{4}{3}}+49x^{\frac{7}{6}}y^{\frac{1}{3}}+14xy$. (5) $x^{\frac{19}{12}}y^{\frac{5}{6}}+x^{\frac{3}{2}}$ $+x^{\frac{13}{12}}y^{\frac{1}{6}}+xy+xy^{-\frac{2}{3}}-x^{\frac{1}{2}}y^{\frac{5}{4}}+x^{\frac{11}{12}}y^{\frac{1}{6}}-y^{\frac{7}{12}}-x^{-\frac{1}{12}}y^{\frac{17}{12}}$.

4. (1) $x+y+2x^{\frac{1}{2}}y^{\frac{1}{2}}$. (2) $x+y+z+2x^{\frac{1}{2}}y^{\frac{1}{2}}+2x^{\frac{1}{2}}z^{\frac{1}{2}}+2y^{\frac{1}{2}}z^{\frac{1}{2}}$.

   (3) $a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}}+2a^{\frac{1}{3}}c^{\frac{1}{3}}+2b^{\frac{1}{3}}c^{\frac{1}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}$ (4) $2a+2(a^2-x^2)^{\frac{1}{2}}$

   (5) $1+9x^{-1}y^2+4x^{-\frac{1}{2}}y^{-2}+6x^{-\frac{1}{2}}y+4x^{-\frac{1}{4}}y^{-1}+12x^{-\frac{3}{4}}$.

5. (1) $x^{\frac{4}{5}}+5a^{\frac{1}{5}}x^{\frac{3}{5}}+6a^{\frac{3}{5}}x^{\frac{1}{5}}+a^{\frac{4}{5}}$. (2) $x-x^{\frac{2}{3}}a^{\frac{1}{3}}+x^{\frac{1}{3}}a^{\frac{2}{3}}-a$.

   (3) $x^{\frac{2}{3}}-a^{\frac{1}{3}}x^{\frac{1}{3}}+a^{\frac{2}{3}}$. (4) $x^{\frac{4}{3}}-7x^{\frac{2}{3}}y^{-\frac{2}{3}}+12y^{-\frac{4}{3}}$.

   (5) $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.

6. (1) $(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})(x^{\frac{1}{3}}-x^{\frac{1}{6}}y^{\frac{1}{6}}+y^{\frac{1}{3}})(x^{\frac{1}{3}}+x^{\frac{1}{6}}y^{\frac{1}{6}}+y^{\frac{1}{3}})$.

   (2) $(x^{-1}+a)(x^{-2}-ax^{-1}+a^2)(x^{-1}-a)$
   $\times(x^{-2}+ax^{-1}+a^2)(x^{-2}+a^2)(x^{-4}-a^2x^{-2}+a^4)$.

   (3) $(x^{\frac{1}{3}}+y^{\frac{1}{3}}+1)(x^{\frac{2}{3}}+y^{\frac{2}{3}}+1-x^{\frac{1}{3}}-y^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}})$.

   (4) $(a^{\frac{3}{5}}+b^{\frac{1}{2}})(a^{\frac{6}{5}}-a^{\frac{3}{5}}b^{\frac{1}{2}}+b)$.

7. (1) $x^{\frac{1}{3}}+2$; (2) $x^{-\frac{1}{3}}+3$.

8. (1) $(3x^{\frac{1}{4}}-1)(2x^{\frac{1}{4}}-3)(2x^{\frac{1}{4}}+9)$; (2) $(a^{\frac{1}{3}}+b^{\frac{1}{3}})(a^{\frac{1}{3}}-b^{\frac{1}{3}})$
   $\times(a^{\frac{2}{3}}+b^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}})$. 9. (1) $x^{\frac{4}{5}}-x^{\frac{7}{5}}a^{-\frac{3}{5}}+a^{\frac{4}{5}}$; (2) $x^{\frac{2}{3}}+x^{-\frac{2}{3}}-2$.

10. $x^{2^{n-1}} - a^{2^{n-2}} x^{2^{n-2}} + a^{2^{n-1}}$. **11.** $(a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}})^2$.

15. (1) 1. (2) -11. (3) 2. (4) 2. (5) $\dfrac{-(b+d)}{a+c}$.

16. (1) $-\frac{4}{5}, \frac{9}{5}$. (2) 11, $6\frac{1}{2}$.

EXERCISE CXIV. (pp. 178-79).

1. (1) $P = a^{10}b^{15}$. (2) $P = a^{12}b^8c^4$. **2.** (1) $\sqrt{20}$. (2) $\sqrt[3]{\frac{2}{9}}$. (3) $\sqrt{a^4b^2c^3}$.
(4) $\sqrt{\dfrac{1-x}{1+x}}$. **3.** (1) $\sqrt[3]{a^{\frac{3}{2}}b^{\frac{9}{2}}c^{\frac{15}{2}}}$. (2) $\sqrt[4]{a^6b^{-2}c^8}$.

4. (1) $\frac{2}{3}\sqrt[4]{\frac{81}{16}ab^2c^3}$. (2) $2abc\sqrt[6]{\dfrac{1}{64a^3b^4c^3}}$.

5. (1) $\sqrt[30]{a^{10}}, \sqrt[30]{b^6}, \sqrt[30]{c^3}$. (2) $\sqrt[12]{16}, \sqrt[12]{27}, \sqrt[12]{16}$. (3) $\sqrt[10]{5^6}$, $\sqrt[10]{2^8}, \sqrt[10]{9^3}$. (4) $\sqrt[12]{2^6}, \sqrt[12]{8^4}, \sqrt[12]{7^3}$.

6. (1) $\sqrt{11}$ smaller, (2) $\sqrt[3]{5}$ greater, (3) in ascending order.

7. (1) $2\sqrt{3}$, (2) $7\sqrt[3]{2}$, (3) $-\frac{7}{3}\sqrt{3}$, (4) $\frac{23}{3}\sqrt{2}$, (5) $10\sqrt[4]{5}$, (6) $39\sqrt{3}$.

9. (1) $84\sqrt{30}$, (2) $\sqrt[6]{392}$, (3) $\sqrt[12]{15125}$, (4) $101\frac{1}{4}$. (5) $5\sqrt[6]{\dfrac{125}{7^4}}$.
(6) $\sqrt[12]{\frac{81}{125}}$.

10. (1) $-7-3\sqrt{3}$, (2) $\sqrt{2}-\sqrt{3}-1$, (3) $6a-6b-5\sqrt{ab}$, (4) $2\sqrt{21}-10$.
(5) $2a$, (6) $a+b-3-2\sqrt{a+b}$.

EXERCISE CXV. (pp. 180-81)

1. (1) $a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{2}{3}}$, (2) $3^{\frac{1}{5}}2^{\frac{2}{3}}7^{\frac{3}{4}}$. **2.** (1) $5\sqrt{2}-5$, (2) $15-8\sqrt{3}$,
(3) $\dfrac{75+11\sqrt{15}}{127}$, (4) $\dfrac{x-\sqrt{x^2-a^2}}{a}$.

3. (1) ·894 (2) 1·633 (3) 1·605 (4) ·289 (5) 9·899 (6) ·205.

5. (1) 1. (2) 0. (3) $\dfrac{4x\sqrt{x^2-a^2}}{a^2}$. (4) $2x^2$.

EXERCISE CXVI. (p. 183).

4. (1) $3+\sqrt{5}$. (2) $\sqrt{6}-1$. (3) $3\sqrt{3}-1$. (4) $\sqrt{3}-\sqrt{2}$.
(5) $\sqrt[4]{5}+\frac{1}{2}\sqrt[4]{125}$. (6) $\sqrt[4]{250}-\sqrt[4]{40}$.

EXERCISE CXVII. (p. 187).

1. 9. **2.** 3. **3.** 7. **4.** 3. **5.** 2. **6.** 3. **7.** $\frac{5}{3}$. **8.** -3.

9. $\dfrac{(a-b)^2}{2b}$. **10.** 4. **11.** 9. **12.** $\frac{2}{5}$. **13.** $\dfrac{m(m+2k)}{2(m+k)}$. **14.** 5.

15. 2. **16.** 5. **17.** 11. **18.** 36. **19.** 7. **20.** $\frac{8}{5}$. **21.** $\frac{16}{25}$. **22.** $-\frac{24}{25}$. **23.** $8\frac{1}{3}$. **24.** 30. **25.** $-\frac{23}{24}$. **26.** 2. **27.** $\frac{5}{4}$. **28.** $\frac{1}{24}(a-25b)$. **29.** [illegible]

30. $\dfrac{a(b^4-c^4)}{b^4+c^4}$.

EXERCISE CXVIII. (p. 189).

1. ± 2. **2.** ± 2. **3.** $\pm\sqrt{\frac{7}{3}}$. **4.** $0, \pm\sqrt{\frac{3}{2}}$. **5.** $\pm$ [illegible].
6. $\pm\sqrt{5}$. **7.** ± 1. **8.** ± 7. **9.** $\pm\sqrt{\frac{39}{10}}$.
10. $\pm a\sqrt{\dfrac{b+2c}{b-2c}}$. **11.** $\pm\sqrt{\dfrac{a-2}{a+4}}$. **12.** $\pm\sqrt{5}, \pm\sqrt{-5}$.

EXERCISE CXIX. (p. 194).

1. 3, 4. **2.** $7, -5$. **3.** $\frac{7}{3}, -\frac{9}{5}$. **4.** $\frac{5}{2}, \frac{3}{7}$. **5.** $\frac{1}{3}, -\frac{7}{5}$.
6. $-\frac{1}{2}, -\frac{9}{4}$. **7.** $\frac{11}{7}, -\frac{7}{11}$. **8.** $\frac{12}{5}, -\frac{17}{7}$. **9.** $\frac{11}{2}, -\frac{17}{5}$.
10. $\frac{11}{19}, -\frac{2}{3}$. **11.** $\frac{21}{2}, -41$. **12.** $47, -\frac{37}{3}$. **13.** $\frac{1}{3}, -\frac{11}{7}$. **14.** $\frac{1}{2}, -\frac{67}{11}$.
15. $\dfrac{b\pm\sqrt{b^2-ac}}{a}$. **16.** $5, -\frac{8}{5}$. **17.** $-2, \frac{13}{7}$. **18.** $\frac{2}{3}, \frac{3}{10}$.

19. (1) $5\left(x+\dfrac{7+\sqrt{229}}{10}\right)\left(x+\dfrac{7-\sqrt{229}}{10}\right)$. (2) $-7\left(x-\dfrac{1+2\sqrt{2}}{7}\right)$
$\times\left(x-\dfrac{1-2\sqrt{2}}{7}\right)$. (3) $2\left(x+\dfrac{7+\sqrt{17}}{4}\right)\left(x+\dfrac{7-\sqrt{17}}{4}\right)$.
(4) $(x-2)(4x-1)$.

20. (1) Real, unequal, rational. (2) Real, equal, rational.
(3) Real, unequal, irrational. (4) Imaginary.

EXERCISE CXX. (p. 198).

1 3. **2.** 5. **3.** 3. **4.** 1. **5.** 4, 8. **6.** $3, \frac{17}{3}$.
7 $\pm\dfrac{a}{2}\sqrt{\dfrac{(a^2-4)}{(a^2-1)}}$. **8.** $\pm\sqrt{a^2+\left(\dfrac{2a-b^3}{3b}\right)^3}$ **9.** $1, -1$. **10.** $\pm\frac{27}{14}$
11. $\pm\dfrac{\sqrt{2ab-1}}{a}$. **12.** $0, \pm\left(\dfrac{2}{ab}-\dfrac{1}{b^2}\right)^{\frac{1}{2}}$. **13.** ± 5.
14. $-1, -\frac{2}{5}$. **15.** $a, -5a$. **16.** $\frac{3}{4}$. **17.** 1. **18.** $3, -\frac{1}{4}$.

EXERCISE CXXI. (pp. 200-1).

1. $\dfrac{a}{b}, \dfrac{c}{d}$. **2.** $\dfrac{ab}{a+b}, \dfrac{ab}{a-b}$. **3.** $\frac{1}{6}\{-9(a-b)\pm(a+b)\sqrt{-15}\}$

4. $0, \dfrac{-(a+b+c)\pm\sqrt{a^2+b^2+c^2-2ab-2bc-2ca}}{[illegible]}$. **5.** $4, -\frac{4}{3}$.

6. $3, -\frac{219}{172}$. **7.** $3, 3$. **8.** $2, 4\frac{6}{13}$. **9.** $\frac{1}{4}, -\frac{409}{127}$. **10.** $2, \frac{41}{50}$.

11. $\pm 1, \pm\frac{1}{2}\sqrt{\frac{3}{2}}$. **12.** $\frac{-1 \pm \sqrt{33}}{2}$.

13. $\frac{(ab+bc+ca) \pm \sqrt{a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)}}{a+b+c}$.

14. $1, \frac{31}{15}$. **15.** $\frac{1}{2}(a+b+c)$. **16.** $\frac{-1 \pm \sqrt{-3}}{2}, \quad \frac{a \pm \sqrt{a^2-4}}{2}$.

17. (1) $2, -\frac{1}{7}$, (2) $17, \frac{2}{3}$. **18.** (1) $2, 3$; (2) $\frac{3}{4}, -\frac{1}{2}$.

EXERCISE CXXII. (pp. 204-6).

1. $\pm 1, \pm\frac{1}{2}$. **2.** $\frac{9}{4}, \frac{9}{100}$. **3.** $\frac{3}{2}, -\frac{2}{5}$. **4.** $\frac{1}{9}, \frac{25}{49}$. **5.** $3^{\frac{5}{2}}, (\frac{7}{6})^{\frac{5}{2}}$.

6. $\sqrt[n]{a}, \sqrt[n]{b}$. **7.** $x^2 = -3a^2 \pm \sqrt{(8a^4+b^4)}$.

8. $\frac{5}{3}, -1, \frac{3 \pm 4\sqrt{3}}{9}$. **9.** $2, -5, \frac{-3 \pm \sqrt{241}}{2}$. **10.** $-1 \pm 4\sqrt{2}, 4, -6$.

11. $\frac{-17 \pm \sqrt{769}}{16}, \frac{-7 \pm \sqrt{739}}{15}$. **12.** $2 \pm \sqrt{(3 \pm \sqrt{\frac{19}{3}})}$.

13. $\pm\frac{n^5+1}{n^5-1}m$. **14.** $7, -2, \frac{3 \pm \sqrt{359}}{5}$. **15.** $a, -9a, (-4 \pm \sqrt{-15})a$.

16. $3, -\frac{17}{6}, \frac{1 \pm \sqrt{-359}}{12}$. **17.** $0, 1, \frac{2 \pm \sqrt{-3}}{4}$. **18.** $\frac{1}{2}, -\frac{9}{2}, -2 \pm \sqrt{-6}$.

19. $3, -1, 1 \pm \sqrt{-28}$. **20.** $-1, -4, \frac{-5 \pm \sqrt{-15}}{2}$. **21.** $-1, -\frac{2}{5}, \frac{5}{2}$.

22. $\frac{3}{4}, \frac{4}{3}, \frac{3}{2}, \frac{2}{3}$. **23.** (1) $-1, \frac{1 \pm \sqrt{-3}}{2}$, (2) $\frac{\pm 1 \pm \sqrt{-1}}{\sqrt{2}}$,

(3) $1, -1, \frac{\pm 1 \pm \sqrt{-3}}{2}$. **24.** $2, -\frac{1}{2}, \frac{2}{5}, -\frac{5}{2}$. **25.** $9, 3 \pm \sqrt{-8}$.

26. $a, \frac{1}{2}\{-(a-6) \pm \sqrt{(-3a^2+12a-8)}\}$. **27.** $-\frac{3}{2}, -1, -\frac{8}{3}$. **28.** $2, 3$.

29. $1, 2$. **30.** $\frac{1}{2}\{-1 \pm \sqrt{(3-2\sqrt{5})}\}, \frac{1}{2}\{-1 \pm \sqrt{(3+2\sqrt{5})}\}$.

31. $\pm 2, \pm\sqrt{-2}$. **32.** $-a, -b$. **33.** $-a, \frac{a(c+1)}{c(2c+3)}$.

34. $\frac{abc}{a+b+c}, \quad \frac{1}{3}[bc+ca+ab \pm \sqrt{b^2c^2+c^2a^2+a^2b^2-abc(a+b+c)}]$.

35. $0, -\frac{1}{2}(a+b)$. **36.** $0, (a+b)$. **37.** $6, -2, 3 \pm \sqrt{21}$.

38. $bc+ca+ab, \pm\sqrt{\{-abc(a+b+c)\}}$. **39.** $a, b, \frac{1}{2}(a+b)$.

40. $\pm 1, \pm\sqrt{-\frac{ac}{bd}}$.

EXERCISE CXXIII. (pp. 207-8).

1. 9, 10, 11, 12. **2.** 20 days. **3.** 7 miles per hour.
4. 6 miles per hour. **5.** 264. **6.** 60. **7.** 9, 27, 81.
8. 229 yds., 165 yds. **9.** 10, 4. **10.** 1, 3, 4.
11. Sugar 4 as., tea 8 as. per seer. **12.** 3, 4, 5.

EXERCISE CXXIV. (p. 217).

8. (i) Transfer the origin to the point ($\frac{5}{6}$, $\frac{71}{12}$) and the equation becomes $y=3x^2$: (ii) Transfer the origin to the point ([illegible], $-$[illegible]) and the equation becomes $y=-5x^2$.

9. (1) Always positive; minimum value 7·9. (2) Negative except when x lies between $-$·67 and 1, maximum value 2·08.

10. *(i)* ·50, ·67, *(ii)* ·80, $-$1·67

EXERCISE CXXV. (p. 223).

2. (1) $x^2+(y-4)^2=25$. (2) $(x-3)^2+(y-2)^2=49$.
(3) $x^2+y^2-15x+y+14=0$. (4) $(x+2)^2+(y-6)^2=36$; or, $(x-4)^2+(y-12)^2=36$.

3. (1) $x=3, 6$; $y=1, -2$. (2) $x=4$, [illegible] $y=-3, -5$. (3) The curves do not intersect.

EXERCISE CXXVI. (pp. 233-34).

2. Transfer the origin, (i) to the point ($\frac{5}{2}$, 2) when the equation becomes $y=\frac{1}{2x}$, (ii) to the point $(-5, 4)$ when the equation becomes $y=-\frac{11}{x}$.

4. (1) $\left.\begin{array}{l} x=5·6, -7·6 \\ y=1·3, -5·3 \end{array}\right\}$ (2) $\left.\begin{array}{l} x=4, -4, -4, 4 \\ y=5, -5, \;\; 5, -5 \end{array}\right\}$ (3) $\left.\begin{array}{l} x=4, \; 3, -4, -3 \\ y=3, \; 4, -3, -4 \end{array}\right\}$
(4) $x=1$, $y=2$.

5. (1) Two straight lines (2) Circle (3) Parabola (4) Ellipse (5) Hyperbola (6) Ellipse (7) Origin (8) Origin (9) Rect. Hyperbola (10) Two straight lines (11) Parabola (12) Circle (13) Ellipse (14) Rect. Hyperbola (15) Hyperbola. (16) Hyperbola.

EXERCISE CXXVII. (p. 236).

1. (1) 137. (2) $\frac{1}{3}(29-2n)$. (3) $\dfrac{n^2-n+1}{n}$.

2. (i) No. (ii) Yes, 17th term (iii) No.

3. (1) First term $=37\frac{4}{5}$, common diff. $=-3\frac{1}{5}$. (ii) first term $=21$, common diff. $=-1$.

4. $13\frac{2}{3}$; $\frac{1}{3}(10n-69)$. **5.** $p+q-m$. **6.** $\dfrac{m+n}{2}$, $m-\dfrac{(m-n)p}{2q}$.

EXERCISE CXXVIII. (p. 239).

1. (1) 156; (2) -470; (3) $62\frac{1}{2}$; (4) 17; (5) -500; (6) $210/a$; (7) $n(x^2+y^2)-n(n-3)xy$; (8) $1325\sqrt{2}$. **3.** 495. **4.** 320
5. 375. **7.** $7+9+11+\ldots$

EXERCISE CXXIX. (p. 241).

1. 475. **2.** $\dfrac{n}{2}(3n-11)$. **3.** $8+18+28+\ldots$ **4.** $p-q+2qr$.

5. $26\frac{8}{15}$. **6.** $-1\frac{8}{33}$. **7.** 9 or 12. **8.** 5. **9.** 7. **10.** 2, 4

EXERCISE CXXX. (p. 243).

1. (1) $-2\frac{2}{5}, \frac{1}{5}, 2\frac{4}{5}$ etc. (2) $7\frac{2}{9}, 6\frac{17}{18}, 6\frac{2}{3}$ etc. **2.** 10. **3.** 8. **4.** 8. **5.** 9

EXERCISE CXXXI. (pp. 246-47).

1. 247050. **2.** 8248500. **3.** 274995000; 275040000.

4. (i) 9555. (ii) $\dfrac{n(n+1)}{2}-\dfrac{m(m-1)}{2}$. **5.** $(n+1)^2-m^2$.

6. (1) $n(n^2+2)$. (2) $(2n-1)(n^2-n+1)$. **7.** 520.

8. (1) $(-1)^{n-1}(4n-3)$; $\frac{1}{2}\{-1+(-1)^n(1-4n)\}$. (2) $(-1)^{n-1}(5n-2)$; $-\frac{1}{4}\{-1+(-1)^n(1+10n)\}$. **9.** 10200. **10.** $\frac{9}{17}$.

14. $m+n$, or, $m+n-1$. **17.** $\dfrac{p(p^2+q^2)}{p^2-q^2}$; $\dfrac{(4pq-p^2-q^2)}{p^2-q^2}$.

19. 5, 8, 11. **20.** 2, 6, 10. **21.** 1, 3, 5, 7, 9. **22.** 1 hr. 24 min.
23. Rs. 6890.

EXERCISE CXXXII. (p. 249).

1. (1) 708588 (2) $\frac{3125}{1024}$ (3) $\dfrac{\sqrt{3}}{3^{n-1}}$. **2.** No.

3. $\sqrt{mn}$; $m(n/m)^{r/2q}$. **4.** (1) $256+128+64+\ldots$ (2) $\frac{1}{128}+\dfrac{1}{64\sqrt{2}}+\frac{1}{64}+\ldots$

EXERCISE CXXXIII (p. 250).

1. (1) 60466175. (2) $\frac{205339}{1701}$. (3) $\frac{133}{243}$. (4) $\frac{9841}{3}(3+\sqrt{3})$.
(5) $\frac{2}{3}(\sqrt{6}-2)\left\{1-(-1)^n(\frac{3}{2})^{n/2}\right\}$.

(6) $2^{(3-n)/2}(\sqrt{2}+1)(2^{n/2}-1.)$. (7) $\frac{1}{2}(4+3\sqrt{2})$
$\times\left\{1-(\sqrt{2}-1)^n\right\}$ (8) $\frac{(a+b)^2}{2b(b-a)}\left\{\left(\frac{a-b}{a+b}\right)^n-1\right\}$.

2. $3^{n+1}-3$. 3. $\frac{3}{2}\{1-(\frac{2}{3})^{15}\}$. 4. $\frac{a(r^n-1)}{r-1}$,

where $r=(P/Q)^{1/(p-q)}$ and $a=P\left(\frac{Q}{P}\right)^{\frac{p-1}{p-q}}$.

5. 2. 6. $2+2^2+2^3+\ldots$

EXERCISE CXXXIV (pp. 251-52).

1. (i) $1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}$. (ii) $\frac{7}{10}, \frac{14}{25}, \frac{56}{125}$. (iii) $4, \frac{16}{5}, \frac{64}{25}$ etc., (iv) $y^{\frac{1}{3}}x^{\frac{2}{3}}, y^{\frac{2}{3}}x^{\frac{1}{3}}$.
2. $\frac{b^2-ac}{a+c-2b}$.

EXERCISE CXXXV (p. 255).

1. (1) $1\frac{1}{2}$; (2) $3\frac{1}{8}$; (3) $\frac{1}{2}(3\sqrt{3}+5)$; (4) $2\frac{1}{4}$. 2. (1) $\frac{3}{9}$ (2) $2\frac{345}{999}$
(3) $5\frac{71424}{99000}$. 3. $\frac{7}{10}$. 4. $1+\frac{1}{2}+\frac{1}{4}+\ldots$

EXERCISE CXXXVI (p. 257).

1. $\frac{n}{1-x}-\frac{x^2(1-x^n)}{(1-x)^2}$ 2. $\frac{x^{2n}-1}{x-1}\left(x^2+\frac{1}{x^{2n}}\right)-2n$.

3. (i) $\frac{70}{81}(10^n-1)-\frac{7n}{9}$; (ii) $\frac{n}{3}-\frac{1}{27}\left(1-\frac{1}{10^n}\right)$. 4. $4\frac{31}{256}$.

5. $2\frac{4093}{4096}$. 6. $\frac{3}{2}\left(3^{-p}-3^{-q}\right)$ where $p=\frac{(n-1)n}{2}$, $q=\frac{n(n+1)}{2}$.

8. $\frac{a}{r-1}\times\left\{\frac{r(r^n-1)}{r-1}-n\right\}$, where a is the first term and r the common ratio.

13. 2, 4, 6, 8. 14. 2, 4, 8, 16.

EXERCISE CXXXVII. (p. 260).

1. (i) $-\frac{6}{7}$. (ii) $\frac{60}{16-n}$. **2.** $1, \frac{2}{3}, \frac{2}{7}, \frac{1}{4}$. **3.** $\frac{ab}{2a-b}$, $\frac{ab}{3a-2b}$, $\frac{ab}{4a-3b}$. **4.** $\frac{2}{5}+\frac{3}{10}+\frac{6}{25}+\ldots$

EXERCISE CXXXVIII (p. 262).

1. (i) $\frac{45}{7}$ (ii) $\frac{21}{29}$ (iii) $\frac{a^4-b^4}{a^2}$. **2.** (i) $\frac{6}{11}, \frac{6}{13}, \frac{2}{5}, \frac{6}{17}, \frac{6}{19}$.

(2) $\frac{3}{13}, \frac{6}{19}, \frac{1}{2}, \frac{6}{5}, -3, -\frac{2}{3}$. **3.** $\frac{ab+bc-2ac}{a+c-2b}$.

4. (i) $a : b = \sqrt{m}+\sqrt{(m-n)} : \sqrt{m}-\sqrt{(m-n)}$
(ii) $a : b = m+\sqrt{(m^2-n^2)} : m-\sqrt{(m^2-n^2)}$.

EXERCISE CXXXIX (p. 266).

1. $\frac{4n}{3}(4n^2-1)$. **2.** $\frac{1}{3}n(8n^2+27n+28)$. **3.** $n^2(2n^2-1)$.

4. $\frac{1}{12}n(n+1)(n+2)(3n+1)$. **5.** $\frac{1}{4}n(n+1)(n+8)(n+9)$.

6. $a^2n+abn(n-1)+\frac{1}{6}b^2n(n-1)(2n-1)$, a being the first term and b the common difference.

7. If $(a+b)$, $(a+2b)$......be the A. P. then $t_n=(a+bn)^3$ $=a^3+3a^2bn+3ab^2n^2+b^3n^3$; hence etc. **8.** (i) $\frac{1}{3}n(3n^4+n^2-1)$ (ii) $\frac{1}{8}n^3(n^2+1)(n^2+3)$. **9.** (i) $\frac{1}{6}n(2n^2+3n+7)$; (ii) $\frac{1}{6}n(2n^2+9n+1$

10. (i) $\frac{1}{4}(3^{n+1}-2n-3)$; (ii) $2(n-1)+\frac{1}{2^{n-1}}$.

EXERCISE CXL. (p. 269).

1. (i) $6-\frac{2n+3}{2^{n-1}}$. (ii) $\frac{25}{16}-\frac{4n+5}{16.5^{n-1}}$. (iii) The series $=1+2x+3x^2+\ldots+nx^{n-1}$, where $x=-1$, the sum $=\frac{1}{4}-(-1)^n\cdot\frac{2n+1}{4}$. (iv) $\frac{1+x-2x^n}{(1-x)^2}-\frac{(2n-1)x^n}{1-x}$. **2.** (i) $\frac{1+2x}{(1-x)^2}$ (ii) $5\frac{1}{4}$ (iii) $\frac{2}{9}$. **3.** $\frac{n}{2n+1}$.

4. $\frac{n}{3n+1}$. **5.** $\frac{n(3n+5)}{4(n+1)(n+2)}$. **6.** $\frac{n(n+3)}{4(n+1)(n+2)}$.

MISCELLANEOUS EXERCISE PAPERS (V.) (pp. 269-74)

PAPER I.

1. (i) ± 1. (ii) b, $b-2a$. **2.** (i) $(a^4-b^4)^{?}$; (ii) x^{-2}. **3.** $(a^{?}-64b^{?})$
4. (i) 12th ; (ii) yes, 16th. **5.** (i) $-\frac{1}{42}$, $(-\frac{1}{4})^{n-1}$. (ii) $(-1)^n.2^{n}$.
8. 4, 7, 10.

PAPER II.

1. $3, -\frac{5}{7}$; $\frac{-1 \pm \sqrt{37}}{6}$. **2.** $4y^2-4y+1$.

3. $2a^{\frac{7}{4}}-7a+11a^{\frac{1}{4}}-7a^{-\frac{1}{2}}-3a^{-\frac{5}{4}}$. **4.** $\frac{b^4}{a}$.

6. (i) -850, (ii) 7407407400. **7.** 4·045.

PAPER III.

1. (i) $a, \frac{1}{a}$, (ii) $-2 \pm \sqrt{-1}$. **2.** a, b, c. **3.** $\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}$.

4. (1) $\frac{17}{7}$, (2) $\frac{1}{x^{m}}$. **5** $x=3, y=3$.

PAPER IV.

1. (i) 1, 4 ; (ii) $\frac{\sqrt{3}+\sqrt{5}}{3}, \frac{\sqrt{3}-\sqrt{5}}{4}$. **3.** 39900. **4** $t^{?}$.

5. (1) $\left\{\frac{n(n+1)}{2}\right\}^2$. (2) $\frac{n(n+1)(4n-1)}{6}$. (3) $\frac{5}{81}(10^{n+1}-10-9n)$

6. $7+\sqrt{15}$. **7.** $\{bc(c-a)+a^3\}^2+(a^2b+c^3)\{ac^2-b^2(c-a)\}=0$.

PAPER V.

1. 1. **2.** (i) $a, \frac{b(2a+b)}{a-b}$ (ii) $-a, b$. **4.** $\frac{1}{6}n(n+1)(7n-4)$

5. $\frac{x}{y}-\frac{y}{x}-\frac{1}{2}$.

PAPER VI.

1. (i) $-1, 1\frac{2}{3}$. (ii) $\sqrt{\frac{3 \pm \sqrt{3}}{2}}$. **2.** $\left(\frac{p}{q}\right)^{p-q}$. **4** 0. **5** 13
7. $\frac{1}{3}n(4n^2-1)$.

Paper VII.

1. (i) $\left(\dfrac{p}{q}\right)^{m-n}$ (ii) 1. **2.** $2x^{\frac{3}{4}}-3+4x^{-\frac{3}{4}}$. **3.** (i) 2, 0 ;
(ii) 7. **4.** (i) $\pm\sqrt{\frac{2}{3}}, \pm\sqrt{\dfrac{-5}{2}}$, (ii) $1, -1, \dfrac{-1\pm\sqrt{-3}}{2}$, $\dfrac{1\pm\sqrt{-3}}{2}$.

6. (i) $\frac{1}{3}n(4n^2+51n+20)$. (ii) $\frac{1}{6}n(n+1)(2n+7)$.

Paper VIII.

1. $\frac{1}{4}x^{\frac{3}{2}}-\frac{1}{3}x^{\frac{1}{2}}-\frac{1}{2}$. **2.** y^2. **3.** $x=\frac{1}{3}$. **4.** (i) -4. (ii) $-\frac{5}{2}$, 2, 3
5. 247050. **7.** (i) $\frac{1}{6}n(2n^2+3n+7)$. (ii) $\dfrac{n}{2(n+2)}$.